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LIMIT THEOREM FOR HIGH LEVEL *A*-UPCROSSINGS BY χ-FIELD

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Abstract. A Poisson limit theorem for A-points of upcrossings of a high level by trajectories of the random field $\chi(\mathbf{t})$ is established. Kallenberg theorem, standard results from asymptotic methods of Gaussian process and fields and Piterbarg result of high level intersection of χ-field are exploited.

1. Introduction. Definitions. Result

Properties of high level intersection sets by trajectories of Gaussian random processes on infinitely increasing time horizon are well elaborated, see [3], [4] and references therein. Many important results in this direction have been obtained for Gaussian fields, [4]. In contrast, there are only few results about limit behavior of the number of large excursions of Gaussian vector processes and fields. First Poisson limit theorem for a-exit points over level u, where Gaussian vector process of arbitrary dimension was investigated, was established in [5]. The present paper deals with A-upcrossing (A-exit) points over high level u. We consider the stationary random field

$$\chi(\mathbf{t}) = \big(X_1^2(\mathbf{t}) + X_2^2(\mathbf{t}) + \cdots + X_n^2(\mathbf{t})\big)^{1/2} = \|\mathbf{X}(\mathbf{t})\|, \quad \mathbf{t} \in \mathbb{R}^m,$$

where $\mathbf{X}(\mathbf{t}) = (X_1(\mathbf{t}), X_2(\mathbf{t}), \ldots, X_n(\mathbf{t}))$ is a Gaussian vector field which components are independent copies of a Gaussian stationary field $X(\mathbf{t})$ with mean zero and covariance function $r(\mathbf{t})$.

Let the collection $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of positive numbers be given, as well as the collection $E = \{e_1, e_2, \ldots, e_k\}$ of positive integers such that $\sum_{i=1}^k e_i = m$. We set $e_0 = 0$. The pair (E, α) is called a *structure*. For any vector $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ the *structural modulus* is defined by

$$|\mathbf{t}|_{E,\alpha} = \sum_{i=1}^{k} \Big(\sum_{j=E(i-1)+1}^{E(i)} t_j^2 \Big)^{\frac{\alpha_i}{2}},$$

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where $E(i) = \sum_{j=0}^{i} e_j$, $i = 1, 2, \ldots, k$. We also denote

$$\mathbf{t}^i = (t_{E(i-1)+1}, \ldots, t_{E(i)}), \quad i = 1, 2, \ldots, k,$$

then $|\mathbf{t}|_{E,\alpha} = \sum_{i=1}^{k} |\mathbf{t}^i|^{\alpha_i}$, where the norm is taken in a Euclidean space of appropriate dimension. For simplicity we will use notation $|\mathbf{t}|_{E,\alpha} = |\mathbf{t}|_\alpha$.

We assume that

$$r(\mathbf{t}) = 1 - |\mathbf{t}|_\alpha + o(|\mathbf{t}|_\alpha) \text{ as } \mathbf{t} \to 0, \text{ for some } \alpha, 0 < \alpha_1 \le 2, \ldots, 0 < \alpha_k \le 2, \quad (1)$$

and

$$r(\mathbf{t}) \log \|\mathbf{t}\| \to 0 \text{ as } \|\mathbf{t}\| \to \infty. \quad (2)$$

From (2) it follows that

$$\text{for any } \delta > 0, \sup_{\|\mathbf{t}\| > \delta} |r(\mathbf{t})| < 1. \quad (3)$$

The aim of this paper is to prove a Poisson limit theorem for A-points of upcrossings of a high level by trajectories of the field $\chi(\mathbf{t})$. Following definitions are needed for the sequel.

Definition 1. A pair (A, ρ_A), $A \subset \mathbb{R}^m$, where A is a bounded Borel set and $\rho_A > 0$, is called a trap if

(a) relations $\mathbf{t} \notin A$, $0 \notin A + \mathbf{t}$ imply $|\mathbf{t}| > \rho_A$;

(b) one can find a point $\mathbf{t}$ in any non-empty closed bounded set $B \subset \mathbb{R}^m$ such that $(A + \mathbf{t}) \cap B = \{\mathbf{t}\}$.

Definition 2. Suppose a set $S \subset \mathbb{R}^m$ and a trap (A, ρ_A) are given. A point $\mathbf{t} \in S$ is called an A-point of the set S if $(A + \mathbf{t}) \cap S = \{\mathbf{t}\}$.

Definition 3. Let a trap (A, ρ_A) is given. A point $\mathbf{t}$ is called an A-upcrossing of the level u by the field $\chi(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^m$, if it is an A-point of the set $\{\mathbf{t} : \chi(\mathbf{t}) \geqslant u\}$.

For simplicity we will use the following notations for maximum distributions,

$$P_X(u, W) = \mathbf{P}\Big(\max_{\mathbf{t} \in W} X(\mathbf{t}) \le u\Big) \quad \text{and} \quad \overline{P}_X(u, W) = 1 - P_X(u, W),$$

where $X(\mathbf{t})$ is a random field.

Let $\nu(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^m$, be a Gaussian field with continuous trajectories, the expected value given by $E\nu(\mathbf{t}) = -|\mathbf{t}|_\alpha$ and the covariance function $\operatorname{cov}(\nu(\mathbf{t}), \nu(\mathbf{s})) = |\mathbf{t}|_\alpha + |\mathbf{s}|_\alpha - |\mathbf{t} - \mathbf{s}|_\alpha$. Let Υ be a compact set, $\Upsilon \subset \mathbb{R}^m$. Denote

$$H_\alpha(\Upsilon) = E \exp(\max_{\Upsilon} \nu(\mathbf{t})).$$

It is proven in [4] that for any $T > 0$ there exists

$$\lim_{T \to \infty} \frac{H_\alpha([0, T]^m)}{T^m} = H_\alpha, \quad 0 < H_\alpha < \infty,$$

H_α is called a Pickands' constant.

Denote

$$\mu(u) = \frac{2^{(3-n)/2}\sqrt{\pi}H_\alpha}{\Gamma(n/2)} u^{n-1}\Psi(u),$$

$$\mu^i(u) = u^{\frac{2}{\alpha_i^*}}(\mu(u))^{\frac{1}{m}}, \quad i = 1, \dots, m, \quad M(u) = \mu^1(u) \cdots \mu^m(u),$$

where Γ is the gamma function, $\Psi(u) = \frac{1}{\sqrt{2\pi}} u^{-1} e^{-u^2/2}$ and α_i^* is the number from the collection $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ corresponding to i-th coordinate of the vector $\mathbf{t}$. It follows from Corollary 7.3 [4], that for any $T > 0$,

$$\overline{P}_\chi(u, [0, T]^m) = T^m M(u)(1 + o(1)) \text{ as } u \to \infty. \tag{4}$$

Let us introduce a transformation of the space $\mathbb{R}^m$,

$$h_u(\mathbf{t}) = (t_1(\mu^1(u))^{-1}, \dots, t_m(\mu^m(u))^{-1}), \quad \mathbf{t} \in \mathbb{R}^m,\ u > 0.$$

By $\mathcal{B}$ denote the σ-algebra of Borel sets on $\mathbb{R}^m$. Let $\eta_{A,u}(B)$, $B \in \mathcal{B}$, be the point field of A-upcrossings of the level u by the field $\chi(\mathbf{t})$. Consider the normalized point field

$$\Phi_u(B) := \eta_{A,u}(h_u B).$$

Let $\Phi(\cdot)$ be the standard Poisson point field on $\mathcal{B}$, that is, stationary with intensity one. Our main result is

Theorem 1. *Let assumptions* (1, 2) *be fulfilled for the stationary random field* $\chi(\mathbf{t})$. *For any trap* (A, ρ_A) *the random point field* $\Phi_u(B)$, $B \in \mathcal{B}$, *converges weakly as* $u \to \infty$ *to the standard Poisson point field* $\Phi(B)$, $B \in \mathcal{B}$.

Let $\mathcal{L}$ be a sub-ring of $\mathcal{B}$, generated by rectangles $\prod_{i=1}^m [t_i, t_i + s_i)$, $s_i > 0$. Let an infinitely divisible point field $\Phi(B)$ on $\mathcal{B}$ be given such that $\Phi(\partial L) = 0$ with probability one for all $L \in \mathcal{L}$. By Kallenberg theorem (Theorem 4.7 [1]), if for any $L \in \mathcal{L}$,

$$\lim_{u\to\infty} \mathbf{P}(\Phi_u(L) = 0) = \mathbf{P}(\Phi(L) = 0) \text{ and } \limsup_{u\to\infty} \mathbf{E}\Phi_u(L) \le \mathbf{E}\Phi(L), \tag{5}$$

then the weak convergence

$$\Phi_u(B) \Rightarrow \Phi(B)$$

takes place. We prove in section 2 the relations of (5) for the Poisson point field $\Phi(B)$.

2. Proofs

Lemma 1. *For the intensity* $\mu_A(u)$ *of the random point process* $\eta_{A,u}(B)$ *we have* $\lim_{u\to\infty} \mu_A(u)/M(u) = 1$.

Proof. It is sufficient to evaluate the asymptotic behavior of the probability $\mathbf{P}(\eta_{A,u}(I) > 0)$ as $u \to \infty$, where $I = [0, 1]^m$. Note that

$$\{\eta_{A,u}(I) > 0\} \subset \Big\{\max_{\mathbf{t}\in I} \chi(\mathbf{t}) \geqslant u\Big\}.$$

On the other hand,

$$\{\eta_{A,u}(I)>0\}\supset\Big\{\max_{\mathbf{t}\in((I\oplus A)\setminus I)}\chi(\mathbf{t})\leqslant u,\max_{\mathbf{t}\in I}\chi(\mathbf{t})>u\Big\},\tag{6}$$

where $\oplus$ is the Minkovsky sum of sets, that is $A\oplus B=\{\mathbf{t}+\mathbf{s}:\mathbf{t}\in A,\mathbf{s}\in B\}$. Further,

$$\begin{aligned}\mathbf{P}&\Big\{\max_{\mathbf{t}\in((I\oplus A)\setminus I)}\chi(\mathbf{t})\leqslant u,\ \max_{\mathbf{t}\in I}\chi(\mathbf{t})>u\Big\}\\&=\bar{P}_\chi(u,I)-\mathbf{P}\Big\{\max_{\mathbf{t}\in((I\oplus A)\setminus I)}\chi(\mathbf{t})>u,\ \max_{\mathbf{t}\in I}\chi(\mathbf{t})>u\Big\}\\&=\bar{P}_\chi(u,I)-\left[\bar{P}_\chi(u,I)-\bar{P}_\chi(u,I\oplus A)+\bar{P}_\chi(u,(I\oplus A)\backslash I)\right].\end{aligned}$$

The expression in square brackets is infinitely smaller then $\bar{P}_\chi(u,I)$ so Lemma follows. ■

LEMMA 2. *For any* $L\in\mathcal{L}$, $\mathbf{P}\left(\Phi_u(L)=0\right)=P_\chi(u,h_uL)+o(1)$ *as* $u\to\infty$.

Proof. Denote $L_u=h_uL$, $L\in\mathcal{L}$. We have, $P_\chi(u,L_u)\leq\mathbf{P}\left(\Phi_u(L)=0\right)$. Further, similarly to (6) we have,

$$\mathbf{P}\left(\Phi_u(L)=0\right)\leq P_\chi(u,L_u)+\mathbf{P}\Big(\max_{\mathbf{t}\in L_u}\chi(\mathbf{t})>u,\max_{(L_u\oplus A)\setminus L_u}\chi(\mathbf{t})>u\Big).$$

We can split the set $(L_u\oplus A)\setminus L_u$ into sets of diameter between $1/2\mathrm{diam}\,A$ and $\mathrm{diam}\,A$. The number of such sets is equal to $O(V((L_u\oplus A)\setminus L_u)=O(V_{m-1}(\partial L_u))$, where V_{m-1} is an $(m-1)$-dimensional volume. We get

$$\mathbf{P}\Big(\max_{L_u}\chi(\mathbf{t})>u,\ \max_{(L_u\oplus A)\setminus L_u}\chi(\mathbf{t})>u\Big)\leq\mathbf{P}\Big(\max_{(L_u\oplus A)\setminus L_u}\chi(\mathbf{t})>u\Big)=o(1),$$

as $u\to\infty$. The Lemma is proven. ■

Introduce a Gaussian random field $Y(\mathbf{t},\mathbf{v})=(\mathbf{X}(\mathbf{t}),\mathbf{v})=\sum_{j=1}^n X_j(\mathbf{t})v_j$, where

$$\mathbf{t}\in\mathbb{R}^m,\mathbf{v}=(v_1,\dots,v_n)\in S_{n-1}:=\{(x_1,x_2,\dots,x_n):x_1^2+x_2^2+\dots+x_n^2=1\}.$$

By duality, for any closed $W\subset\mathbb{R}^m$,

$$\max_{\mathbf{t}\in W}\chi(\mathbf{t})=\max_{(\mathbf{t},\mathbf{v})\in W\times S_{n-1}}Y(\mathbf{t},\mathbf{v}),$$

in particular, $\chi(\mathbf{t})=\max_{\mathbf{v}\in S_{n-1}}Y(\mathbf{t},\mathbf{v})$. By Bunyakovsky, it is easy to see that the last maximum is attained at a unique point of the sphere S_{n-1}, which corresponds to a unit vector directed as $\mathbf{X}(\mathbf{t})$. Random field $Y(t,\mathbf{v})$, $(\mathbf{t},\mathbf{v})\in\mathbb{R}^m\times S_{n-1}$ is homogeneous according to the group of rotations on the sphere.

LEMMA 3. *For every* $L\in\mathcal{L}$ *and any* $\epsilon>0$ *one can find such* $b>0$, $u_0>0$, $K=K(\epsilon)>0$ *and a grid* $\mathcal{R}_{b,u,\epsilon}$ *on the manifold* $L_u\times S_{n-1}$, *that for all* $u\geq u_0$,

$$P_Y(u,(L_u\times S_{n-1})\cap\mathcal{R}_b)-P_Y(u,(L_u\times S_{n-1}))\leq K\epsilon.$$

Proof. We show that the assertion holds true for $L = [0,T]^m$. Since in general L consists of finite number of rectangles, the proof for general L will obviously follow. First we partition the sphere S_{n-1} onto $N(\epsilon)$ parts $A_1, \dots, A_{N(\epsilon)}$ in the following way. Consider polar coordinates on the sphere S_{n-1},

$$(x_1, x_2, \dots, x_n) = S(\varphi_1, \varphi_2, \dots \varphi_{n-1}),$$

$\varphi_1, \varphi_2, \dots \varphi_{n-2} \in [0,\pi)$, $\varphi_{n-1} \in [0, 2\pi)$ and divide the interval $[0,\pi]$ to intervals of length ϵ (or less for the last interval), do the same for the interval $[0, 2\pi]$. This partition of the parallelepiped $[0,\pi]^{n-2} \times [0, 2\pi]$ generates the partition $A_1, \dots, A_{N(\epsilon)}$ of the sphere. Now we construct the grid $\mathcal{R}_{b,u,\epsilon}$. Choose in any A_j an inner point B_j and consider the tangent plane to the manifold $[0,T]^m \times S_{n-1}$ at the point $\mathbf{O} \times B_j$, where $\mathbf{O}$ is origin in $\mathbb{R}^m$. Introduce in the tangent plane rectangular coordinates, with origin at the tangent point, the first coordinates are according to vector $\mathbf{t}$, so that the plane becomes $\mathbb{R}^{m+n-1}$, and consider the grid

$$\mathcal{R}^j(b) := \mathcal{R}_{b,u,\epsilon^{j,P}} := (bk_1 u^{-2/\alpha_1^*}, bk_2 u^{-2/\alpha_2^*}, \dots, bk_m u^{-2/\alpha_m^*}, bl_1 u^{-1}, \dots, bl_{n-1} u^{-1}),$$

$j = 1, 2, \dots, N(\epsilon)$, $(k_1, k_2, \dots, k_m, l_1, l_2, \dots, l_{n-1}) \in \mathbb{Z}_+^m \times \mathbb{Z}^{n-1}$.

Suppose that ϵ is so small that the orthogonal projection of all A_j onto corresponding tangent planes are one-to-one. Denote by $\mathcal{R}^j_{b,u,\epsilon}$, the prototype of $\mathcal{R}^{j,P}_{b,u,\epsilon}(b)$ under this projection. We show that the grid

$$\mathcal{R}_b := \mathcal{R}_{b,u,\epsilon} := \bigcup_{j=1}^{N(\epsilon)} \mathcal{R}^j_{b,u,\epsilon},$$

with appropriate choice of its parameters, satisfies the assertion of the lemma.

We have,

$$\begin{aligned} &P_Y(u, (L_u \times S_{n-1}) \cap \mathcal{R}_b) - P_Y(u, L_u \times S_{n-1}) \\ &\quad = \mathbf{P}\Big(\bigcap_{j=1}^{N(\epsilon)} \Big(\max_{(L_u \times A_j) \cap \mathcal{R}_b} Y(\mathbf{t}, \mathbf{v}) \le u \Big) \cap \bigcup_{j=1}^{N(\epsilon)} \Big(\max_{L_u \times A_j} Y(\mathbf{t}, \mathbf{v}) > u \Big) \Big) \\ &\quad \le \sum_{j=1}^{N(\epsilon)} \mathbf{P}\Big(\max_{(L_u \times A_j) \cap \mathcal{R}_b} Y(\mathbf{t}, \mathbf{v}) \le u, \ \max_{L_u \times A_j} Y(\mathbf{t}, \mathbf{v}) > u \Big). \end{aligned} \tag{7}$$

Denote by A_j^P projection of part $A_j$at the tangent plane, and denote by $\mathbf{w}$ projection of the point $\mathbf{v}$ from A_j at the tangent plane. (We shall use same notation for corresponding vectors.) From the geometry of sphere, it follows that

$$\sup_{\substack{\mathbf{v}_1, \mathbf{v}_2 \in A_j \\ j=1,2,\dots,N(\epsilon)}} \frac{\|\mathbf{w_1} - \mathbf{w_2}\|}{\|\mathbf{v_1} - \mathbf{v_2}\|} \geqslant 1 - 2\epsilon.$$

So, from (1) it follows that for all sufficiently small ϵ there exist $\delta(\epsilon) > 0$ such that for every $j = 1, 2, \ldots, N(\epsilon)$ for the covariance function $r_j((\mathbf{t}, \mathbf{w}_1), (\mathbf{s}, \mathbf{w}_2))$ of the Gaussian field $Z_j(\mathbf{t}, \mathbf{w}) = Y(\mathbf{t}, \mathbf{v})$, $\mathbf{v} \in A_j$ the following holds true,

$$(1-2\epsilon)\Big(|\mathbf{t}-\mathbf{s}|_\alpha + \frac{1}{2}\|\mathbf{w}_1-\mathbf{w}_2\|^2\Big) \le 1 - r_j((\mathbf{t}, \mathbf{w}_1), (\mathbf{s}, \mathbf{w}_2)) \le (1+2\epsilon)\Big(|\mathbf{t}-\mathbf{s}|_\alpha + \frac{1}{2}\|\mathbf{w}_1-\mathbf{w}_2\|^2\Big),$$

where $\mathbf{t}, \mathbf{s} \in [0, \delta(\epsilon)]^m$, $\mathbf{w}_1, \mathbf{w}_2 \in A_j^P$. Partitioning the rectangle $L_u = h_u[0, T]^m$ onto cubes of the length $\delta(\epsilon)$ and using that Y is stationary with respect to $\mathbf{t}$, we get that the last sum in (7) does not exceed

$$\frac{2M(u)^{-1}T^m}{\delta^m(\epsilon)} \sum_{j=1}^{N(\epsilon)} \mathbf{P}\Big(\max_{([0,\delta(\epsilon)]^m \times A_j)\cap \mathcal{R}_b} Y(\mathbf{t}, \mathbf{v}) \le u, \max_{[0,\delta(\epsilon)]^m \times A_j} Y(\mathbf{t}, \mathbf{v}) > u\Big)$$
$$= \frac{2M(u)^{-1}T}{\delta^m(\epsilon)} \sum_{j=1}^{N(\epsilon)} \mathbf{P}\Big(\max_{([0,\delta(\epsilon)]^m \times A_j^p)\cap \mathcal{R}^j(b)} Z_j(\mathbf{t}, \mathbf{w}) \le u, \max_{[0,\delta(\epsilon)]^m \times A_j^p} Z_j(\mathbf{t}, \mathbf{w}) > u\Big). \tag{8}$$

Let $\xi(\mathbf{t}, \mathbf{w})$, $\mathbf{t} \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^{m+n-1}$, be a stationary Gaussian field with zero mean and covariance function $\rho(\mathbf{t}, \mathbf{w}) = e^{-|\mathbf{t}|_\alpha - \frac{1}{2}\|\mathbf{w}\|^2}$. Below we use the following operation of multiplication of a vector by a set. Let A be a set in $\mathbb{R}^{m+n-1}$and $\mathbf{b}$ be a vector in $\mathbb{R}^n$ with positive coordinates. Then we set $\mathbf{b}A := (\mathbf{x} = (x_1, \ldots, x_{m+n-1}) : (x_1/b_1, \ldots, x_{m+n-1}/b_{m+n-1}) \in A)$. Further, by Slepian's inequality, (8) can be estimated from above by

$$\frac{2M(u)^{-1}T^m}{\delta^m(\epsilon)} \sum_{j=1}^{N(\epsilon)} \Bigg\{\mathbf{P}\Big(\max_{\mathbf{c}(-\epsilon)[([0,\delta(\epsilon)]^m \times A_j^P)\cap \mathcal{R}^j(b)]} \xi(\mathbf{t}, \mathbf{w}) \le u, \max_{\mathbf{c}(-\epsilon)[([0,\delta(\epsilon)]^m \times A_j^P)]} \xi(\mathbf{t}, \mathbf{w}) > u\Big) + \Big[\mathbf{P}\Big(\max_{\mathbf{c}(-\epsilon)[([0,\delta(\epsilon)]^m \times A_j^P)]} \xi(\mathbf{t}, \mathbf{w}) \le u\Big) - \mathbf{P}\Big(\max_{\mathbf{c}(\epsilon)[([0,\delta(\epsilon)]^m \times A_j^P)]} \xi(\mathbf{t}, \mathbf{w}) \le u\Big)\Big]\Bigg\} \tag{9}$$

where $\mathbf{c}(\pm\epsilon) = ((1 \pm 2\epsilon)^{1/\alpha_1^*}, \ldots, (1 \pm 2\epsilon)^{1/\alpha_m^*}, (1 \pm 2\epsilon)^{1/2}, \ldots, (1 \pm 2\epsilon)^{1/2})$ and ϵ is sufficiently small. First we estimate the second sum in the right-hand part of (9). By Lemma 6.1 [4], for sufficiently large u, one can find constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ such that the sum can be estimated by

$$\frac{2M(u)^{-1}T^m}{\delta^m(\epsilon)} C_1 u^{n-1} \prod_{i=1}^{m} u^{2/\alpha_i^*} \Psi(u) \delta^m(\epsilon) \cdot \epsilon \sum_{j=1}^{N(\epsilon)} V(A_j^P) \le C_2 \epsilon,$$

where V denotes the volume in corresponding dimension. Turn to the first sum in the right-hand part of (9). Denote

$$\mathbf{g}_u = ((1-2\epsilon)^{1/\alpha_1^*} u^{-2/\alpha_1^*}, (1-2\epsilon)^{1/\alpha_m^*} u^{-2/\alpha_m^*}, (1-2\epsilon)^{1/2} u^{-1}, \ldots, (1-2\epsilon)^{1/2} u^{-1}),$$

$S = bN$, N is an integer, $K = [0, S]^{m+n-1}$. Since ξ is stationary, we have, partitioning the set $\mathbf{c}(-\epsilon)[([0, \delta(\epsilon)]^m \times A_j^P)]$ onto parallelepipeds, equal to $\mathbf{g}_u K$, that the first sum can be estimated by

$$\frac{2T^m \prod_{i=1}^{m} u^{2/\alpha_i^*}}{M(u) S^{m+n-1} u^{1-n}} \sum_{j=1}^{N(\epsilon)} \mathbf{P}\left(\max_{\mathbf{g}_u[K \cap \mathcal{R}^j(b)]} \xi(\mathbf{t}, \mathbf{w}) \leq u, \max_{\mathbf{g}_u K} \xi(\mathbf{t}, \mathbf{w}) > u \right) V(A_j^P). \quad (10)$$

Let $\theta(\mathbf{t}, \mathbf{w})$, $\mathbf{t} \in \mathbb{R}^m$, $\mathbf{w} \in \mathbb{R}^{n-1}$ be a Gaussian separable field with parameters

$$\mathbf{E}\theta(\mathbf{t}, \mathbf{w}) = (1 - 2\epsilon)(|\mathbf{t}|_\alpha + \frac{1}{2}\|\mathbf{w}\|^2),$$

$$\mathbf{Var}(\theta(\mathbf{t}, \mathbf{w}_1) - \theta(\mathbf{s}, \mathbf{w}_2)) = 2(1 - 2\epsilon)(|\mathbf{t} - \mathbf{s}|_\alpha + \frac{1}{2}\|\mathbf{w}_1 - \mathbf{w}_2\|^2).$$

Following evaluations in the proof of Lemma 6.1 from [4] one can show that

$$\lim_{u \to \infty} \sqrt{2\pi} u e^{u^2/2} \mathbf{P}\left(\max_{\mathbf{g}_u[K \cap \mathcal{R}^j(b)]} \xi(\mathbf{t}, \mathbf{w}) \leq u, \max_{\mathbf{g}_u K} \xi(\mathbf{t}, \mathbf{w}) > u \right)$$
$$= \int_0^\infty e^s \mathbf{P}\left(\max_K \theta(\mathbf{t}, \mathbf{w}) > s,\ \max_{K \cap b\mathbb{Z}^n} \theta(\mathbf{t}, \mathbf{w}) \leq s \right) ds.$$

Since trajectories of θ are a.s. continuous, the probability under the integral tends to zero as $b \to 0$, for any s (for a fixed $S = Nb$). By Theorem of dominated convergence, the last integral tends to zero when $b \to 0$. Thus for sufficiently large u and sufficiently small b, the first sum in the right-hand part of (10) can be bounded by ϵ, so Lemma follows. ■

Let $L \in \mathcal{L}$ and T is as large as $L \subset [0, T]^m = \Pi$. Let δ be a positive number less then one. Let us divide each edge of the rectangle Π_u onto "large" segments of length one alternated by the "small" ones of length δ. This partition induces a partition of Π_u on the union π_u of cubes with edges of length one and with distance between them not less than δ, and on the complement of this union. Denote also by λ_u, the union of all the cubes which are contained in L_u, denote by N, the number of all the cubes in the L_u. We will use the notation

$$\mathcal{R}(b) = \{bu^{-2/\alpha_1^*}\mathbb{Z}\} \times \cdots \times \{bu^{-2/\alpha_m^*}\mathbb{Z}\}$$

in m-dimension case as well.

Lemma 4. *For any $L \in \mathcal{L}$ with $\mu(L) > 0$ and every $\varepsilon > 0$ one can find $\delta > 0$ such that for all sufficiently large u, $P_\chi(u, \lambda_u \cap \mathcal{R}(b)) - P_\chi(u, L_u \cap \mathcal{R}(b)) \leq \varepsilon$.*

Proof. Let $L \subset [0, T]^m$, using (4) for all sufficiently large u, we have,

$$P_\chi(u, \lambda_u \cap \mathcal{R}(b)) - P_\chi(u, L_u \cap \mathcal{R}(b)) = \mathbf{P}\left(\max_{\lambda_u \cap \mathcal{R}(b)} \chi(\mathbf{t}) \leq u,\ \max_{L_u \cap \mathcal{R}(b)} \chi(\mathbf{t}) > u \right)$$
$$\leq \overline{P}_\chi(u, L_u \setminus \lambda_u) \leq 2\frac{M(u)^{-1} T^m}{(1 + \delta)^m} \overline{P}_\chi(u, [0, \delta]^m) \leq \frac{\delta^m T^m}{(1 + \delta)^m} \leq \varepsilon,$$

by obvious appropriate choice of δ. The Lemma is proven. ■

Denote by K_j, $j = 1, 2, \ldots$, the cubes with edges of length one from the π_u. Consider infinitely many independent copies $Y_j(\mathbf{t}, \mathbf{v})$ of the Gaussian field $Y(\mathbf{t}, \mathbf{v})$, $(\mathbf{t}, \mathbf{v}) \in K_j \times S_{n-1}$, $j = 1, 2, \ldots$, and introduce the Gaussian random field $Y_0(\mathbf{t}, \mathbf{v}) = Y_j(\mathbf{t}, \mathbf{v})$ for $(\mathbf{t}, \mathbf{v}) \in K_j \times S_{n-1}$. By standard technics, using Berman inequality, we can establish the following

Lemma 5. *For any $L \in \mathcal{L}$,*

$$P_Y(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b) - P_{Y_0}(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b) \to 0 \text{ as } u \to \infty.$$

Proof. By Berman's inequality, we have,

$$\begin{aligned}
&|P_Y(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b) - P_{Y_0}(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b)| \\
&\quad \le \frac{1}{\pi} \sum_{\substack{(\mathbf{t},\mathbf{v}_1),(\mathbf{s},\mathbf{v}_2) \in (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b \\ (\mathbf{t},\mathbf{v}_1) \neq (\mathbf{s},\mathbf{v}_2)}} |r_Y((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2)) - r_{Y_0}((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2))| \\
&\qquad \times \int_0^1 (1 - r_h((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2))^{-1/2} \exp\left(-\frac{u^2}{1 + r_h((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2))}\right) dh \\
&\quad = \frac{1}{\pi} \sum_{i \neq j} \sum_{\substack{(\mathbf{t},\mathbf{v}_1) \in (K_i \times S_{n-1}) \cap \mathcal{R}_b \\ (\mathbf{s},\mathbf{v}_2) \in (K_j \times S_{n-1}) \cap \mathcal{R}_b}} |r_Y((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2)| \\
&\qquad \times \int_0^1 (1 - h r_Y((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2))^{-1/2} \exp\left(-\frac{u^2}{1 + h r_Y((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2))}\right) dh,
\end{aligned}$$

where

$$r_h((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2)) = h r_Y((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2)) + (1 - h) r_{Y_0}((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2)),$$

$\mathbf{t}, \mathbf{s} \in \mathbb{R}^m$, $\mathbf{v}_1, \mathbf{v}_2 \in S_{n-1}$, r_Y is the covariance function of the field Y, r_{Y_0} is the covariance function of the field Y_0. Now we are in a position to estimate the last sum to show that it tends to zero. To begin with, note that from the equality

$$r_Y((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2)) = r(\mathbf{t} - \mathbf{s})\Big(1 - \frac{1}{2}\|\mathbf{v}_1 - \mathbf{v}_2\|^2\Big)$$

and (3) it follows that there is γ_2, $0 < \gamma_2 < 1$ such that

$$|r_Y((\mathbf{t}, \mathbf{v}_1), (\mathbf{s}, \mathbf{v}_2))| \le 1 - \gamma_2$$

as $\|\mathbf{t} - \mathbf{s}\| \ge \delta$. Further, consider first "not too outstanding" $\mathbf{t}$ and $\mathbf{s}$, that is, $\mathbf{t} \in K_i$, $\mathbf{s} \in K_j$, and $d(K_i, K_j) \le M(u)^{-\gamma_1}$, where $\gamma_1 \in (0, 1 + \frac{1}{2}\gamma_2)$. (We define $d(K_i, K_j) := \sup\{\|\mathbf{t} - \mathbf{s}\| : \mathbf{t} \in K_i, \mathbf{s} \in K_j\}$.) Denoting by Σ_1 the part of the sum over such $\mathbf{t}, \mathbf{s}, \mathbf{v}_1, \mathbf{v}_2$ we get,

$$\begin{aligned}
\Sigma_1 &\leq C_1 \sum_{\substack{i \neq j \\ d(K_i,K_j) \leq M(u)^{-\gamma_1}}} \sum_{\substack{(\mathbf{t},\mathbf{v}_1) \in (K_i \times S_{n-1}) \cap \mathcal{R}_b \\ (\mathbf{s},\mathbf{v}_2) \in (K_j \times S_{n-1}) \cap \mathcal{R}_b}} \exp\left(-\frac{u^2}{1 + |r_Y((\mathbf{t},\mathbf{v}_1),(\mathbf{s},\mathbf{v}_2))|}\right) \\
&\leq C_2 \sum_{\substack{i \neq j \\ d(K_i,K_j) \leq M(u)^{-\gamma_1}}} \sum_{\substack{(\mathbf{t},\mathbf{v}_1) \in (K_i \times S_{n-1}) \cap \mathcal{R}_b \\ (\mathbf{s},\mathbf{v}_2) \in (K_j \times S_{n-1}) \cap \mathcal{R}_b}} \exp\left(-\frac{u^2}{2}(1 + \frac{\gamma_2}{2})\right) \\
&= O\left(\mu(u)^{-1} \epsilon^{\frac{n(n-1)}{2}} u^{n-1+2/\alpha} \mu(u)^{-\gamma_1} u^{n-1+2/\alpha} \epsilon^{\frac{n(n-1)}{2}} e^{-\frac{u^2}{2}(1+\frac{\gamma_2}{2})}\right) \\
&= o(1)
\end{aligned}$$

as $u \to \infty$, where C_1 and C_2 are constants. We have used above that the volume of every A_j has order $\epsilon^{\frac{n(n-1)}{2}}$, for small ϵ.

Turn now to those $\mathbf{t}, \mathbf{s}, \mathbf{v}_1, \mathbf{v}_2$ for which $d(K_i, K_j) \geq M(u)^{-\gamma_1}$, $\mathbf{t} \in K_i$, $\mathbf{s} \in K_j$. Denote the corresponding part of the sum by Σ_2. From (2) we get in this case,

$$\sup_{\|\mathbf{t}-\mathbf{s}\| \geq M(u)^{-\gamma_1}, \mathbf{v}_1, \mathbf{v}_2 \in S_{n-1}} r_Y((\mathbf{t},\mathbf{v}_1),(\mathbf{s},\mathbf{v}_2)) := \kappa(u) = o(u^{-2})$$

as $u \to \infty$, so that

$$\begin{aligned}
\Sigma_2 \leq C_3 \kappa(u) &\sum_{\substack{i \neq j \\ d(K_i,K_j) \geq M(u)^{-\gamma_1}}} \\
&\sum_{\substack{(\mathbf{t},\mathbf{v}_1) \in (K_i \times S_{n-1}) \cap \mathcal{R}_b \\ (\mathbf{s},\mathbf{v}_2) \in (K_j \times S_{n-1}) \cap \mathcal{R}_b}} \exp\left(-\frac{u^2}{1 + |r_Y((\mathbf{t},\mathbf{v}_1),(\mathbf{s},\mathbf{v}_2))|}\right) \\
\leq C_4 \kappa(u) e^{-u^2} &\sum_{\substack{i \neq j \\ d(K_i,K_j) \geq M(u)^{-\gamma_1}}} \\
&\sum_{\substack{(\mathbf{t},\mathbf{v}_1) \in (K_i \times S_{n-1}) \cap \mathcal{R}_b \\ (\mathbf{t},\mathbf{v}_2) \in (K_j \times S_{n-1}) \cap \mathcal{R}_b}} \exp\left(-\frac{|r_Y((\mathbf{t},\mathbf{v}_1),(\mathbf{s},\mathbf{v}_2))| u^2}{1 + |r_Y((\mathbf{t},\mathbf{v}_1),(\mathbf{s},\mathbf{v}_2))|}\right) \\
= O &\left(\left(M(u)^{-1} \epsilon^{\frac{n(n-1)}{2}} u^{n-1+2/\alpha}\right)^2 \kappa(u) e^{-u^2}\right) = O(u^2 \kappa(u)) = o(1)
\end{aligned}$$

as $u \to \infty$, where C_3 and C_4 are constants. Thus the lemma is proven. ∎

Proof of Theorem 1. Note that Lemma 3 holds true also for the field Y_0, with the same grid. It is easy to see that for any $\varepsilon > 0$ one can chose δ sufficiently small in order to have $|N \cdot M(u) - V(L)| \leq \varepsilon$, $V(\cdot)$ is a propriety measure. From here we get,

$$P_{Y_0}(u, \lambda_u \times S_{n-1}) = (1 - \overline{P}_{Y_0}(u, \lambda_u \times S_{n-1}))^N \to e^{-V(L)}, \text{ as } u \longrightarrow \infty.$$

Taking into account Lemma 2, Lemma 3, Lemma 4, Lemma 5, we obtain the first relation in (5). It is easy to see that the second assertion of the (5) follows from the equivalence of $M(u)$ and $\mu_A(u)$. Now the proof of Theorem 1 is completed. ∎

REFERENCES

[**1**] O. Kallenberg, *Random measures*, Academic Press, 1983.

[**2**] D. Konstant, V. Piterbarg, S. Stamatovic, *Limit theorems for cyclo-stationary* χ^2 *processes*, Lithuanian Mathematical Journal, **44**(2) (2004), 196–208.

[**3**] M. R. Leadbetter, G. Lindgren, H. Rootzen, *Extremes and Related Properties of Random Sequences and Processes*, Springer, 1986.

[**4**] V. Piterbarg, *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, AMS, Providence, 1996.

[**5**] V. Piterbarg, S. Stamatovic, *Limit theorem for high level* $a-$*upcrossings by* χ*-process*, Theory Probab. Appl., **48**(4) (2003), 734–741.

[**6**] S. Stamatovic, *Limit theorem for high level* A*-upcrossings by* χ*-process*, Math. Balkanica, **18** (1–2) (2004), 205–214.

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