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THE MEASURE OF NONCOMPACTNESS OF MATRIX TRANSFORMATIONS ON THE SPACES $c^p(\Lambda)$ AND $c^p_\infty(\Lambda)$ $(1 < p < \infty)$

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Abstract. We study linear operators between certain sequence spaces X and Y when X is $C^p(\Lambda)$ or $C^p_\infty(\Lambda)$ and Y is one of the spaces: c, c_0, l_∞, $c(\mu)$, $c_0(\mu)$, $c_\infty(\mu)$, that is, we give necessary and sufficient conditions for A to map X into Y and after that necessary and sufficient conditions for A to be a compact operator. These last conditions are obtained by means of the Hausdorff measure of noncompactness and given in the form of conditions for the entries of an infinite matrix A.

1. Introduction

Let ω be the set of all complex sequences, Φ be the set of all finite sequences and X and Y be subsets of ω. We write l_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively. By e and $e^{(n)}$ $(n \in N_0)$ we denote the sequences such that: $e_k = 1$ for all k and $e_k^{(n)} = \begin{cases} 1, & k = n \\ 0, & k \neq n. \end{cases}$ A sequence $(b_n)_{n=0}^\infty$ in a linear metric space X is called Schauder basis if for each $x \in X$, there is a unique sequence $(\lambda_n)_n$ of scalars with $\sum_{n=0}^\infty \lambda_n b_n$, that is $\lim_{m\to\infty} \sum_{n=0}^m \lambda_n b_n = x$.

As mentioned in the abstract, the aim of this paper is the characterization of matrix transformations between some sequence spaces and the main tool in this is the theory of FK and BK spaces.

An FK space is a complete metric sequence space with the property that convergence implies coordinatewise convergence; a BK space is a normed FK space. An FK space $X \supset \Phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \sum_{k=0}^\infty x_k e^{(k)}$, that is $\lim_{n\to\infty} \sum_{k=0}^n x_k e^{(k)} = x$.

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Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex entries. We write

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad \text{and} \quad A(x) = (A_n(x))_{n=0}^{\infty};$$

then

$A \in (X, Y)$ if and only if $A_n(x)$ converges for all $x \in X$ and all n and $A(x) \in Y$.

Furthermore,

$$X^{\beta} = \{ a \in \omega \mid \sum_k a_k x_k \text{ converges for all } x \in X \}$$

denotes the β-dual of X. The set $X_A = \{a \in \omega \mid Ax \in X\}$ is called the matrix domain of A in X. We also write

$$xy = (x_k y_k)_{k=0}^{\infty}, \quad x^{-1} * Y = \{a \in \omega \mid ax \in Y\}$$

and

$$M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{ a \in \omega \mid ax \in Y \text{ for all } x \in X \}$$

is called the multiplier space of X and Y.

If $X \supset \Phi$ is a BK space and $a \in \omega$ we write

$$\|a\|_X^* = \sup\{ \,|\sum_{k=0}^{\infty} a_k x_k| \mid \|x\| = 1 \}.$$

2. The spaces $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$ $(1 < p < \infty)$ and their β-duals

The case when $p = 1$ was investigated by E.Malkowsky and V.Rakočević (see [3]).

Let $1 \leq p$ and $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be a non-decreasing sequence of positive reals tending to infinity. We write

$$c_\infty^p(\Lambda) = \{ x \in \omega \mid \sup_n \frac{1}{\lambda_n^p} \sum_{k=0}^{n} |\lambda_k x_k - \lambda_{k-1} x_{k-1}|^p < \infty \},$$

$$c_0^p(\Lambda) = \{ x \in \omega \mid \lim_{n \to \infty} \frac{1}{\lambda_n^p} \sum_{k=0}^{n} |\lambda_k x_k - \lambda_{k-1} x_{k-1}|^p = 0 \},$$

and

$$c^p(\Lambda) = \{ x \in \omega \mid x - le \in c_0^p(\Lambda) \text{ for some } l \in C \}$$

for the sets of sequences that are Λ-strongly bounded, Λ-strongly convergent to zero and Λ-strongly convergent, respectively.

We say that a non-decreasing sequence $\Lambda = (\lambda_k)_{k=0}^{\infty}$ of positive reals tending to infinity is exponentially bounded if there are reals s and t with $0 < s \leq t < 1$

such that for some subsequence $(\lambda_{k(\nu)})_{\nu=0}^{\infty}$ of Λ, we have $s \le \frac{\lambda_{k(\nu)}}{\lambda_{k(\nu+1)}} \le t$ for all ν; such a subsequence $(\lambda_{k(\nu)})_{\nu=0}^{\infty}$ is called an associated subsequence. If $(k(\nu))_{\nu=0}^{\infty}$ is a strictly increasing sequence of nonnegative integers, then we write $K^{\langle\nu\rangle}$ for the set of all integers k with $k(\nu) \le k \le k(\nu+1)-1$, and $\sum_{\nu}$ and $\max_{\nu}$ for the sum and maximum taken over all k in $K^{\langle\nu\rangle}$.

In our further consideration, let $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be an exponentially bounded sequence of positive reals and $(\lambda_{k(\nu)})_{\nu=0}^{\infty}$ be an associated subsequence.

Proposition 2.1.([4]) *The spaces $c^p(\Lambda)$ and $c_{\infty}^p(\Lambda)$ are BK spaces. The space $c_0^p(\Lambda)$ is also a BK space with a Schauder basis $(c^{(k)})_{k=0}^{\infty}$ where $c^{(k)} = (\frac{1}{\Lambda})b^{(k)}$, and $b^{(k)}$ is defined by* $b_j^{(k)} = \begin{cases} 0, & j < k \\ 1, & j \le k. \end{cases}$

What is the main reason to obtain the β-dual of an arbitrary sequence space X? β-duals are very important in the characterization of matrix classes (X, Y) since $A \in (X, Y)$ can only hold if $A_n(x)$ converges for all $x \in X$ and for each n, that is $A_n \in X^{\beta}$.

We write E for the matrix with entries $e_{nk} = \begin{cases} 1, & n \ge k \\ 0, & n < k \end{cases}$ and put

$$W^p(\Lambda) = \{\, a \in \omega \mid \sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} (\sum_{\nu} |a_k|^q)^{\frac{1}{q}} < \infty \,\}$$

and $\|a\|_{W^p(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} (\sum_{\nu} |a_k|^q)^{\frac{1}{q}}$ for $1 < p < \infty$ and $q = \frac{p}{p-1}$.

Theorem 2.2. ([1, Theorem 3]) *Let $1 < p < \infty$, $q = \frac{p}{p-1}$ and the sequence $b = (b_n)_n$ be defined by:*

$$b_n = \sum_{\nu=0}^{\nu(n)-1} \lambda_{k(\nu+1)} (k(\nu+1) - k(\nu))^{\frac{1}{q}} + \lambda_{k(\nu(n)+1)} (n - k(\nu(n)) + 1)^{\frac{1}{q}}.$$

Then

$$(c_{\infty}^p(\Lambda))^{\beta} = (\frac{1}{\Lambda})^{-1} * (W^p(\Lambda) \cap (b^{-1} * c_0))_E,$$

that is $a = (a_k)_{k=0}^{\infty} \in (c_{\infty}^p(\Lambda))^{\beta}$ if and only if $\sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} (\sum_{\nu} |\sum_{j=k}^{\infty} \frac{a_j}{\lambda_j}|^q)^{\frac{1}{q}} < \infty$ and $b(\sum_{j=k}^{\infty} \frac{a_j}{\lambda_j})_k \in c_0$. Furthermore, $(c_0^p(\Lambda))^{\beta} = (c^p(\Lambda))^{\beta} = (\frac{1}{\Lambda})^{-1} * (W^p(\Lambda) \cap (b^{-1} * \ell_{\infty}))_E$.

3. Matrix transformations on the spaces $c^p(\Lambda)$ and $c_{\infty}^p(\Lambda)$

We need the following known results.

Theorem 3.1. ([2, Theorem 1.23]) *Let X and Y be FK spaces. Then $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$ where $L_A x = Ax, x \in X$. If $(b^{(k)})_{k=0}^{\infty}$ is a Schauder basis for X, and Y_1 a closed FK space in Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A(b^{(k)}) \in Y_1$ for all k.*

THEOREM 3.2. ([4, Proposition 3.2]) *Let $X \supset \Phi$ and Y be BK spaces. Then we have $A \in (X, l_\infty)$ if and only if*

$$\|A\|_X^* = \sup_n \|A_n\|_X^* < \infty.$$

Furthermore, if $A \in (X, l_\infty)$ then it follows that $\|L_A\| = \|A\|_X^*$.

Now, let us consider the classes $(c_\infty^p(\Lambda),\ \ell_\infty)$, $(c^p(\Lambda),\ \ell_\infty)$, $(c^p(\Lambda), c)$, and $(c^p(\Lambda), c_0)$. Since $c^p(\Lambda)$ has a Schauder basis, we can apply Theorem 3.1 and obtain conditions for their characterization. ($\{e\} \cup \{c^{(k)}\}_k$ is a Schauder basis for $c^p(\Lambda)$). Hence, we have

$$A \in (c^p(\Lambda), c_0) \ \Leftrightarrow\ A \in (c^p(\Lambda), \ell_\infty) \wedge A(c^{(k)}) \in c_0 \wedge A(e) \in c_0,$$
$$A \in (c^p(\Lambda), c) \ \Leftrightarrow\ A \in (c^p(\Lambda), \ell_\infty) \wedge A(c^{(k)}) \in c \wedge A(e) \in c.$$

What are necessary and sufficient conditions for A to be an element of $(c^p(\Lambda), \ell_\infty)$? We give one more useful result.

THEOREM 3.3. ([1, Theorem 4]) *Let $Y \subset \omega$ be a linear space.*

(a) *Then $A \in (c_\infty^p(\Lambda), Y)$ if and only if $\begin{cases} R^A(\Lambda) \in (\omega_\infty^p(\Lambda), Y) \\ R_n^A(\Lambda) \in b^{-1} * c_0 \text{ for all } n \end{cases}$, where $r_{nk}^A(\Lambda) = \sum_{j=k}^\infty \frac{a_{nj}}{\lambda_j}$ for all n, k and*

$$\omega_\infty^p(\Lambda) = \{\, x \in \omega \mid \sup_n \frac{1}{\lambda_n^p} \sum_{k=0}^n |x_k|^p < \infty \}.$$

(b) *Let $\overline{\lim}_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1$. Then $A \in (c^p(\Lambda), Y)$ if and only if*

$$\begin{cases} R^A(\Lambda) \in (\omega_0^p(\Lambda), Y) \\ R_n^A(\Lambda) \in b^{-1} * \ell_\infty \text{ for all } n \\ A(e) \in Y, \end{cases}$$

where $\omega_0^p(\Lambda) = \{\, x \in \omega \mid \lim_{n\to\infty} \frac{1}{\lambda_n^p} \sum_{k=0}^n |x_k|^p = 0 \}$.

PROPOSITION 3.4. *Let Y denote any of the spaces c, c_0, ℓ_∞. If $A \in (c^p(\Lambda), Y)$, then $\|L_A\| = \|A\|_{(c^p(\Lambda),\ell_\infty)}$ and*

$$\|A\|_{(c^p(\Lambda),\ell_\infty)} = \sup_n (\sum_{\nu=0}^\infty \lambda_{k(\nu+1)} (\sum_\nu |\sum_{j=k}^\infty \tfrac{a_j}{\lambda_j}|^q)^{\frac{1}{q}}), q = \frac{p}{p-1}.$$

Now we have

COROLLARY 3.5.

$$A \in (c_\infty^p(\Lambda), \ell_\infty) \ \Leftrightarrow\ \begin{cases} \sup_n \sum_{\nu=0}^n \lambda_{k(\nu+1)} (\sum_\nu |\sum_{j=k}^\infty \tfrac{a_{nj}}{\lambda_j}|^q)^{\frac{1}{q}} \} < \infty \\ \lim_{k\to\infty} b_k \sum_{j=k}^\infty \tfrac{a_{nj}}{\lambda_j} = 0. \end{cases}$$

Corollary 3.6.

$$A \in (c^p(\Lambda), \ell_\infty) \;\Leftrightarrow\; \begin{cases} \sup_n \sum_{\nu=0}^{n} \lambda_{k(\nu+1)} (\sum_\nu |\sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j}|^q)^{\frac{1}{q}}\} < \infty \\ \sup_k |b_k \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j}| < \infty \\ \sup_n |\sum_{k=0}^{\infty} a_{nk}| < \infty. \end{cases}$$

Corollary 3.7.

$$A \in (c^p(\Lambda), c) \;\Leftrightarrow\; \begin{cases} \sup_n \sum_{\nu=0}^{n} \lambda_{k(\nu+1)} (\sum_\nu |\sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j}|^q)^{\frac{1}{q}}\} < \infty \\ \sup_k |b_k \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j}| < \infty \\ \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \textit{ for some } \alpha \\ \lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} = \alpha_k \textit{ for each } k. \end{cases}$$

Corollary 3.8.

$$A \in (c^p(\Lambda), c_0) \;\Leftrightarrow\; \begin{cases} \sup_n \sum_{\nu=0}^{n} \lambda_{k(\nu+1)} (\sum_\nu |\sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j}|^q)^{\frac{1}{q}}\} < \infty \\ \sup_k |b_k \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j}| < \infty \\ \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0 \\ \lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} = 0. \end{cases}$$

We have seen transformations (X, Y) and their necessary and sufficient conditions but in the cases when Y is one of the classical sequence spaces, i.e. ℓ_∞, c, c_0. In our further studies, we will find necessary and sufficient conditions for classes: $(c^p(\Lambda),\, c_\infty(\mu))$, $(c^p(\Lambda), c_0(\mu))$, $(c^p(\Lambda), c(\mu))$, $(c^p_\infty(\Lambda), c_\infty(\mu))$.

Let us put

$$\|A\|_{(X,c_\infty(\mu))} = \sup_{m\ge 0}\big(\max_{N_m \subset \{0,\dots,m\}} \|\frac{1}{\mu_m} \sum_{n\in N_m} (\mu_n A_n - \mu_{n-1}A_{n-1})\|^*_D\big)$$

where X is an arbitrary FK space and $\mu = (\mu_n)_{n=0}^\infty$ is a nondecreasing sequence of positive reals tending to infinity.

Theorem 3.9. ([3, Corollary]) *$A \in (X, c_\infty(\mu))$ if and only if $\|A\|_{(X,c_\infty(\mu))} < \infty$ for some $D > 0$. Further, if $\{b^{(k)}\}_k$ is basis of X, then*

$$A \in (X, c_0(\mu)) \;\Leftrightarrow\; \begin{cases} \|A\|_{(X,c_\infty(\mu))} < \infty \textit{ for some } D > 0 \\ \lim_{m\to\infty} (\frac{1}{\mu_m} \sum_{n=0}^{m} |\mu_n A_n(b^{(k)}) - \mu_{n-1}A_{n-1}(b^{(k)})|) = 0 \\ \textit{for all } k. \end{cases}$$

$A \in (X, c(\mu))$ if and only if

$$\begin{cases} \|A\|_{(X,c_\infty(\mu))} < \infty \textit{ for some } D > 0 \\ (\exists l_k \in C) \lim_{m\to\infty} (\frac{1}{\mu_m} \sum_{n=0}^{m} |\mu_n A_n(b^{(k)} - l_k) - \mu_{n-1}A_{n-1}(b^{(k)} - l_k)|) = 0 \\ \textit{for all } k. \end{cases}$$

Finally, if X is p-normed and $A \in (X,Y)$ for $Y \in \{c_\infty(\mu), c_0(\mu), c(\mu)\}$, then, for

$$\|A\|^*_{(X,c_\infty(\mu))} = \sup_{m\geq 0}\big(\max_{N_m\subset\{0,\dots,m\}} \|\frac{1}{\mu_m}\sum_{n\in N_m}(\mu_n A_n - \mu_{n-1}A_{n-1})\|^*\big)$$

we have $\|A\|^*_{(X,c_\infty(\mu))} \leq \|L_A\| \leq 4\|A\|^*_{(X,c_\infty(\mu))}$.

Corollary 3.10. *$A \in (c^p_\infty(\Lambda), c_\infty(\mu))$ if and only if $\|A\|_{(c^p_\infty(\Lambda),c_\infty(\mu))} < \infty$, where*

$$\|A\|_{(c^p_\infty(\Lambda),c_\infty(\mu))} =$$

$$\sup_{m\geq 0}\{\max_{N_m\subset\{0,\dots,m\}} \sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)}(\sum_{\nu}|\tfrac{1}{\mu_m}\sum_{n\in N_m}(\mu_n \sum_{j=k}^{\infty}\tfrac{a_{nj}}{\lambda_j} - \mu_{n-1}\sum_{j=k}^{\infty}\tfrac{a_{n-1,j}}{\lambda_j})|^q)^{\frac{1}{q}}\}.$$

As we said before, if X is a BK space and $a \in \omega$ then we put

$$\|a\|^* = \sup\{|\sum_{k=0}^{\infty} a_k x_k| \mid \|x\| = 1\},$$

provided the term on the right side exists and is finite. This is the case whenever $a \in X^\beta$. Hence, we have one more condition for $A_n \in c^p_\infty(\Lambda))^\beta$, namely

$$\lim_{k\to\infty}(b_k \sum_{j=k}^{\infty}\tfrac{a_{nj}}{\lambda_j}) = 0.$$

Let us mention that we could obtain the same conditions also by means of Theorem 3.3.

Corollary 3.11. *$A \in (c^p(\Lambda), c_\infty(\mu))$ if and only if $\|A\|_{(c^p_\infty(\Lambda),c_\infty(\mu))} < \infty$, i.e.*

$$\sup_{m\geq 0}\{\max_{N_m\subset\{0,\dots,m\}} \sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)}(\Sigma_{\nu}|\tfrac{1}{\mu_m}\sum_{n\in N_m}(\mu_n \sum_{j=k}^{\infty}\tfrac{a_{nj}}{\lambda_j} - \mu_{n-1}\sum_{j=k}^{\infty}\tfrac{a_{n-1,j}}{\lambda_j})|^q)^{\frac{1}{q}}\} < \infty$$

and $\sup_k |b_k \sum_{j=k}^{\infty}\frac{a_{nj}}{\lambda_j}| = 0$ *and*

$$\sup_m \frac{1}{\mu_m}\sum_{n=0}^{m} |\mu_n \sum_k a_{nk} - \mu_{n-1}\sum_k a_{n-1,k}| < \infty.$$

Corollary 3.12. *$A \in (c^p(\Lambda), c_0(\mu))$ if and only if $\|A\|_{(c^p_\infty(\Lambda),c_\infty(\mu))} < \infty$ and* $\sup_k |b_k \sum_{j=k}^{\infty}\frac{a_{nj}}{\lambda_j}| = 0$ *and*

$$\sup_m \frac{1}{\mu_m}\sum_{n=0}^{m} |\mu_n \sum_k a_{nk} - \mu_{n-1}\sum_k a_{n-1,k}| < \infty,$$
$$\lim_{m\to\infty}(\frac{1}{\mu_m}\sum_{n=0}^{m} |\mu_n \sum_{j=k}^{\infty}\tfrac{a_{nj}}{\lambda_j} - \mu_{n-1}\sum_{j=k}^{\infty}\tfrac{a_{n-1,j}}{\lambda_j}|) = 0,$$
$$\lim_{m\to\infty}(\frac{1}{\mu_m}\sum_{n=0}^{m} |\mu_n \sum_{j=0}^{\infty} a_{nj} - \mu_{n-1}\sum_{j=0}^{\infty} a_{n-1,j}|) = 0.$$

Corollary 3.13. *$A \in (c^p(\Lambda), c(\mu))$ if and only if $\|A\|_{(c^p_\infty(\Lambda), c_\infty(\mu))} < \infty$ and $\sup_k |b_k \sum_{j=k}^\infty \frac{a_{nj}}{\lambda_j}| = 0$ and*

$$\sup_m \frac{1}{\mu_m} \sum_{n=0}^m |\mu_n \sum_k a_{nk} - \mu_{n-1} \sum_k a_{n-1,k}| < \infty,$$

$$\lim_{m\to\infty} (\frac{1}{\mu_m} \sum_{n=0}^m (\mu_n (\sum_{j=k}^\infty \frac{a_{nj}}{\lambda_j} - l_k) - \mu_{n-1} (\sum_{j=k}^\infty \frac{a_{n-1,j}}{\lambda_j} - l_k))) = 0 \text{ for some } l_k,$$

$$\lim_{m\to\infty} (\frac{1}{\mu_m} \sum_{n=0}^m (\mu_n (\sum_{j=0}^\infty a_{nj} - l) - \mu_{n-1} (\sum_{j=0}^\infty a_{n-1,j} - l)) = 0 \text{ for some } l.$$

4. The Hausdorff measure of noncompactness and matrix transformations

Let X and Y be metric spaces and $f \colon X \longrightarrow Y$. We say that f is a compact map if $f(Q)$ is a relatively compact subset of Y for each bounded subset Q of X. (A set K is said to be relatively compact if $\overline{K}$ is a compact set). In this section, we will consider an operator L_A with the aim to find conditions for A to be a compact operator. For this purpose we will use the Hausdorff measure of noncompactness. Recall that if Q is a bounded subset of a metric space X, then the Hausdorff measure of noncompactness of Q is denoted by $\chi(Q)$ where

$$\chi(Q) = \inf\{ \epsilon > 0 \mid Q \text{ has a finite } \epsilon\text{-net in X } \}$$

(For properties of χ see [5]).

If Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d), then we have

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is a totally bounded set,}$$
$$\chi(Q) = \chi(\overline{Q}),$$
$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \le \chi(Q_2),$$
$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$$

and

$$\chi(Q_1 \cap Q_2) \le \min\{\chi(Q_1), \chi(Q_2)\}.$$

If Q, Q_1 and Q_2 are bounded subsets of a normed space X, then we have

$$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2),$$
$$\chi(Q + x) = \chi(Q) \ (x \in X)$$

and

$$\chi(\lambda Q) = |\lambda| \chi(Q) \text{ for all } \lambda \in C.$$

As we can measure the noncompactness of a bounded subset of a metric space, we can also measure the noncompactness of an operator. Let X and Y be normed spaces and $A \in B(X,Y)$. The Hausdorff measure of noncompactness of A is defined by

$$\|A\|_\chi = \chi(AK),$$

where $K = \{x \in X \mid \|x\| \le 1\}$ is the unit ball in X (see [5]). Further, A is compact if and only if $\|A\|_\chi = 0$. It holds: $\|A\|_\chi \le \|A\|$. Let us recall some well-known results(see [2]) which will be useful for our investigation.

Theorem 4.1. [Goldenstein, Gohberg, Markus] ([2, Theorem 2.23]) *Let X be a Banach space with Schauder basis $\{e_1, e_2, \dots\}$, Q be a bounded subset of X, and $P_n\colon X \to X$ be the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$. Then we have*

$$\frac{1}{a}\limsup_{n\to\infty}(\sup_{x\in Q}\|(I-P_n)x\|) \le \chi(Q) \le \limsup_{n\to\infty}(\sup_{x\in Q}\|(I-P_n)x\|),$$

where $a = \limsup_{n\to\infty}\|I - P_n\|$.

Theorem 4.2. *Let A be an infinite matrix, $1 < p < \infty$, $q = \frac{p}{p-1}$ and for any integers n, r, $n > r$ we write*

$$\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)} = \sup_{n>r}(\sum_{\nu=0}^{\infty}\lambda_{k(\nu+1)}(\sum_{\nu}|\sum_{j=k}^{\infty}\tfrac{a_{nj}}{\lambda_j}|^q)^{\frac{1}{q}}).$$

a) *If $A \in (c^p(\Lambda), c_0)$, then $\|L_A\|_\chi = \lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)}$.*

b) *If $A \in (c^p(\Lambda), c)$, then $\frac{1}{2}\lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)} \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)}$.*

c) *If $A \in (c^p(\Lambda), \ell_\infty)$, then $0 \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)}$.*

d) *If $A \in (c^p_\infty(\Lambda), \ell_\infty)$, then $0 \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)}$.*

Proof. Let us remark that the limits in a), b) and c) exist. Set $K = \{x \in c^p(\Lambda) \mid \|x\| \le 1\}$.

a) By Theorem 4.1, we have

$$\|L_A\|_\chi = \chi(AK) = \lim_{r\to\infty}(\sup_{x\in K}\|(I-P_r)Ax\|),$$

$(P_r : c_0 \longrightarrow c_0, P_r(x) = (x_0, x_1, \dots, x_r, 0, 0, \dots), a = 1)$. By Proposition 3.4, we have $\|L_A\| = \|A\|_{(c^p(\Lambda),\ell_\infty)}$. Now, let $A_{(r)} = (\widetilde{a_{nk}})$ be the infinite matrix defined by

$$\widetilde{a_{nk}} = \begin{cases} 0, & 0 \le n \le r \\ a_{nk}, & n > r. \end{cases}$$

We have $\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)} = \|A_{(r)}\|_{(c^p(\Lambda),\ell_\infty)} = \|L_{A_{(r)}}\|$ and

$$L_{A_{(r)}}(x) = A_{(r)}(x) = (I - P_r)Ax.$$

Hence, we have $\|L_{A_{(r)}}\| = \sup_{x\in K}\|(I-P_r)Ax\|$ and therefore

$$\|L_A\|_\chi = \chi(AK) = \lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)}.$$

b) Let $x \in c$ (that means that x has a unique representation $x = le + \sum_{k=0}^{\infty}(x_k - l)e^{(k)}$ where l is such that $x - le \in c_0$), and let us define the projector $P_r\colon c \longrightarrow c$ by $P_r(x) = le + \sum_{k=0}^{r}(x_k - l)e^{(k)}$ (let us remark that $a = 2$ for $P_r\colon c \longrightarrow c$). Now we have

$$\frac{1}{2}\limsup_{r\to\infty}(\sup_{x\in K}\|(I-P_r)Ax\|) \le \|L_A\|_\chi \le \limsup_{r\to\infty}(\sup_{x\in K}\|(I-P_r)Ax\|).$$

As in (a), we can prove that $\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)} = \sup_{x\in K}\|(I-P_r)Ax\|$ and therefore we have

$$\frac{1}{2}\lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)} \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)}.$$

c) (We can prove (d) in the same way.) Let us define $P_r\colon \ell_\infty \longrightarrow \ell_\infty$ by $P_r(x) = (x_0, x_1, \dots, x_r, 0, 0, , \dots)$. It is obvious that we cannot use Theorem 4.1 because ℓ_∞ has no Schauder basis.

We have $AK \subset P_r(AK) + (I - P_r)AK$ (in case (d), in the definition of K, $c^p(\Lambda)$ is replaced by $c^p_\infty(\Lambda)$). Applying the properties of χ, we have

$$\begin{aligned}\chi(AK) &\le \chi(P_r(AK)) + \chi((I-P_r)AK) = \chi((I-P_r)AK)\\ &= \|(I-P_r)A\|_\chi \le \|(I-P_r)A\| \le \sup_{x\in K}\|(I-P_r)Ax\|.\end{aligned}$$

Therefore, by (a):

$$\chi(AK) \le \sup_{x\in K}\|(I-P_r)Ax\| = \|L_{A_{(r)}}\| \quad \text{and} \quad 0 \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)}. \quad \blacksquare$$

Corollary 4.3. (i) *If either* $A \in (c^p(\Lambda), c_0)$ *or* $A \in (c^p(\Lambda), c)$*, then* L_A *is compact if and only if*

$$\lim_{r\to\infty}\|A\|^{(r)}_{(c^p(\Lambda),\ell_\infty)} = 0.$$

(ii) *If* $A \in (c^p_\infty(\Lambda), \ell_\infty)$ *or* $A \in (c^p(\Lambda), \ell_\infty)$*, then* L_A *is compact if*

$$\lim_{r\to\infty}\|A\|^{(r)}_{(c^p_\infty(\Lambda),\ell_\infty)} = 0.$$

We wonder if the equivalence holds. The following example will give the answer: it is possible for L_A in Theorem 4.2(c) to be compact but $\lim_{r\to\infty}\|A\|^{(r)}_{(c^p_\infty(\Lambda),\ell_\infty)} \neq 0$.

Example 4.4. Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix such that $A_n = e^{k(0)}$, i.e.

$$a_{nk} = \begin{cases} 1, & k = k(0)\\ 0, & k \neq k(0).\end{cases}$$

By Corollary 3.5, $A \in (c^p_\infty(\Lambda), \ell_\infty)$ and by Corollary 3.6, $A \in (c^p(\Lambda), \ell_\infty)$. Furthermore, we obtain

$$\begin{aligned}\|A\|^{(r)}_{(c^p_\infty(\Lambda),\ell_\infty)} &= \sup_{n>r}\sum_{\nu=0}^{\infty}\lambda_{k(\nu+1)}(\sum_{\nu}|\sum_{j=k}^{\infty}\tfrac{a_{nj}}{\lambda_j}|^q)^{\frac{1}{q}})\\ &= \sup_{n>r}\lambda_{k(1)}(\sum_{k=k(0)}^{k(1)-1}|\sum_{j=k}^{\infty}\tfrac{a_{nj}}{\lambda_j}|^q)^{\frac{1}{q}} = \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \neq 0.\end{aligned}$$

Also, putting $x = e = (1, 1, \dots)$, we see that L_A is a compact operator. Hence, the equivalence in (ii) does not hold.

It remains to "measure" the noncompactness of operators $A \in (X, Y)$ where X is $c^p(\Lambda)$ or $c^p_\infty(\Lambda)$ and Y is one of the spaces $c_\infty(\mu)$, $c_0(\mu)$, $c(\mu)$. The next theorem is of great importance for the mentioned task.

THEOREM 4.5. *Let A be an infinite matrix, $1 < p < \infty$, $q = \frac{p}{p-1}$ and for any integers $m, r \in N$, $m > r$, we put*

$$\|A\|^{(r)}_{c_\infty} = \sup_{m>r}\{\max_{N_{r,m}\subset\{r+1,\dots,m\}} \sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} \times$$
$$\times (\sum_\nu |\tfrac{1}{\mu_m} \sum_{n\in N_m} (\mu_n \sum_{j=k}^{\infty} \tfrac{a_{nj}}{\lambda_j} - \mu_{n-1} \sum_{j=k}^{\infty} \tfrac{a_{n-1,j}}{\lambda_j})|^q)^{\frac{1}{q}}\}.$$

(a) *If $A \in (c^p(\Lambda), c_0(\mu))$, then* $\lim_{r\to\infty} \|A\|^{(r)}_{c_\infty} \le \|L_A\|_\chi \le 4 \lim_{r\to\infty} \|A\|^{(r)}_{c_\infty}$.

(b) *If $A \in (c^p(\Lambda), c(\mu))$, then* $\frac{1}{2} \lim_{r\to\infty} \|A\|^{(r)}_{c_\infty} \le \|L_A\|_\chi \le 4 \lim_{r\to\infty} \|A\|^{(r)}_{c_\infty}$.

(c) *If $A \in (c^p(\Lambda), c_\infty(\mu))$, then* $0 \le \|L_A\|_\chi \le 4 \lim_{r\to\infty} \|A\|^{(r)}_{c_\infty}$.

(d) *If $A \in (c^p_\infty(\Lambda), c_\infty(\mu))$, then* $0 \le \|L_A\|_\chi \le 4 \lim_{r\to\infty} \|A\|^{(r)}_{c_\infty}$.

Proof. (a) Let $P_r : c_0(\mu) \to c_0(\mu)$ be defined by $P_r(x) = (x_0, x_1, \dots, x_r, 0, 0, \dots)$ (this is possible because $c_0(\mu)$ has AK and every $x = (x_k)_k \in c_0(\mu)$ has a unique representation $x = \sum_{k=0}^\infty (x_k) e^{(k)}$). We have (see [4])

$$\|(I - P_r)x\| = \|(0, 0, \dots, 0, x_{r+1}, x_{r+2}, \dots)\| = \{(I - P_r)x \in c_0(\mu), p = 1\}$$
$$= \sup_k(\frac{1}{\mu_k} \sum_{j=0}^{k} |\mu_j x_j - \mu_{j-1} x_{j-1}|) = \sup_k(\frac{1}{\mu_{r+k}} \sum_{j=r+1}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1}|)$$
$$= \sup_k(\frac{1}{\mu_{r+k}}(|\mu_{r+1} x_{r+1} - 0| + \sum_{j=r+2}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1}|))$$
$$= \sup_k(\frac{1}{\mu_{r+k}}(|\mu_{r+1} x_{r+1}| + \sum_{j=r+2}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1}|)).$$

Since

$$|\mu_{r+1} x_{r+1}| \le |\mu_{r+1} x_{r+1} - \mu_r x_r| +$$
$$+ |\mu_r x_r - \mu_{r-1} x_{r-1}| + \dots + |\mu_1 x_1 - \mu_0 x_0| + |\mu_0 x_0|$$

it follows

$$|\mu_{r+1} x_{r+1}| + \sum_{j=r+2}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1}| \le \sum_{j=0}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1}|,$$

$(\mu_{-1} = 0)$ and

$$\|(I - P_r)x\| \le \sup_k \frac{1}{\mu_{r+k}} \sum_{j=0}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1}|.$$

As we know, $\|x\| = \sup_k \frac{1}{\mu_k}\sum_{j=0}^{k} |\mu_j x_j - \mu_{j-1}x_{j-1}|$ for $x = (x_k)_k \in c_0(\mu)$ and therefore we have

$$\|(I-P_r)x\| \le \|x\|.$$

That implies $\|I - P_r\| \le 1$. Since $I - P_r$ is a projector, $\|I - P_r\| \ge 1$ and finally, $\|I - P_r\| = 1$. Let K be defined as in Theorem 4.2. By Theorem 4.1, we have

$$\|L_A\|_\chi = \chi(AK) = \limsup_{r\to\infty}(\sup_{x\in K}\|(I-P_r)Ax\|).$$

For given $\epsilon > 0$, there is $x \in K$ such that

$$\|(I-P_r)Ax\| \ge \|(I-P_r)A\| - \frac{\epsilon}{2},$$

i.e.

$$\sup_n \frac{1}{\mu_n}\sum_{i=r+1}^{n} |\mu_i A_i x - \mu_{i-1}A_{i-1}x| \ge \|(I-P_r)A\| - \tfrac{\epsilon}{2}. \qquad (*)$$

In the proof, we need the next lemma.

Lemma ([2]) *Let $a_0, a_1, \dots a_n \in C$. Then,*

$$\sum_{k=0}^{n} |a_k| \le 4 \max_{N\subset\{0,\dots,n\}} |\sum_{k\in N} a_k|.$$

By $(*)$, there is an integer $k(x) > r$ such that

$$\frac{1}{\mu_{k(x)}}\sum_{i=r+1}^{k(x)} |\mu_i A_i x - \mu_{i-1}A_{i-1}x| \ge \|(I-P_r)A\| - \frac{\epsilon}{2}.$$

By the lemma, we have

$$\sum_{i=r+1}^{k(x)} |\mu_i A_i x - \mu_{i-1}A_{i-1}x| \le 4 \max_{N\subset\{r+1,\dots,k(x)\}} |\sum_{i\in N} \mu_i A_i x - \mu_{i-1}A_{i-1}x|$$

and therefore

$$4 \max_{N\subset\{r+1,\dots,k(x)\}} \frac{1}{\mu_{k(x)}} |\sum_{i\in N} \mu_i A_i x - \mu_{i-1}A_{i-1}x| \ge$$
$$\ge \frac{1}{\mu_{k(x)}}\sum_{i=r+1}^{k(x)} |\mu_i A_i x - \mu_{i-1}A_{i-1}x| \ge \|(I-P_r)A\| - \epsilon. \qquad (**)$$

So, for arbitrary $\epsilon > 0$ and $x \in K$ we obtain that $(**)$ holds. Hence, $(**)$ holds for each r and we have

$$\|(I-P_r)A\| \le 4\sup_{k>r}(\max_{N_{r,k}\subset\{r+1,\dots,k\}} \|\frac{1}{\mu_k}\sum_{i\in N_{r,k}} \mu_i A_i - \mu_{i-1}A_{i-1}\|). \qquad (***)$$

By $(***)$ and the definition of $\|L_A\|_\chi$, we have

$$\|L_A\|_\chi \le 4\lim_{r\to\infty}(\sup_{k>r}(\max_{N_{r,k}\subset\{r+1,\dots,k\}} \|\frac{1}{\mu_k}\sum_{i\in N_{r,k}} \mu_i A_i - \mu_{i-1}A_{i-1}\|)),$$

i.e. $\|L_A\|_\chi \le 4\lim_{r\to\infty}\|A\|_{c_\infty}^{(r)}$.

We have proved the righthand side in (a). It remains to prove the lefthand one. Suppose that $x \in K$, $r \in N$, $k > r$ and $N_{r,k} \subset \{r+1, \dots, k\}$. Then we have

$$|\frac{1}{\mu_k} \sum_{i \in N_{r,k}} \mu_i A_i x - \mu_{i-1} A_{i-1} x| \le \frac{1}{\mu_k} \sum_{i \in N_{r,k}} |\mu_i A_i x - \mu_{i-1} A_{i-1} x|$$
$$\le \frac{1}{\mu_k} \sum_{i=r+1}^{k} |\mu_i A_i x - \mu_{i-1} A_{i-1} x| \le \|(I - P_r) A x\|.$$

Since $x \in K$, $r \in N$, $k > r$ were arbitrary, we conclude that for each r and $k > r$ we have

$$\|\frac{1}{\mu_k} \sum_{i \in N_{r,k}} \mu_i A_i x - \mu_{i-1} A_{i-1}\| \le \|(I - P_r) A x\|.$$

Hence , $\lim_{r \to \infty} \|A\|_{c_\infty}^{(r)} \le \|L_A\|_\chi$. Thus we have proved Part (a).

(b) Let $x \in c(\mu)$ (that means that x has a unique representation $x = le + \sum_{k=0}^{\infty} (x_k - l) e^{(k)}$ where l is such that $x - le \in c_0(\mu)$). Let us define the projector $P_r : c(\mu) \to c(\mu)$ by $P_r(x) = le + \sum_{k=0}^{r} (x_k - l) e^{(k)}$, i.e. $P_r(x) = (x_0, x_1, \dots, x_r, l, l, \dots)$. Similarly as in the case (a) we conclude

$$\|(I - P_r)x\| = \|(0, 0, \dots, 0, x_{r+1} - l, x_{r+2} - l, \dots)\|$$
$$= \sup_k \Big(\frac{1}{\mu_{r+k}} \sum_{j=r+1}^{r+k} |\mu_j (x_j - l) - \mu_{j-1}(x_{j-1} - l)| \Big)$$
$$= \sup_k \Big(\frac{1}{\mu_{r+k}} (|\mu_{r+1}(x_{r+1} - l)| + \sum_{j=r+2}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1} - l(\mu_j - \mu_{j-1})|) \Big)$$
$$\le |l| + \sup_k \Big\{ \frac{1}{\mu_{r+k}} (|\mu_{r+1} x_{r+1}| + \sum_{j=r+2}^{r+k} |\mu_j x_j - \mu_{j-1} x_{j-1}|) \Big\}.$$

By (a), we have

$$\|(I - P_r)x\| \le |l| + \|x\|. \tag{$\diamond$}$$

We apply a result from [6] and obtain

$$0 \le \Big| |l| - \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k x_k - \mu_{k-1} x_{k-1}| \Big| \le \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k (x_k - l) - \mu_{k-1}(x_{k-1} - l)|.$$

If n tends to infinity, the righthand side tends to zero (since $x - le \in c_0(\mu)$) and we obtain

$$|l| = \lim_{n \to \infty} \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k x_k - \mu_{k-1} x_{k-1}|.$$

Hence, $|l| \le \|x\|$ and from ($\diamond$) we have $\|I - P_r\| \le 2$. Applying Theorem 4.2, we obtain (b).

(c) This part is proved similarly as Theorem 4.2(c), because the space $c_\infty(\mu)$ does not have a Schauder basis either. ∎

Corollary 4.6. (i) *If either* $A \in (c^p(\Lambda), c_0(\mu))$ *or* $A \in (c^p(\Lambda), c(\mu))$ *then*

$$L_A \text{ is compact if and only if } \lim_{r\to\infty} \|A\|^{(r)}_{c_\infty} = 0.$$

(ii) *If either* $A \in (c^p_\infty(\Lambda), c_\infty(\mu))$ *or* $A \in (c^p(\Lambda), c_\infty(\mu))$ *then*

$$L_A \text{ is compact if } \lim_{r\to\infty} \|A\|^{(r)}_{c_\infty} = 0.$$

Let us remark that the converse of (ii) does not hold in general. The next example illustrates this.

Example 4.7. Let $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix as in Example 4.4. By Corollary 3.10, $A \in (c^p_\infty(\Lambda), c_\infty(\mu))$ and by Corollary 3.11, $A \in (c^p(\Lambda), c_\infty(\mu))$. Putting $x = e = (1, 1, \dots)$, we see that L_A is a compact operator. On the other side, we have

$$\begin{aligned}
\|A\|^{(r)}_{c_\infty} &= \sup_{m>r} \max_{N_{r,m}\subset\{r+1,\dots,m\}} \{\sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} \times \\
&\quad \times (\sum_{\nu} |\frac{1}{\mu_m} \sum_{n\in N_m} (\mu_n \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} - \mu_{n-1} \sum_{j=k}^{\infty} \frac{a_{n-1,j}}{\lambda_j})|^q)^{\frac{1}{q}}\} \\
&= \sup_{m>r} \max_{N_{r,m}\subset\{r+1,\dots,m\}} \{\lambda_{k(1)} (|\frac{1}{\mu_m} \sum_{n\in N_m} (\mu_n \frac{1}{\lambda_{k(0)}} - \mu_{n-1} \frac{1}{\lambda_{k(0)}})|^q)^{\frac{1}{q}}\} \\
&= \sup_{m>r} \max_{N_{r,m}\subset\{r+1,\dots,m\}} \{\lambda_{k(1)} \frac{1}{\lambda_{k(0)}} \frac{1}{\mu_m} \sum_{n\in N_m} (\mu_n - \mu_{n-1})\} \\
&= \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \sup_{m>r} \max_{N_{r,m}\subset\{r+1,\dots,m\}} (\frac{1}{\mu_m} \sum_{n\in N_m} (\mu_n - \mu_{n-1})) \\
&= \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \sup_{m>r} \frac{1}{\mu_m} (\mu_m - \mu_{r+1}) = \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \sup_{m>r} (1 - \frac{\mu_{r+1}}{\mu_m}) = \frac{\lambda_{k(1)}}{\lambda_{k(0)}} > 0.
\end{aligned}$$

We have $\lim_{r\to\infty} \|A\|^{(r)}_{c_\infty} \neq 0$ and L_A is a compact operator. Hence, the equivalence in (ii) does not hold.

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