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THE GENERALIZED COHOMOLOGY THEORIES, BRUMFIEL-MADSEN FORMULA AND TOPOLOGICAL CONSTRUCTION OF BGG-TYPE OPERATORS

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Abstract. In this work, we investigate the topological construction of BGG-type operators, giving details about complex orientable theories, Becker-Gottlieb transfer and a formula of Brumfiel-Madsen. We generalize the BGG operators on the Morava K-theory and the others $\mathbb{F}_p$-generalized cohomology theories.

1. Introduction

In this work, we will discuss the generalized complex-oriented cohomology and homology theories of the flag space G/B, and the classical BGG and Kac operators will be constructed topologically using the transfer map for compact fiber bundles.

In order to do this, we will give some topological notations.

In the third section, we will discuss the Becker-Gottlieb map and transfer map for a fiber bundle $\pi\colon E \to B$ with the fiber F, which is a compact differentiable G-manifold for a compact Lie group G.

In the fourth section, we will examine the Brumfiel-Madsen formula for the transfer map.

In the last section, we will give the results of this work.

2. Topological preliminaries

The reference for this section is [1].

2.1. Generalities on generalized cohomology. A generalized cohomology theory $h^*(\,)$ is a contravariant functor from topological spaces to graded abelian groups which satisfies all the Eilenberg-Steenrod axioms except the dimension axiom. That is, the coefficients $h^* = h^*(pt)$ need not be concentrated in a single degree. We will always assume that h^* is multiplicative, and that the associated

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ring structure is commutative in the graded sense. Then for a topological space X, $h^*(X)$ is a h^*-module. The first example is ordinary cohomology with coefficients in $\mathbb{Z}$.

We take $H^i(X) = H^i(X, \mathbb{Z}) = [X, K(\mathbb{Z}, i)]$, where $K(\mathbb{Z}, i)$ is an Eilenberg-Maclane space, and $[X, Y]$ denotes homotopy classes of based maps from X to Y for X and Y topological spaces with based points.

For a generalized theory $h^*(\,)$, there is a spectral sequence which computes $h^*(X)$ in terms of $H^*(X; h^*)$. This spectral sequence is called the *Atiyah-Hirzebruch spectral sequence*, and details can be found in [1].

THEOREM 2.1. *There is a spectral sequence with E_2 term $H^p(X, h^q(pt)) \Longrightarrow h^{p+q}(X)$. The differential d_r is of bi-degree $(r, 1-r)$.*

COROLLARY 2.2. *Suppose that X has no odd dimensional cells and $h^q(pt) = 0$ for q odd. Then the Atiyah-Hirzebruch spectral sequence collapses at the E_2 term.*

Now we define reduced cohomology. Let $i\colon pt \to X$ be the inclusion of a point and $\pi\colon X \to pt$ be the collapsing map. Then $\pi \circ i = id$, so $i^* \circ \pi^* = id$ on $h^*(pt)$. Let $\tilde{h}^*(X) = \ker i^*$ be the reduced cohomology of X. Then, as a h^*-module,

$$h^*(X) = \tilde{h}^*(X) \oplus h^*.$$

2.2. Classifying spaces. In this section, we give some facts about the construction of universal bundles and classifying spaces of groups. The general reference for this section is [12]. Let G be a compact Lie group. There is a universal space EG with a free right G-action and homotopy groups $\pi_i(EG) = 0$ for all $i > 0$. Moreover, EG is a limit of Stiefel manifolds with the inductive limit topology. For example, for $G = U(n)$, the unitary group,

$$EU(n) = \lim_{m\to\infty} V_n(\mathbb{C}^{n+m}),$$

where $V_n(\mathbb{C}^{n+m}) = \frac{U(n+m)}{U(m)}$ is a Stiefel manifold. The classifying space BG is defined as EG/G. For $G = U(n)$,

$$BG \cong \lim_{m\to\infty} G_n(\mathbb{C}^{n+m}),$$

the Grassmannian manifold of n-planes.

We have the universal bundle (EG, p, BG) where $EG \xrightarrow{p} BG$ is the obvious projection map. Then BG has the following universal property.

THEOREM 2.3. *Let $P \xrightarrow{\rho} B$ be a right principal G-bundle. Then there exists a unique (up to homotopy) classifying map $f\colon B \to BG$ such that $f^*(EG) \cong P$ as principal G-bundles over B.*

As a consequence,

COROLLARY 2.4. *BG is well-defined up to homotopy and classifies induced vector bundles.*

Let $P \xrightarrow{\rho} B$ be a right principal G-bundle. Then, if F is a finite dimensional representation of G, $E = P \times_G F$ is the associated vector bundle over B with structure group G, where

$$E = P \times_G F = P \times F/\sim$$

is the space obtained as the quotient of the product space $P \times F$ by the relation

$$(x, y) \sim (xt, t^{-1}y), \quad t \in G, x \in P, y \in F.$$

THEOREM 2.5. *Let $E \to B$ be a vector bundle associated to the fibre F with structure group G. Then there exists $f \colon B \to BG$ with $f^*(EG \times_G F) \cong E$ as vector bundles over B.*

Consider the special case of the classifying space for a complex line bundle. The appropriate structure group is $U(1)$, so the appropriate classifying space is $BU(1)$. By the above construction,

$$BU(1) = \lim_{m \to \infty} \mathbb{C}P^m = \mathbb{C}P^\infty.$$

We know from [12] that $H^*(BU(1), \mathbb{Z}) = \mathbb{Z}[x]$, where $\mathbb{Z}[x]$ is the graded ring of polynomials in one variable with coefficients in $\mathbb{Z}$ and $\deg x = 2$. Let $T = \prod_{i=1}^{l} U(1)$ be a torus. Then,

$$BT = \prod_{i=1}^{l} BU(1),$$

and since $H^*(BU(1), \mathbb{Z})$ is torsion-free, by the Künneth formula, we have

$$H^*(BT, \mathbb{Z}) \cong \bigotimes_{i=1}^{l} H^*(BU(1), \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_l],$$

where $\mathbb{Z}[x_1, \dots, x_l]$ is the graded ring of polynomials in l variables with coefficients in the ring $\mathbb{Z}$.

2.3. Complex orientable cohomology theories. We follow [1] in this discussion.

Let $i \colon \mathbb{C}P^1 \to \mathbb{C}P^\infty = BU(1)$ be the inclusion.

DEFINITION 2.6. We say that the multiplicative cohomology theory h^* is *complex oriented* if there exists a class $x \in \tilde{h}^*(\mathbb{C}P^\infty)$ such that $i^*(x)$ is a generator of $\tilde{h}^*(\mathbb{C}P^1)$ over the ring $h^*(pt)$. Such a class x is called a *complex orientation.*

$\tilde{h}^*(\mathbb{C}P^1) \cong \tilde{h}^*(\mathbb{S}^2)$ is generated by one element over $h^*(pt)$.

As an example, if $h^* = H^*$, then x can be taken as a ring generator of $H^*(\mathbb{C}P^\infty, \mathbb{Z})$, so $x \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$. $\mathbb{C}P^\infty$ has a universal line bundle L_λ given as follows. Let e^λ be the one-dimensional representation of $\mathbb{T} = U(1)$ given by

$$\mathrm{e}^\lambda(\mathrm{e}^{i\theta}) \cdot v = \mathrm{e}^{i\theta} \cdot v,$$

where $\lambda \in \mathrm{Lie}(T)$ is a fundamental weight. Then, for a complex orientable theory h^* with orientation given by x, the first Chern class is given by $x = c_1(L_\lambda)$, where L_λ is the line bundle associated to e^λ. Let T be an l-dimensional torus.

THEOREM 2.7. [1] *With the above notation, we have isomorphisms of graded h^*-algebras*

$$\begin{aligned} h^*(\mathbb{C}P^\infty) &\cong h^*(pt)[[x]], \\ h^*(BT) &\cong h^*(pt)[[x_1, \dots, x_l]], \\ h^*(\mathbb{C}P^n) &\cong h^*(pt)[[x]]/(x^{n+1}), \\ h^*(\prod_{i=1}^{l} \mathbb{C}P^{n_i}) &\cong h^*(BT)/(x_1^{n_1+1}, \dots, x_l^{n_l+1}). \end{aligned}$$

Now let $\pi \colon L \to X$ be a line bundle over X. Then L induces a classifying map $\theta \colon X \to \mathbb{C}P^\infty$. Then the first Chern class of L is $c_1(L) = \theta^*(x)$. Next we define the top Chern class of a vector bundle.

DEFINITION 2.8. Let $\pi \colon E \to X$ be a vector bundle. If there is a space Y and a map $f \colon Y \to X$ such that $f^* \colon h^*(X) \to h^*(Y)$ is injective and $f^*(E) \cong \bigoplus L_i$, where L_i are line bundles on Y, f is called a *splitting map* for π.

From [12],

THEOREM 2.9. *If $\pi \colon E \to X$ is a vector bundle, there exists a splitting map of π.*

Then,

DEFINITION 2.10. The top Chern class $c_n(E)$ where $\dim E = n$, which also will be referred as the Euler class $\chi(E)$, is defined by the formula

$$f^*(c_n(E)) = \prod_i c_1(L_i),$$

where f is a splitting map for π.

2.4. Formal group laws. Let $\mathbb{F}$ be a commutative ring with unit.

DEFINITION 2.11. A *formal group law* over $\mathbb{F}$ is a power series $F(x, y)$ over $\mathbb{F}$ that satisfies the following conditions:

1. $F(x, 0) = F(0, x) = x$,
2. $F(x, y) = F(y, x)$,
3. $F(F(x, y), z) = F(x, F(y, z))$,
4. there exists a series $i(x)$ such that $F(x, i(x)) = 0$.

From [17], we have

THEOREM 2.12. *In a complex oriented theory, for line bundles L, M, we have*

$$c_1(L \otimes M) = F(c_1(L), c_1(M))$$

where F is a formal group law over the coefficient ring h^*.

A line bundle L over a space X is equivalent to a homotopy class of maps $f_L: X \to \mathbb{C}P^\infty$. Let L and M be two line bundles. Then we have

$$f_L \times f_M : X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty .$$

$\mathbb{C}P^\infty$ has an H-space structure $m: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. The homotopy class of $m \circ (f_L \times f_M)$ is then equivalent to the tensor product $L \otimes M$. There is an induced map $m^*: h^*(\mathbb{C}P^\infty) \to h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Since, $h^*(\mathbb{C}P^\infty) \cong h^*(pt)[[x]]$ and $h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong h^*(pt)[[x_1, x_2]]$, m^* has the form,

$$m^*(x) = \sum a_{ij} {x_1}^i {x_2}^j = F(x_1, x_2).$$

Then $c_1(L \otimes M) = F(c_1(L), c_1(M))$. As an example, if L and M are line bundles, we see in ordinary cohomology $H^*(\quad)$ that

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

The complex cobordism MU^* is the universal cohomology theory with respect to push-forwards. From [1],

Theorem 2.13. *The complex cobordism ring MU^* is $\mathbb{Z}[x_1, x_2, \dots]$ where $\dim x_i = 2i$. The formal group law of MU^* is the Lazard's universal formal group law.*

3. The Becker-Gottlieb map and transfer.

The general reference for this section is [2].

Let $\pi: E \to B$ be a fiber bundle with the fiber F, which is a compact differentiable G-manifold for a compact Lie group G. For any cohomology theory h^* we have the induced map $\pi^*: h^*(B) \to h^*(E)$. A transfer map is a backwards map $h^*(E) \to h^*(B)$. Here, we will give a technique for producing a transfer map.

Definition 3.1. Let $\xi \to B$ be a vector bundle. Let $D(\xi) = \{x \in \xi : |x| \leqslant 1\}$ and $S(\xi) = \{x \in \xi : |x| = 1\}$ be the disk and sphere bundles respectively. Then, $B\xi = D(\xi)/S(\xi)$ is called the *Thom space* of the vector bundle ξ.

Now we give the useful propositions from [12],

Proposition 3.2. *If $\xi \to B$ is a trivial n dimensional vector bundle, then the Thom space $B\xi = \Sigma^n B^+$, where B^+ is the union of B with a point.*

Proposition 3.3. *If ξ and η are two vector bundles over B, then $B\xi \wedge B\eta = B(\xi \oplus \eta)$.*

We define transfer for the map from the fiber F to a point. We can embed F equivariantly into a real G-representation V of dimension r such that $r \gg \dim F$. Let $N \to F$ be the normal bundle of the embedding. By the tubular neighbourhood theorem, we can identify the normal bundle N with a neighbourhood U of F

by a diffeomorphism φ. There is an associated Pontryagin-Thom collapsing map $c\colon S_V \to F_N$, where S_V is the one point compactification of V, defined by

$$c(x) = \begin{cases} \text{base point of } F^N & \text{if } x \notin U, \\ \varphi(x) & \text{if } x \in U. \end{cases}$$

Let $T(F)$ be the tangent bundle of F. Then we can identify $T(F) \oplus N$ with the trivial bundle $F \times V$. There is an inclusion $i\colon N \to N \oplus T(F) \cong F \times V$ and hence we have an inclusion of Thom spaces $i\colon F_N \to S_V \wedge F^+$.

DEFINITION 3.4. The transfer τ to a point is the composition $\tau = i \circ c$.

Let $\pi\colon E \to B$ be a fiber bundle associated to the principle G-bundle $p\colon P \to B$. Then the transfer to a point gives a map

$$\mathrm{Id} \times \tau\colon P \times_G S_V \to P \times_G (F \times V)^+.$$

When we collapse the section at ∞ to a point, which is equivalent to taking Thom spaces, we get a map $t\colon B\xi \to B\pi^*(\xi)$ where ξ is a vector bundle associated to the representation V. Then there is a map $t \wedge \mathrm{Id}\colon B\xi \wedge B\overline{\xi} \to B\pi^*(\xi) \wedge B\overline{\xi}$ where $\overline{\xi}$ is the complementary bundle of of ξ. If we restrict to the diagonal Δ in $B \times B$, we have transfer map

$$\tau(\pi)\colon \Sigma^m B^+ \to \Sigma^m E^+.$$

4. The Brumfiel-Madsen formula for transfer

The general reference for this section is [6].

Let G be a compact connected semi-simple Lie group with maximal torus T. Let H be a closed connected subgroup of G containing the maximal torus T. Let W_G and W_H be the Weyl groups of G and H respectively. Suppose that $P \to B$ is a principal G-bundle. We have associated bundles

$$\pi_1\colon E_1 = P \times_G G/T \to B$$
$$\pi_2\colon E_2 = P \times_G G/H \to B.$$

Then there is a fibration $\pi\colon E_1 \to E_2$ with the fiber H/T. Since the Weyl group W_G acts on G/T, W_G also acts on E_1. The Weyl group W_H of H also acts on E_1 over E_2. Thus, cosets $w \in W_G/W_H$ define maps $\pi \circ w$ on E_1.

THEOREM 4.1. *We have*

$${\pi_1}^* \circ \tau(\pi_2)^* = \sum_{w \in W_G/W_H} w \circ \pi^*.$$

COROLLARY 4.2. *If we choose $H = T$, we get*

$${\pi_1}^* \circ \tau(\pi_1)^* = \sum_{w \in W_G} w.$$

Although Brumfiel and Madsen were the first to assert that Theorem 4.1 is true, their proof was wrong. In [10] Feshbach, and in [15] Lewis, May and Steinberger have given different proofs of Theorem 4.1. Since EG is the universal space for G, we have the principle bundle $EG \to BG$.

COROLLARY 4.3. *Let $BT \to BG$ be the fiber bundle with the fiber G/T. Then*

$$\pi^* \circ \tau(\pi)^* = \sum_{w \in W_G} w.$$

For a compact semi-simple Lie group G, any root α defines a subgroup $M_\alpha = K_\alpha \cdot T$ such that the complexified Lie algebra $\boldsymbol{m}_\alpha$ contains the root spaces $\boldsymbol{g}_\alpha$ and $\boldsymbol{g}_{-\alpha}$ where K_α already has been defined in [13]. The induced fiber bundle $\pi_i\colon BT \to BM_i$ has fiber $M_i/T \cong SU_2/T \cong \mathbb{C}P^1$. Then

COROLLARY 4.4.

$$\pi_i{}^* \circ \tau(\pi_i)^* = 1 + r_{\alpha_i},$$

if r_{α_i} is the reflection to corresponding to the simple root α_i.

5. The transfer and the Gysin homomorphism

Let $\xi\colon E \to X$ be a vector bundle and h^* be the complex oriented theory. Then there is the associated Thom class $u \in h^*(B\xi)$. From [4], we have

THEOREM 5.1. *The Thom map $\Phi\colon h^*(X) \to h^*(X\xi)$ given by $\Phi(x) = u \cdot \pi^*(x)$ is an isomorphism.*

Let $\pi\colon E \to B$ be a fiber bundle with compact smooth f-dimensional fiber F. Suppose that the tangent bundle $TF \to F$ is a complex vector bundle. Then we have the Gysin homomorphism $\pi_*\colon h^k(E) \to h^{k-f}(B)$. Since the tangent bundle $T(F)$ has complex structure, so does the tangent space along the fibers T_π. Hence, in the complex orientable theory h^*, T_π has an Euler class, so $\chi(T_\pi) = c_n(T_\pi)$.

THEOREM 5.2. (see [2]) *The transfer $\tau(\pi)^*\colon h^k(E) \to h^k(B)$ is given by $\tau(\pi)^*(x) = \pi_*(x \cdot \chi(T_\pi))$.*

Let L_α be the line bundle on BT associated to the character e^α where α is a root. We want to determine when its characteristic classes are not zero divisors. We know that the characters e^α do not usually generate the representation ring $R(T)$. Let λ_i be the fundamental weight corresponding to the simple root α_i such that $\lambda_i(h_{\alpha_i}) = 1$, where h_{α_i} is the coroot. Then

THEOREM 5.3. [12] *These e^{λ_i} generate the representation ring $R(T)$.*

By Theorem 2.7,

$$h^*(BT) \cong h^*(pt)[[c_1(L_{\lambda_1}), \cdots, c_1(L_{\lambda_l})]]$$

where l is the rank of the compact Lie group G. Since $\chi(L_{\lambda_i})$ are generators of $h^*(BT)$, the $\chi(L_{\lambda_i})$ are not zero-divisors in $h^*(BT)$. This implies that $\chi(L_{\lambda_i})$ is not nilpotent. We know that for any weight $\lambda \in \boldsymbol{h}^*$, λ can be written as $\lambda = \sum_{i=1}^{l} n_i \lambda_i$, where n_i is the multiplicity number. Using the formal group law in h^*, the Euler class $\chi(L_\lambda)$ of the line bundle L_λ in h^* is equal to

$$\sum_{i=1}^{l} n_i c_1(L_{\lambda_i}) + \text{higher order terms}.$$

If n_i is not a zero-divisor in $h^*(pt)$, then $\chi(L_\lambda)$ is not a zero-divisor in $h^*(BT)$. If the weight λ is a root corresponding to the adjoint representation, the multiplicity numbers n_i in the sum are the Cartan integers. By an examination of the Cartan matrices, we have

PROPOSITION 5.4. *If $p \geqslant 3$ is a prime, there is some n_i such that p does not divide n_i.*

COROLLARY 5.5. *If $h^*(pt)$ has no 2-torsion and 3-torsion, then the Euler class $\chi(L_{\alpha_i})$ is not a zero-divisor for any simple root α_i.*

Since every root is the image of a simple root by an element of the Weyl group W_G and the Weyl group acts by automorphism on $h^*(BT)$, we have

COROLLARY 5.6. *If $h^*(pt)$ has no 2-torsion and 3-torsion, then the Euler class $\chi(L_\alpha)$ is not a zero-divisor for any root α.*

Now we want to give the Brumfiel-Madsen formula for the Gysin map of the fibration $\pi\colon BT \to BG$ with the fiber G/T.

We need a complex structure on G/T. We know that the smooth manifold G/T is diffeomorphic to the complexified space $G_{\mathbb{C}}/B$ where B is a Borel group. Then we can determine the tangent bundle of the fiber $G_{\mathbb{C}}/B$. The tangent bundle $T(G_{\mathbb{C}}/B)$ is isomorphic to $G \times_T \boldsymbol{g}/\boldsymbol{b}$,where $mathbfg$ is the complexified Lie algebra of G and $\boldsymbol{b}$ is the Borel subalgebra of $\boldsymbol{g}$. Using the adjoint representation of T, we have

$$\boldsymbol{g} = \boldsymbol{b} \oplus \bigoplus_{\alpha \in \Delta^+} \boldsymbol{g}_{-\alpha},$$

where Δ^+ is the set of positive roots corresponding to B. Thus

$$\boldsymbol{g}/\boldsymbol{b} = \bigoplus_{\alpha \in \Delta^+} \boldsymbol{g}_{-\alpha}.$$

Therefore the tangent bundle along the fiber G/T is

$$T_\pi = EG \times_T \boldsymbol{g}/\boldsymbol{b} \cong \bigoplus_{\alpha \in \Delta^+} L_{-\alpha},$$

where $L_{-\alpha}$ is as above. We know that

$$\chi^n(T_\pi) = \prod_{\alpha \in \Delta^+} c_1(L_{-\alpha}),$$

where $\prod$ is the cup product in any complex orientable theory h^*. By Theorem 5.2, we have

$$\pi^* \circ \tau(\pi)^*(x) = \pi^* \circ \pi_*(x \cdot \chi(T_\pi))$$

for $x \in h^*(BT)$. Since $\chi(T_\pi)$ is a product of the non-zero divisors in $h^*(BT)$, we have

THEOREM 5.7. (see [4]) *For* $x \in h^*(BT)$,

$$\pi^* \circ \pi_*(x) = \sum_{w \in W} w \left(\frac{x}{\prod \chi(L_{-\alpha})} \right),$$

here the right hand side is in a localization $h^*(BT)[\frac{1}{\prod \chi(L_{-\alpha})}]$.

But since the left hand side preserves the subring $h^*(BT)$, it may be regarded as an identity on $h^*(BT)$.

COROLLARY 5.8. *If* $\chi(L_{-\alpha})$ *is a non-zero divisor, for the fibration* $\pi_i \colon BT \to BM_i$ *with the fiber* M_i/T,

$$D_i(x) = {\pi_i}^* \circ \pi_{i*}(x) = (1 + r_i) \left(\frac{x}{\chi(L_{-\alpha})} \right).$$

Let h^* be the ordinary cohomology with complex coefficients. From chapter 1 of [16], we know that there is an isomorphism $\Theta \colon \boldsymbol{h}^* \to H^2(BT, \mathbb{C})$ given by $\lambda \to \chi(L_\lambda)$, where $\boldsymbol{h}^*$ is the dual Cartan subalgebra of semi-simple Lie algebra.

The isomorphism Θ extends to an inclusion of the symmetric algebra $R = S(\boldsymbol{h}^*)$ into $H^*(BT, \mathbf{C})$. Then

$$H^*(BT, \mathbb{C}) \cong \mathbb{C}[\lambda_1, \ldots, \lambda_l]$$

under the identification $\chi(L_{\lambda_i}) = \lambda_i$.

In [3], Bernstein-Gelfand-Gelfand introduced certain operators

$$\frac{1}{\alpha}(r_i - 1) \colon H^k(BT) \to H^{k-2}(BT)$$

where r_i is the simple reflection associated to the simple root α_i.

When $G = M_i$,

COROLLARY 5.9.

$$D_i = \frac{1}{\alpha}(r_i - 1)$$

is just the classical BGG operator.

If we apply Theorem 5.7 to K-theory, for $G = M_i$, the formula $D_i = {\pi_i}^* \circ \pi_{i*}$ in K-theory gives the Demazure operator. We map the representation ring $R(T)$ to $K(BT)$ by mapping e^λ to $[L(\lambda)]$, the class of the line bundle defined by λ. In K-theory, we can take $\chi L(\lambda) = [1] - [L(\lambda)]$, $[1]$ is the class of the trivial line bundle. In the case where $G = M_i$ is rank one, D_i is the Demazure operator.It has the form

$$D_i = \frac{1}{1 - \mathrm{e}^{-\alpha_i}}(1 - \mathrm{e}^{-\alpha_i}\, r_i).$$

Now, we will apply this result to BP-theory and Morava K-theory. In order to do this, we will give some definitions. The BP-theory began with Brown and Peterson [5] who showed that after localization at any prime p, the MU spectrum splits into an infinite wedge suspension of identical smaller spectra subsequently called BP. We have

$$BP^* = \mathbb{Z}_p[v_1, v_2, \ldots,]$$

where $\mathbb{Z}_p$ denotes the integers localized at p and $\dim v_n = 2(p^n - 1)$.

DEFINITION 5.10. A formal group law over a torsion free $\mathbb{F}_p$-algebra is *p-typical* if its logarithm has the form $\sum_{i \geqslant 0} \lambda_i x^{p^i}$ with $\lambda_0 = 1$.

This definition works even when the $\mathbb{F}_p$-algebra R has torsion. A formal group law F over a $\mathbb{F}_p$-algebra is *p-typical* if $f_q(x) = 0$ for all primes $q \neq p$.

By a Cartier Theorem [7], we have

THEOREM 5.11. *Every formal group law over a $\mathbb{F}_p$-algebra is canonically strictly isomorphic to a p-typical one.*

Let F be a formal group law over commutative ring with unit R.

DEFINITION 5.12. For each n, the *n-series* $[n](x)$ of F is given by

$$\begin{aligned} [1](x) &= x, \\ [n](x) &= F(x, [n-1](x)) \quad \text{for } n > 1, \\ [-n](x) &= i([n](x)). \end{aligned}$$

Of particular interest is the p-series, where p is a prime. In characteristic p it always has leading term ax^q where $q = p^h$ for some integer h. This leads to the following.

DEFINITION 5.13. Let $F(x, y)$ be a formal group law over an $\mathbb{F}_p$-algebra. If $[p](x)$ has the form

$$[p](x) = ax^{p^h} + \text{higher terms}$$

with a invertible, then we say that F has *height* h at p. If $[p](x) = 0$ then the height is infinity.

Now we give the construction the universal p-typical formal group law.

THEOREM 5.14. *Let $V = \mathbb{F}_p[v_1, v_2, \ldots,]$ with $\dim v_n = 2(p^n - 1)$. Then there is a universal p-typical formal group law F over V.*

THEOREM 5.15. *The height of a formal group law is an isomorphism invariant.*

Suppose that h^* is an $\mathbb{F}_p$-algebra and the formal group law F has the height h. Since the elements $x = \chi(L_{\lambda_i}) \in h^*(BT)$ are non-zero divisors, $[p](x)$ has the form

$$[p](x) = ax^{p^h} + \text{higher terms}, \quad (a \text{ is a unit.})$$

This leads us to mod p K-theory and the Morava K-theories. The reference for these cohomology theories is [17]. We give some imporant special cases of Corollaries 6 and 7.

THEOREM 5.16. *For any prime p, in $K(n)^*(BT)$, the Euler class $\chi(L_{\alpha_i})$ is not a zero divisor for any simple root α_i.*

THEOREM 5.17. *For any prime p, in $K(n)^*(BT)$, the Euler class $\chi(L_\alpha)$ is not a zero divisor for any root α.*

Let $\pi\colon BT \to BG$ is a fiber bundle with the fiber G/T.

THEOREM 5.18. *For $x \in K(n)^*(BT)$,*

$$\pi^* \circ \pi_*(x) = \sum_{w \in W} w\left(\frac{x}{\prod \chi(L_{-\alpha})}\right),$$

here the right hand side is in a localization $K(n)^*(BT)[\frac{1}{\prod \chi(L_{-\alpha})}]$.

COROLLARY 5.19. *Let $\pi_i\colon BT \to BM_i$ be a fiber bundle with the fiber M_i/T. For $x \in K(n)^*(BT)$,*

$$D_i(x) = {\pi_i}^* \circ \pi_{i*}(x) = (1 + r_i)\left(\frac{x}{\chi(L_{-\alpha})}\right).$$

Of course, these results can be generalized to a $\mathbb{F}_p$-algebra h^* which has a formal group law F with the height n.

In this section, so far we have concentrated our attention on BT. Now, we will give some interesting results about the flag variety G/T. Since the cohomology of G/T vanishes in odd degrees, Corollary 1 gives

CORROLAY 5.20. *Let h^* be any complex oriented cohomology theory. Then the Atiyah-Hirzebruch spectral sequence for G/T collapses at the E_2-term.*

Let $\pi_i\colon BT \to BM_i$. Since G/T is a T-principal bundle, there is a classifying map $\theta\colon G/T \to BT$. Similarly there is a classifying map $\theta_i\colon G/M_i \to BM_i$. The following diagram is a cartesian square.

$$\begin{array}{ccc} G/T & \xrightarrow{\theta} & BT \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \pi_i} \\ G/M_i & \xrightarrow{\theta_i} & BM_i \end{array}$$

Let $C_i = {p_i}^* \circ p_{i*}$. Then $\theta^* \circ D_i = C_i \circ \theta^*$. The following theorem gives a topological description of the operator C_i. From [9],

THEOREM 5.21. *If $h^*(pt)$ contains $\mathbb{Z}[\frac{1}{|W_G|}]$, then θ^* is surjective.*

DEFINITION 5.22. For $i = 1, \ldots, l$, let D_i be the linear operator associated to the simple root α_i. Then we say that D_i satisfy *braid relations* if

$$(D_i D_j D_i)^{m_{ij}} = (D_j D_i D_j)^{m_{ij}},$$

where m_{ij} is the number of factors in each side for all pairs i and j.

For any Kac-Moody group, since T and M_i are still finite dimensional compact groups, we can define D_i operators on $h^*(BT)$.

THEOREM 5.23. *Let G be a Kac-Moody group and let h^* be torsion-free. Then the operators D_i satisfy braid relations if and only if the formal group law is polynomial.*

Proof. There are three cases to consider. These cases are when two non-orthogonal roots α_i and α_j have $m_{ij} = 3, 4$, or 6. In the finite dimensional case, the reference for case $m_{ij} = 3$ is [4] and for the remaining two cases the reference is [11]. In the affine case, it can be done similar way. ∎

This theorem tells that the ordinary cohomology and K-theory satisfy braid relations but cobordism and elliptic cohomology and another complex oriented cohomology theories do not satisfy.

By Theorems 5.21 and 5.23, we have

THEOREM 5.24. *The operators C_i satisfy braid relations for ordinary cohomology and K-theory.*

Now we will give our result about the infinite dimensional flag variety. Let G be an affine Kac-Moody group and K be the unitary form of G. For every simple root α_i, let $M_i = K_i \cdot T$. We have a principal M_i-bundle $K \to K/M_i$, and the associated fiber bundle $K/T \to K/M_i$ with fiber M_i/T. M_i/T is diffeomorphic to the complex projective space $\mathbb{C}P^1$.

THEOREM 5.25. *Let $\pi_i \colon K/T \to K/M_i$ be the fiber bundle with the compact fiber $\mathbb{C}P^1$ and let $\mathbb{F}$ be a commutative ring with unit. For $x \in H^*(K/T, \mathbb{F})$,*

$$O_i(x) = \pi_i^* \circ \pi_{i*}(x) = -(1 + r_i)\left(\frac{x}{\varepsilon^{r_i}}\right),$$

here the right hand side is in the localization $H^*(BT)[\frac{1}{\prod \chi(L_{-\alpha})}]$. In fact O_i is the Kac operator which was introduced in [13].

Proof. By the Brumfiel-Madsen formula and Theorem [10], we have the following identity

$$\pi_i^* \circ \tau(\pi_i)^*(x) = \pi_i^* \circ \pi_{i*}(\psi(-\chi_i) \cdot x) = (1 + r_i)(x),$$

where r_i is the simple reflection associated to α_i and χ_i is the fundamental weight corresponding to the simple root α_i. Let $x \in H^*(K/T, \mathbb{F})$. We know from [14] that $\psi(\chi_i) = \varepsilon^{r_i}$ where $\psi \colon S(\boldsymbol{h}^*) \to H^*(K/T, \mathbb{F})$. In $H^*(K/T, \mathbb{F})$, we know that the element ε^{r_i} is a non zero-divisor, so we can define the local ring $H^*(K/T,\mathbb{F})[\frac{1}{\varepsilon^{r_i}}]$. Then, we have the following identity in the local ring $H^*(K/T,\mathbb{F})[\frac{1}{\varepsilon^{r_i}}]$,

$$\pi_i^* \circ \pi_{i*}(x) = -(1+r_i)\left(\frac{x}{\varepsilon^{r_i}}\right).$$

Since the left hand side of the identity is an element of $H^*(K/T,\mathbb{F})$, we are done. ■

We know from [13] that the Kac operators satisfy braid relations for all affine Kac-Moody groups.

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