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OPTIMALITY CONDITIONS AND TOLAND'S DUALITY FOR A NONCONVEX MINIMIZATION PROBLEM

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Abstract. This paper studies necessary and sufficinet conditions and provides a duality theory for a wide class of problems arising in nonconvex optimization, such as minimizing a difference of two convex functions subject to a convex vector constraint taking values in an ordered topological vector space. These results are then used to study a problem of nondifferentiable optimization.

1. Introduction

Many of the nonconvex minimization problems arising in applied mathematics, operations research and mathematical programming can be formulated as the following nonconvex problem

$$(\mathcal{P}) \quad \begin{cases} \inf f_1(x) - f_2(x) \\ h(x) \in -Y_+ \\ x \in C, \end{cases}$$

where X and Y are two real topological vector spaces and Y is equipped with a preorder induced by a convex cone Y_+, $f_1, f_2 \colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$ are two convex functions and $h\colon X \longrightarrow Y \cup \{+\infty\}$ is a convex mapping and C is a nonempty convex subset of X. The problem $(\mathcal{P})$ includes a wide family of DC-minimization problems subject to vector constraint taking values in finite or infinite dimensional spaces. The purpose of this paper is to present the optimality conditions related to the problem $(\mathcal{P})$ and to formulate its Toland's dual problem.

The main tools used to deal with this class of problems are the formulas of the subdifferential and Legendre-Fenchel conjugate function of the composition of a nondecreasing convex function with a convex mapping taking values in an ordered topological vector space (see [1] and [2]).

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This paper is organised as follows. In section 2 we give some notations and recall some definitions and results. Section 3 studies necessary and sufficient optimality conditions linked to problem $(\mathcal{P})$. In Section 4, we deal with the dual problem of $(\mathcal{P})$ in the sense of Toland's duality. In Section 5, we give an applications to nondifferentiable fractional programming problem.

2. Notations and Preliminaries

All the vector spaces introduced in the sequel are real spaces. By X and Y we denote two topological vector spaces with respective topological duals X^* and Y^*. The canonical bilinear form on $X^* \times X$ (resp. $Y^* \times Y$) is denoted by $\langle \cdot, \cdot \rangle$. The space Y will be assumed to be equipped with a preorder $\leq_Y$ induced by closed convex cone Y_+ as follows

$$y_1 \leq_Y y_2 \iff y_2 - y_1 \in Y_+,$$

and an abstract maximal element $+\infty$ will be adjoined to Y. By Y_+^* we denote the dual positive cone, i.e., the cone of positive linear functionals on Y given by

$$Y_+^* := \{ y^* \in Y^* : \langle y^*, y \rangle \geq 0, \quad \forall y \in Y_+ \}.$$

Recall that an operator $H \colon X \longrightarrow Y \cup \{+\infty\}$ is said to be Y_+-convex if

$$H(\alpha x_1 + (1-\alpha)x_2) \leq_Y \alpha H(x_1) + (1-\alpha)H(x_2)$$

for each x_1, x_2 in X and for each $\alpha \in [0,1]$. By $\operatorname{dom} H := \{ x \in X : H(x) \in Y \}$, $\operatorname{Epi} H := \{ (x,y) \in X \times Y : H(x) \leq_Y y \}$ and $\operatorname{Im} H := H(\operatorname{dom} H)$ we denote, respectively, the effective domain, the epigraph and the effective range of H. A function $G \colon Y \longrightarrow \mathbf{R} \cup \{+\infty\}$ is said to be Y_+-nondecreasing on Y if

$$y_1 \leq_Y y_2 \implies G(y_1) \leq G(y_2).$$

Given $F \colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$, the subdifferential of F at $\overline{x} \in \operatorname{dom} F$, denoted by $\partial F(\overline{x})$, is defined as

$$\partial F(\overline{x}) := \{ x^* \in X^* : F(x) \geq F(\bar{x}) + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X \}$$

and its conjuguate function $F^* \colon X^* \longrightarrow \mathbf{R} \cup \{+\infty\}$ is defined by

$$F^*(x^*) := \sup_{x \in X} \{ \langle x^*, x \rangle - F(x) \}.$$

If C is a nonempty subset of X, then the cone that it generates is

$$\mathbf{R}_+ C := \bigsqcup_{\lambda \geq 0} \lambda C,$$

its indicator function is $\delta_C \colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$ defined for every $x \in X$ by

$$\delta_C(x) := \begin{cases} 0, & \text{if } x \in C \\ +\infty, & \text{otherwise,} \end{cases}$$

and its support function δ_C^* defined on the dual space X^* is

$$\delta_C^*(x^*) := \sup_{x \in C} \langle x^*, x \rangle.$$

Also, we define the cone of normal directions to C at $x_0 \in C$ by

$$N_C(\bar{x}) := \{ x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq 0, \ \forall x \in C \}.$$

Throughout, we will adopt two conventions; the first is $(+\infty) - (+\infty) = +\infty$ and the other is linked to the composite function, that is

$$x \in X \mapsto (G \circ H)(x) := \begin{cases} G(H(x)), & \text{if } x \in \operatorname{dom} H \\ +\infty, & \text{otherwise.} \end{cases}$$

Now, let us recall some definitions and results that will be used throughout the paper.

DEFINITION 2.1. [4]

(i) A function $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ is called a DC-function if it can be represented as a difference of two convex functions defined on X.

(ii) $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ is called a polyhedral convex function if

$$f(x) = \max_{1 \leq i \leq n} \{ \langle x_i^*, x \rangle + b_i \}, \quad \forall x \in X$$

where $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $b_1, b_2, \ldots, b_n \in \mathbf{R}$.

In order to establish our main results we shall need the following results due respectively to J. B. Hiriarty Urruty [4] and J. F. Toland [5]. These results established respectively the local optimality conditions and Toland's dual problem related to an unconstrained DC-mathematical programming problem.

PROPOSITION 2.1. [4] *(i) Let us consider two convex functions f_1 and $f_2 : X \longrightarrow \mathbf{R} \cup \{+\infty\}$. The condition $\partial f_2(\overline{x}) \subset \partial f_1(\overline{x})$ is necessary for $\overline{x}$ being a local minimum of $f = f_1 - f_2$ on X.*

(ii) If, furthermore, we assume that f_2 is a polyhedral function, then the condition $\partial f_2(\overline{x}) \subset \partial f_1(\overline{x})$ is necessary and sufficient for $\bar{x}$ being a local minimum of $f = f_1 - f_2$ on X.

PROPOSITION 2.2. [5] *Let $f_1 : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be any function and $f_2 : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function. We have*

$$\inf_{x \in X} \{ f_1(x) - f_2(x) \} = \inf_{x^* \in X^*} \{ f_2^*(x^*) - f_1^*(x^*) \}.$$

We finish this section by recalling some formulas established in [1] (see also [2]) by C. Combari, M. Laghdir and L. Thibault, concerning the computation of the subdifferential and conjugate function of the composition of a nondecreasing convex function with a convex mapping taking values in a partially ordered topological vector space. For this, let us consider the following constraint qualification

$$(C.Q.A_0.B_0)\quad \begin{cases} X \text{ and } Y \text{ are Fréchet spaces,} \\ G \text{ is convex, proper, lower semicontinuous and} \\ \quad Y_+\text{-nondecreasing on } \operatorname{Im} h + Y_+, \\ H \text{ is } Y_+\text{-convex, proper with closed epigrah in } X \times Y, \\ \mathbf{R}_+[\operatorname{dom} G - \operatorname{Im}(domF \cap \operatorname{dom} H)] \text{ is a closed vector subspace.} \end{cases}$$

THEOREM 2.1. [1] *If the condition* $(C.Q.A_0.B_0)$ *holds then we have:*
i) $\partial(F + G \circ H)(\bar{x}) = \bigsqcup_{y^* \in \partial G(H(\bar{x}))} \partial(F + y^* \circ H)(\bar{x})$, *for any* $\bar{x} \in X$ *and*
ii) $(F + G \circ H)^*(x^*) = \min_{y^* \in Y_+^*} [G^*(y^*) + (F + y^* \circ H)^*(x^*)]$, *for any* $x^* \in X^*$.

REMARK 2.1. The above proposition holds also (see [1]), if we suppose, instead of the condition $(C.Q.A_0.B_0.)$, that X and Y are locally convex topological real linear spaces and G is finite and continuous at some point $\bar{y} = h(\bar{x})$ where $\bar{x} \in C \cap \operatorname{dom} f_1 \cap \operatorname{dom} h$.

3. Optimality conditions

In order to establish the optimality conditions related to problem $(\mathcal{P})$ we will need the following constraint qualification

$$(C.Q.A_1.B_1) \begin{cases} X \text{ and } Y \text{ are Fréchet spaces,} \\ f_1 \text{ and } f_2 \text{ are convex, proper and lower semicontinuous} \\ h \text{ is convex and } \operatorname{Epi} h \text{ is a closed subset of } X \times Y \\ \mathbf{R}_+[Y_+ + h(\operatorname{dom} f_1 \cap C \cap \operatorname{dom} h)] \text{ is a closed vector subspace.} \end{cases}$$

Now, we are ready to state the necessary optimality conditions.

PROPOSITION 3.1. *Let* $\bar{x}$ *be a feasible point of* $(\mathcal{P})$, *i.e.,* $\bar{x} \in C \cap h^{-1}(-Y_+)$. *We suppose that the constraint qualification* $(C.Q.A_1.B_1)$ *is satisfied. If* $\bar{x}$ *is a local minimum for the problem* $(\mathcal{P})$ *then for each* $x^* \in \partial f_2(\bar{x})$ *there exists some* $y^* \in Y_+^*$ *satisfying* $\langle y^*, h(\bar{x})\rangle = 0$ *and* $x^* \in \partial(f_1 + \delta_C + y^* \circ h)(\bar{x})$.

Proof. First, let us notice that the function $y \mapsto \delta_{-Y_+}(y)$ is convex and Y_+-nondecreasing on the whole space (see also [1]) and with the convexity of mapping h we obtain that the composite function is also convex.

It is easy to see that a feasible point $\bar{x}$ of $(\mathcal{P})$ is a local minimum of $(\mathcal{P})$ if and only if $\bar{x}$ is a local minimum of the following unconstrained DC-minimization problem $(\mathcal{Q})$

$$(\mathcal{Q})\quad \inf_{x \in X} \{f_1(x) + \delta_C(x) + (\delta_{-Y_+} \circ h)(x) - f_2(x)\}.$$

Hence, by applying Proposition 2.1, it follows that

$$\partial f_2(\bar{x}) \subset \partial(f_1 + \delta_C + \delta_{-Y_+} \circ h)(\bar{x}),$$

and according to Theorem 2.1, the condition $(C.Q.A_1.B_1)$ ensures that

$$\partial f_2(\bar{x}) \subset \bigsqcup_{\substack{y^* \in N(h(\bar{x}), -Y_+) \\ \langle y^*, h(\bar{x})\rangle = 0}} \partial(f_1 + \delta_C + y^* \circ h)(\bar{x}),$$

i.e., for each $x^* \in \partial f_2(\bar{x})$, there exists some $y^* \in Y_+^*$ such that $\langle y^*, h(\bar{x})\rangle = 0$ and $x^* \in \partial(f_1 + \delta_C + y^* \circ h)(\bar{x})$. ■

By taking into account Remark 2.1 we obtain the same result as above in the setting of locally convex topological real vector spaces.

PROPOSITION 3.2. *Let X and Y be two locally convex topological real vector spaces, $f_1, f_2 \colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$ are two proper convex functions with f_2 lower semicontinuous, $h \colon X \longrightarrow Y \cup \{+\infty\}$ is a proper and Y_+-convex mapping and C is a closed convex subset of X. If there exists some $\bar{x} \in C \cap \operatorname{dom} f_1 \cap \operatorname{dom} h$ such that $\bar{y} := h(\bar{x}) \in -\operatorname{int} Y_+$ where $\operatorname{int} Y_+$ denotes the topological interior of the cone Y_+, then for each $x^* \in \partial f_2(\bar{x})$ there exists some $y^* \in Y_+^*$ such that $\langle y^*, h(\bar{x})\rangle = 0$ and $x^* \in \partial(f_1 + \delta_C + y^* \circ h)(\bar{x})$.*

In order to establish the sufficient conditions linked to problem $(\mathcal{P})$, let us denote by $I(\bar{x})$ the subset of $\{1, 2, \dots, n\} \subset \mathbf{N}^*$ given by

$$I(\bar{x}) := \{\, i \in \{1, 2, \dots, n\} : f_2(\bar{x}) = \langle x_i^*, \bar{x}\rangle + b_i \,\}$$

where $f_2(x) := \max_{1 \le i \le n}\{\langle x_i^*, x\rangle + b_i\}$ is a polyhedral function with x_1^*, x_2^*, ... , $x_n^* \in X^*$ and $b_1, b_2, \dots, b_n \in \mathbf{R}$.

PROPOSITION 3.3. *If we assume that f_2 is a polyhderal convex fuction, then under the same assumptions as in Proposition 3.1 or Proposition 3.2 we obtain that a feasible point $\bar{x}$ of $(\mathcal{P})$ is a local minimum for the problem $(\mathcal{P})$ if and only if for each $i \in I(\bar{x})$ there exists some $y_i^* \in Y_+^*$ such that $\langle y_i^*, h(\bar{x})\rangle = 0$ and $x_i^* \in \partial(f_1 + \delta_C + y_i^* \circ h)(\bar{x})$.*

Proof. By proceeding as in the proof of Proposition 3.1 and by combining assertion ii) of Proposition 2.1 and Proposition 2.2, we obtain that a feasible point $\bar{x} \in X$ is a local minimum of the problem $(\mathcal{P})$ if and only if

$$\partial f_2(\bar{x}) \subset \partial(f_1 + \delta_C + \delta_{-Y_+} \circ h)(\bar{x}) = \bigsqcup_{\substack{y^* \in N(h(\bar{x}), -Y_+) \\ \langle y^*, h(\bar{x})\rangle = 0}} \partial(f_1 + \delta_C + y^* \circ h)(\bar{x}).$$

As f_2 is the supremum of a finite family of affine functions, thus it is easy to check (or see [1]) that

$$\partial f_2(\bar{x}) = \operatorname{co}\{\, x_i^* : i \in I(\bar{x}) \,\} \tag{3.1}$$

where co stands for the convex hull. Since the subdifferential is a convex subset, we deduce that $\bar{x}$ is a local minimum of $(\mathcal{P})$ if and only if for each $i \in I(\bar{x})$ there exists some $y_i^* \in Y_+^*$ such that $\langle y_i^*, h(\bar{x})\rangle = 0$ and $x_i^* \in \partial(f_1 + \delta_C + y_i^* \circ h)(\bar{x})$. ∎

COROLLARY 3.1. *If, in addition to the assumptions of the above Proposition 3.3, we assume that f_1 and h are finite and continuous at some point of C, then a feasible point $\bar{x}$ is a local minimum of $(\mathcal{P})$ if and only if for each $i \in I(\bar{x})$ there exists some $y_i^* \in Y_+^*$ such that $\langle y_i^* h(\bar{x})\rangle = 0$ and $x_i^* \in \partial f_1(\bar{x}) + N(\bar{x}, C)\partial(y_i^* \circ h)(\bar{x})$.*

4. Toland's dual problem associated to $(\mathcal{P})$

Let us consider the following functions $f_1, f_2 \colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$, $g \colon Y \longrightarrow \mathbf{R} \cup \{+\infty\}$ and $h \colon X \longrightarrow Y \cup \{+\infty\}$. By $(\mathcal{L})$ we shall mean the following minimization problem

$$(\mathcal{L}): \quad \inf_{x \in X} \{(f_1 + g \circ h)(x) - f_2(x)\}.$$

By virtue of Proposition 2.2, if we assume that $f_2 = f_2^{**}$ then

$$\inf_{x \in X} \{(f_1 + g \circ h)(x) - f_2(x)\} = \inf_{x^* \in X^*} \{f_2^*(x^*) - (f_1 + g \circ h)^*(x^*)\}$$

where f_1, g and h are arbitrary functionals. Hence the dual problem $(\mathcal{L}^*)$ associated to $(\mathcal{L})$ takes the following form

$$(\mathcal{L}^*): \quad \inf_{x^* \in X^*} \{f_2^*(x^*) - (f_1 + g \circ h)^*(x^*)\}.$$

Now, we can state the following duality result.

PROPOSITION 4.1. *Let us assume that f_1 and $f_2 \colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$ are convex, proper and lower semicontinuous, $g \colon Y \longrightarrow \mathbf{R} \cup \{+\infty\}$ is Y_+-nondecreasing, proper, convex and lower semicontinuous and $h \colon X \longrightarrow Y \cup \{+\infty\}$ is Y_+-convex and proper with closed epigraph. If $\mathbf{R}_+[\operatorname{dom} g - h(\operatorname{dom} f_1 \cap \operatorname{dom} f_2)]$ is a closed vector subspace of Y then we have $\inf(\mathcal{L}) = \inf(\mathcal{L}^*)$ where*

$$\inf(\mathcal{L}^*) = \inf_{x^* \in X^*} \max_{y^* \in Y_+^*} \{f_2^*(x^*) - g^*(y^*) - (f_+ y^* \circ h)^*(x^*)\}.$$

Proof. We have already pointed out that the following equality

$$\inf_{x \in X} \{(f_1 + g \circ h)(x) - f_2(x)\} = \inf_{x^* \in X^*} \{f_2^*(x^*) - (f_1 + g \circ h)^*(x^*)\}$$

holds under the assumption $f_2 = f_2^{**}$ which is fulfilled since f_2 is proper, convex and lower semicontinuous. By applying again Theorem 2.1, we get

$$(f_1 + g \circ h)^*(x^*) = \min_{y^* \in Y_+^*} \{g^*(y^*) + (f_1 + y^* \circ h)^*(x^*)\},$$

and thus we may write that

$$\begin{aligned}
&\inf_{x \in X} \{(f_1 + g \circ h)(x) - f_2(x)\} \\
&\qquad = \inf_{x^* \in X^*} \{f_2^*(x^*) - \min_{y^* \in Y_+^*} \{g^*(y^*) + (f_1 + y^* \circ h)^*(x^*)\}\} \\
&\qquad\qquad = \inf_{x^* \in X^*} \max_{y^* \in Y_+^*} \{f_2^*(x^*) g^*(y^*) - (f_1 + y^* \circ h)^*(x^*)\}. \quad ∎
\end{aligned}$$

Now, let us go back to our main problem $(\mathcal{P})$,

$$(\mathcal{P}) \quad \begin{cases} \inf f_1(x) - f_2(x) \\ h(x) \in -Y_+ \\ x \in C. \end{cases}$$

From the previous results the Toland's dual problem associated to $(\mathcal{P})$ is defined by

$$(\mathcal{P}^*) \qquad \inf_{x^* \in X^*} \{f_2^*(x^*) - (f_1 + \delta_C + \delta_{-Y_+} \circ h)^*(x^*)\}.$$

It was mentioned previously that when f_2 is convex, proper and lower semicontinuous we have $\inf(\mathcal{P}) = \inf(\mathcal{P}^*)$.

PROPOSITION 4.2. *Let us consider two convex, proper and lower semicontinuous functions f_1 and $f_2 : X \longrightarrow \mathbf{R} \cup \{+\infty\}$, a Y_+-convex and proper mapping $h\colon X \longrightarrow Y \cup \{+\infty\}$ with closed epigraph and let C be a nonempty closed and convex subset of X. If $\mathbf{R}_+[Y_+ + h(\operatorname{dom} f_1 \cap C)]$ is a closed vector subspace of Y, then we have $\inf(\mathcal{P}) = \inf(\mathcal{P}^*)$ with*

$$\inf(\mathcal{P}^*) = \inf_{x^* \in X^*} \max_{y^* \in Y_+^*} \{f_2^*(x^*) - (f_1 + \delta_C + y^* \circ h)^*(x^*)\}.$$

Proof. It suffices to observe that $(\mathcal{P})$ has the form of the problem $(\mathcal{L})$, i.e., $(\mathcal{P})$ is equivalent to

$$\inf_{x \in X} \{(f_1 + \delta_C + \delta_{-Y_+} \circ h)(x) - f_2(x)\},$$

and that the function $g\colon Y \longrightarrow \mathbf{R} \cup \{+\infty\}$ given by $g(y) = \delta_{-Y_+}(y)$ is convex, proper, lower semicontinuous and Y_+-nondecreasing on the whole space Y (see [1]) and $g^* = \delta^*_{-Y_+} = \delta_{Y_+^*}$ and therefore by applying Proposition 4.1 we obtain there desired result. ∎

Under the conditions of Remark 2.1 we obtain the above result in the setting of locally convex topological vector spaces.

PROPOSITION 4.3. *Let X and Y be two locally convex topological vector spaces, $f_1, f_2\colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$ two convex proper functions, $h\colon X \longrightarrow Y \cup \{+\infty\}$ is a Y_+-convex and proper mapping and C is a nonempty convex subset of X. If we assume that there exists some $\bar{x} \in C \cap \operatorname{dom} f_1 \cap \operatorname{dom} h$ such that $\bar{y} := h(\bar{x}) \in \operatorname{int} Y_+$, then we have $\inf(\mathcal{P}) = \inf(\mathcal{P}^*)$ with*

$$\inf(\mathcal{P}^*) = \inf_{x^* \in X^*} \max_{y^* \in Y_+^*} \{f_2^*(x^*) - (f_+ \delta_C + y^* \circ h)^*(x^*)\}. \tag{4.1}$$

COROLLARY 4.1. *If we assume, in addition to the assumptions of Proposition 4.2 or Proposition 4.3, that f_1 and h are finite and continuous at some point of C then $\inf \mathcal{P} = \inf \mathcal{P}*$ with*

$$\inf(\mathcal{P}^*) = \inf_{x^* \in X^*} \max_{(y^*, u^*, v^*) \in Y_+^* \times X^* \times X^*} \{f_2^*(x^*) - f_1^*(x^* - u^* - v^*) - \delta_C^*(u^*) - (y^* \circ h)(x^*)\}.$$

Proof. Since h is finite and continuous at some point $\bar{x} \in \operatorname{dom} f_1 \cap C$, we have according to [1] that

$$(f_1 + \delta_C + y^* \circ h)^*(x^*) = \min_{u^* \in X^*} \{(f_1 + \delta_C)^*(x^* - u^*) + (y^* \circ h)^*(u^*)\}.$$

Also, since f_1 is finite and continuous at the same point $\bar{x}$, we obtain

$$(f_1+\delta_C+y^*\circ h)^*(x^*) = \min_{u^* \in X^*} \min_{v^* \in X^*} \{f_1^*(x^*-u^*-v^*)+\delta_C^*(v^*)+(y^*\circ h)^*(u^*)\}. \quad (4.2)$$

Substituting (4.2) in (4.1) we get

$$\inf(\mathcal{P}^*) = \inf_{x^* \in X^*} \max_{y^* \in Y_+^*} \{f_2^*(x^*) - (f_1 + \delta_C + y^* \circ h)^*(x^*)\} =$$

$$\inf_{x^* \in X^*} \max_{(y^*, u^*, v^*) \in Y_+^* \times X^* \times X^*} \{f_2^*(x^*) - f_1^*(x^* - u^* - v^*) - \delta_C^*(v^*) - (y^* \circ h)^*(v^*)\}.$$

■

5. Application to fractional mathematical programming problems

In this section, we illustrate the previous results by deriving optimality conditions for a scalar nondifferentiable fractional programming problem subject to a convex vector constraint. Let us consider the following optimization problem

$$(\mathcal{H}) \quad \begin{cases} \inf \frac{f_1(x)}{f_2(x)} \\ h(x) \in -Y_+ \\ x \in C, \end{cases}$$

where $f_1, f_2 \colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$ are two convex functions with $f_1(x) \geq 0$ and $f_2(x) > 0$ for all $x \in X$, $h \colon X \longrightarrow Y \cup \{+\infty\}$ is a Y_+-convex mapping and C is a nonempty convex closed subset of X. By $(\mathcal{H}_{\bar{s}})$ we denote the following minimization problem

$$(\mathcal{H}_{\bar{s}}) \quad \begin{cases} \inf\{f_1(x) - \bar{s} f_2(x)\} \\ h(x) \in -Y_+ \\ x \in C \end{cases}$$

where $\bar{s}$ is a real number.

REMARK 5.1. Since f_1 and f_2 may take both the value $+\infty$, we adopt throughout this section the convention $(+\infty) \times 0 = 0$.

The following result from [3] allows us to write equivalently the fractional programming problem as a DC-minimization problem.

PROPOSITION 5.1. [3] *The point $\bar{x}$ is a local (resp. global) solution of the problem $(\mathcal{H})$ if and only if $\bar{x}$ is a local (resp. global) solution of the problem $(\mathcal{H}_{\bar{s}})$ with $\bar{s} = \dfrac{f_1(\bar{x})}{f_2(\bar{x})}$.*

Let us consider the following constraint qualification

$$(C.Q.A_2.B_2)\ \begin{cases} X \text{ and } Y \text{ are Fréchet spaces,} \\ f_1 \text{ and } f_2 \text{ are convex, proper and lower semicontinuous,} \\ \operatorname{Epi} h \text{ is a closed subset of } X \times Y, \\ \mathbf{R}_+[Y_+ + h(\operatorname{dom} f_1 \cap C \cap \operatorname{dom} h)] \text{ is a closed vector subspace.} \end{cases}$$

Now, we state the necessary optimality condition for the scalar fractional programming problem $(\mathcal{H})$.

PROPOSITION 5.2. *Let $\bar{x}$ be a feasible point of $(\mathcal{H})$, i.e., $\bar{x} \in C \cap h^{-1}(-Y_+)$. If we assume that the condition $(C.Q.A_2.B_2)$ is satisfied and $\bar{x}$ is a local minimum for the problem $(\mathcal{H})$, then we have: for any $x^* \in \partial f_2(\bar{x})$ there exists some $y^* \in Y_+^*$ verifying $\langle y^*, h(\bar{x})\rangle = 0$ and $\bar{s}x^* \in \partial(f_1 + \delta_C + y^* \circ h)(\bar{x})$.*

Proof. For proving the necessity of optimality conditions we distinguish two cases. The first is when $\bar{s} > 0$. For this, from Proposition 5.1, a feasible point $\bar{x}$ of $(\mathcal{H})$ is a local minimum if and only if $\bar{x}$ is a local minimum of $(\mathcal{H}_{\bar{s}})$. As $\partial(\bar{s}f_2) = \bar{s}\partial f_2(\bar{x})$ and by applying Proposition 3.1 we get under the condition $(C.Q.A_2.B_2)$ that for each $x^* \in \partial f_2(\bar{x})$ there exists some $y^* \in Y_+^*$ satisfying $\langle y^*, h(\bar{x})\rangle = 0$ and

$$\bar{s}x^* \in \partial(f_1 + \delta_C + y^* \circ h)(\bar{x}).$$

The case when $\bar{s} = 0$ implies $f_1(\bar{x}) = 0$ and by virtue of Remark 5.1 and the fact that $\bar{x}$ is a local minimum of $(\mathcal{H})$ we obtain

$$0 \in \partial(f_1 + \delta_C + \delta_{-Y+} \circ h)(\bar{x}),$$

which ensures under the condition $(C.Q.A_2.B_2)$ the existence of $y^* \in Y_+^*$ satisfying $\langle y^*, h(\bar{x})\rangle = 0$ and0 $\in \partial(f_1 + \delta_C + y^* \circ h)(\bar{x})$. This completes the proof. ■

COROLARRY 5.1. *If we assume, in addition to the assumptions of the above Proposition 5.2, that f_1 and h are finite and continuous at some point of C, that the condition $(C.Q.A_2.B_2)$ holds and $\bar{x}$ is a local minimum of $(\mathcal{H})$ then we obtain: for any $x^* \in \partial f_2(\bar{x})$ there exists some $y^* \in Y_+^*$ satisfying $\langle y^*, h(\bar{x})\rangle = 0$ and*

$$\bar{s}x^* \in \partial f_2(\bar{x}) + N_C(\bar{x}) + \partial(y^* \circ h)(\bar{x}).$$

The related sufficient optimality conditions are given by

PROPOSITION 5.3. *If we suppose that f_2 is a polyhedral convex function, then under the same assumptions in Proposition 5.2 we obtain that a feasible point $\bar{x}$ of $(\mathcal{H})$ is a local minimum for the problem $(\mathcal{H})$ if and only if for each $i \in I(\bar{x})$ there exists some $y_i^* \in Y_+^*$ such that $\langle y_i^*, h(\bar{x})\rangle = 0$ and*

$$\bar{s}x_i^* \in \partial(f_1 + \delta_C + y_i^* \circ h)(\bar{x}).$$

Proof. We apply Proposition 3.3 and Proposition 5.2. ■

COROLLARY 5.2. *If, under the assumptions of Propositions 5.3, f_1 and h are assumed to be finite and continuous at some point of C, then $\bar{x}$ is a local solution of $(\mathcal{H})$ if and only if for each $i \in I(\bar{x})$, there exists some $y_i^* \in Y_+^*$ such that $\langle y_i^*, h(\bar{x})\rangle = 0$ and $\bar{s}x_i^{\in}\partial f_1(\bar{x}) + N_C(\bar{x}) + \partial(y_i^* \circ h)(\bar{x})$.*

Proof. We use the same arguments as in Corollary 3.1. ■

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