МАТЕМАТИЧКИ ВЕСНИК
57 (2005), 109–112

UDK 517.984
оригинални научни рад
research paper

# COMPACT COMPOSITION OPERATORS ON LORENTZ SPACES

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**Abstract.** We give a necessary and sufficient condition for the compactness of composition operators on the Lorentz spaces.

Let $X = (X, \Sigma, \mu)$ be a $\sigma$-finite measure space. By $L(\mu)$, we denote the linear space of all equivalence classes of $\Sigma$-measurable functions on $X$. Let $T \colon X \to X$ be a non-singular measurable transformation. Then $T$ induces a composition transformation $C_T$ from $L(\mu)$ into itself defined by

$$C_T f(x) = f(T(x)), \quad x \in X, \quad f \in L(\mu).$$

Here, the non-singularity of $T$ guarantees that the operator $C_T$ is well defined as a mapping of equivalence classes of functions into itself. If $C_T$ maps a Lorentz space $L(pq, \mu)$ into itself, then we call $C_T$ a composition operator on $L(pq, \mu)$ induced by $T$.

Composition operators are simple operators, but have wide range of applications in ergodic theory, dynamical systems etc., see [7]. Composition operators have been studied mostly on $H^2$-spaces, $L^p$-spaces and Orlicz spaces (cf. [2–7]). So, it is natural to extend the study of composition operators to other measurable function spaces. In this paper, we have initiated the study of composition operators on Lorentz spaces, which generalize $L^p$ spaces. See [1] for details on Lorentz spaces.

For $f \in L(\mu)$, we define the distribution function of $|f|$ on $0 < \lambda < \infty$ by

$$\mu_f(\lambda) = \mu\{x \in X : |f(x)| > \lambda\},$$

and the non-increasing rearrangement of $f$ on $(0, \infty)$ by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\} = \sup\{\lambda > 0 : \mu_f(\lambda) > t\}.$$

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*AMS Subject Classification*: Primary 47B33, 46E30; Secondary 47B07, 46B70.

*Keywords and phrases*: Compact operator, composition operators, Lorentz spaces, measurable transformation.

First named author is supported by CSIR–grant (sanction no. 9/100 (96)/2002-EMR-I, dated–13-5-2002).

The Lorentz spaces $L(pq,\mu)$ are defined by

$$L(pq,\mu) = \{f \in L(\mu) : \|f\|_{L(pq)} < \infty\},$$

where

$$\|f\|_{L(pq)} = \begin{cases} [\int_0^\infty (t^{1/p} f^*(t))^q dt/t]^{1/q}, & \text{if } 1 < p < \infty,\ 1 \le q < \infty, \\ \sup_{0<t<\infty} t^{1/p} f^*(t), & \text{if } 1 < p < \infty,\ q = \infty. \end{cases}$$

The Lorentz spaces $(L(pq,\mu,\|.\|_{L(pq)})$ are Banach spaces for $1 \le q \le p < \infty$, or $p = q = \infty$.

The following theorem is easy to prove.

Theorem 1. *Let $T\colon X \to X$ be a non-singular measurable transformation. Then $T$ induces a composition operator $C_T$ on $L(pq,\mu)$, $1 \le q \le p < \infty$ if and only if there exists some $M > 0$ such that*

$$\mu \circ T^{-1}(A) \le M\mu(A), \quad \text{for each } A \in \Sigma.$$

*Moreover,*

$$\|C_T\| = \sup_{A\in\Sigma, 0<\mu(A)<\infty} \left(\frac{\mu \circ T^{-1}(A)}{\mu(A)}\right)^{1/p}.$$

Compact composition operators on $L^p$-spaces were studied in [4], [5], [6] and [8]. For the compactness of these operators on Orlicz spaces, see [2]. Now we give a necessary and sufficient condition for the compactness of composition operators on $L^{pq}(\mu)$, $1 \le q \le p < \infty$.

Theorem 2. *Let $(X,\Sigma,\mu)$ be a $\sigma$-finite measure space and $T\colon X \to X$ be a non-singular measurable transformation. Let $\{A_n\}$ be all the atoms of $X$ and assume that $\mu(A_n) = a_n > 0$, for each $n$. Then $C_T$ is a compact composition operator on Lorentz spaces $L(pq,\mu)$, $1 \le q \le p < \infty$ if and only if the measure space $(X,\Sigma,\mu)$ is purely atomic and*

$$b_n = \frac{\mu T^{-1}(A_n)}{\mu(A_n)} \to 0.$$

*Proof.* Suppose $C_T$ is compact. Then, using the similar techniques as in [6], it is easy to show that $\mu$ is purely atomic.

Next, we claim that $b_n \to 0$. Suppose the contrary. Then there exists some $\epsilon > 0$ and a subsequence $\{b_{n_k}\}_{k\ge1}$ of the sequence $\{b_n\}_{n\ge1}$ such that $b_{n_k} \ge \epsilon$, for all $k \in \mathbf{N}$.

Let $X = \bigcup_{n=1}^\infty A_n$, where $A_n$'s are atoms. For each $n \in \mathbf{N}$, define

$$f_n = \frac{\chi_{A_n}}{\|\chi_{T^{-1}(A_n)}\|_{L(pq)}}.$$

So, for $1 \le q \le p < \infty$, we have

$$\|f_{n_k}\|_{L(pq)} = \frac{\|\chi_{A_{n_k}}\|_{L(pq)}}{\|\chi_{T^{-1}(A_{n_k})}\|_{L(pq)}} = \left(\frac{\mu(A_{n_k})}{\mu(T^{-1}(A_{n_k}))}\right)^{1/p} = 1/b_{n_k}^{1/p} \le 1/\epsilon^p,$$

for each $k \in \mathbf{N}$.

Let $g_{nm} = C_T f_{k_n} - C_T f_{k_m}$, where $\{f_{k_n}\}_{n\ge1}$ is the subsequence of the sequence $\{f_k\}_{k\ge1}$. For $n \ne m$, three cases arise: either $\mu(T^{-1}(A_{k_n})) < \mu(T^{-1}(A_{k_m}))$ or $\mu(T^{-1}(A_{k_n})) = \mu(T^{-1}(A_{k_m}))$; the third case is similar to the first one, with $m$ and $n$ interchanging places.

In the first case, we have $\|\chi_{T^{-1}(A_{k_n})}\|_{L(pq)} < \|\chi_{T^{-1}(A_{k_m})}\|_{L(pq)}$ and

$$\mu_{g_{nm}}(\lambda) = \mu\left\{ x \in X : \left| \frac{\chi_{T^{-1}(A_{k_n})(x)}}{\|\chi_{T^{-1}(A_{k_n})}\|_{L(pq)}} - \frac{\chi_{T^{-1}(A_{k_m})(x)}}{\|\chi_{T^{-1}(A_{k_m})}\|_{L(pq)}} \right| > \lambda \right\}$$
$$= \begin{cases} \mu T^{-1}(A_{k_m}) + \mu T^{-1}(A_{k_n}), & \text{if } 0 < \lambda < \frac{1}{\|\chi_{T^{-1}(A_{k_m})}\|_{L(pq)}}, \\ \mu T^{-1}(A_{k_n}), & \text{if } \frac{1}{\|\chi_{T^{-1}(A_{k_m})}\|_{L(pq)}} \le \lambda < \frac{1}{\|\chi_{T^{-1}(A_{k_n})}\|_{L(pq)}}, \\ 0, & \text{if } \lambda \ge \frac{1}{\|\chi_{T^{-1}(A_{k_n})}\|_{L(pq)}}. \end{cases}$$

So, we have

$$g^*_{nm}(t) = \inf\{\lambda > 0 : \mu_{g_{nm}}(\lambda) \le t\}$$
$$= \begin{cases} \frac{1}{\|\chi_{T^{-1}(A_{k_n})}\|_{L(pq)}}, & \text{if } 0 \le t < \mu T^{-1}(A_{k_n}), \\ \frac{1}{\|\chi_{T^{-1}(A_{k_m})}\|_{L(pq)}}, & \text{if } \mu T^{-1}(A_{k_n}) \le t < \mu T^{-1}(A_{k_n}) + \mu T^{-1}(A_{k_m}), \\ 0, & \text{if } t \ge \mu T^{-1}(A_{k_m}) + \mu T^{-1}(A_{k_n}). \end{cases}$$

Thus,

$$\|g_{nm}\|^q_{L(pq)} = 1 + \left( \frac{\left(\mu T^{-1}(A_{k_m}) + \mu T^{-1}(A_{k_n})\right)^{q/p} - \left(\mu T^{-1}(A_{k_n})\right)^{q/p}}{(\mu T^{-1}(A_{k_m}))^{q/p}} \right) > 1,$$

which contradicts the compactness of $C_T$.

In the second case, $1/c = \|\chi_{T^{-1}(A_{k_n})}\|_{L(pq)} = \|\chi_{T^{-1}(A_{k_m})}\|_{L(pq)}$, we have

$$\|g_{nm}\|_{L(pq)} = c(\,p/q\,)^{1/q}(\,2\,\mu T^{-1}(A_{k_n})\,)^{1/p} = 2^{1/p} > 0$$

which contradicts the compactness of $C_T$. Hence $b_n \to 0$.

Conversely, let $(X, \Sigma, \mu)$ be atomic with atoms $A_n$ and $b_n \to 0$. Note that $f$ and $\sum f(A_n)\chi_{A_n}$ are equal $\mu$-a.e. For each $N \in \mathbf{N}$, define $C_T^{(N)}$ by $C_T^{(N)} f = \sum_{n\le N} f(A_n)\chi_{T^{-1}(A_n)}$. Then for each $\lambda > 0$, we have

$$\mu_{(C_T - C_T^{(N)})f}(\lambda) \le \sum_{n>N, |f(A_n)|>\lambda} \mu(T^{-1}(A_n))$$
$$\le (\sup_{n>N} b_n) \sum_{|f(A_n)|>\lambda} \mu(A_n) = (\sup_{n>N} b_n\,)\mu_f(\lambda).$$

Therefore

$$\|C_T - C_T^{(N)}\|_{L(pq)\mapsto L(pq)} \leq (\sup_{n>N} b_n)^{1/p} \to 0$$

as $N \to \infty$. Since $C_T$ is the limit of finite rank operators $C_T^{(N)}$, it is compact. ■

ACKNOWLEDGEMENTS. The authors are grateful to the referee for the valuable suggestions and comments.

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(received 06.08.2004, in revised form 17.08.2005)

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