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ON PSEUDO-SEQUENCE COVERINGS, π-IMAGES OF METRIC SPACES

Ying Ge

Abstract. In this paper, we prove that a space X is a pseudo-sequence-covering, π-image of a metric space if and only if X has a point-star network consisting of *wcs*-covers, which answers a conjecture posed by Lin affirmatively. As an application of this result, we have that a space is a pseudo-sequence-covering, π-image of a separable metric space is characterized as a sequentially-quotient, π-image of a separable metric space.

1. Introduction

A study of images of metric spaces under certain π-mappings is an important question in general topology [2, 5, 7, 8, 13]. In recent years, sequence-covering (resp. pseudo-sequence-covering, sequentially-quotient), π-images of metric spaces cause attention once again [4, 10, 16, 17]. Lin proved that a space is a sequence-covering (resp. sequentially-quotient), π-image of a metric space if and only if it has a point-star network consisting of *cs*-covers (resp. cs^*-covers) [10]. However, the following question is still open.

QUESTION 1.1. What is a similar characterization of a pseudo-sequence-covering, π-image of a metric space?

Note that sequence-covering mapping $\Longrightarrow$ pseudo-sequence-covering mapping $\Longrightarrow$ (if the domain is metric) sequentially-quotient mapping. Lin raised the following conjecture in a personal communication.

CONJECTURE 1.2. There exists a class of covers, which is between *cs*-covers and cs^*-covers, such that a space is a pseudo-sequence-covering, π-image of a metric space if and only if it has a point-star network consisting of such covers.

On the other hand, Tanaka and Ge proved that every space with a point-star network consisting of point-countable cs^*-covers is a pseudo-sequence-covering, π-image of a metric space [17]) More precisely, Lin obtained the following result.

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Theorem 1.3. [10] *A space is a pseudo-sequence-covering (or sequentially-quotient), s and π-image of a metric space if and only if it has a point-star network consisting of point-countable cs^*-covers.*

In this paper, we introduce wcs-covers, which are between cs-covers and cs^*-covers, and prove that a space is a pseudo-sequence-covering, π-image of a metric space if and only if it has a point-star network consisting of wcs-covers, which gives an affirmative answer to Conjecture 1.2. Moreover, we establish a relation between point-countable wcs-covers and point-countable cs^*-covers and generalize Theorem 1.3. As an application of these results, we prove that a space is a pseudo-sequence-covering, π-image of a separable metric space if and only if it is a sequentially-quotient, π-image of a separable metric space.

Throughout this paper, all spaces are assumed to be Hausdorff, and all mappings are continuous and onto. $\mathbf{N}$ denotes the set of all natural numbers. Let X be a space and let A be a subset of X. We say that a sequence $\{x_n : n \in \mathbf{N}\} \cup \{x\}$ in X converging to x is eventually in A if $\{x_n : n > k\} \cup \{x\} \subset A$ for some $k \in \mathbf{N}$. Let $\mathcal{P}$ be a family of subsets of X and let $x \in X$. $\bigcup\mathcal{P}$, $st(x,\mathcal{P})$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$, the union $\bigcup\{P \in \mathcal{P} : x \in P\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$, respectively. If $f \colon X \longrightarrow Y$ is a mapping, $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$.

2. π-images of metric spaces

Definition 2.1. Let (X,d) be a metric space, and let $f \colon X \longrightarrow Y$ be a mapping. f is called a π-mapping [13], if for every $y \in Y$ and for every neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$.

Remark 2.2. Recall that a mapping $f \colon X \longrightarrow Y$ is a compact mapping (resp. an s-mapping), if $f^{-1}(y)$ is a compact subset (resp. a separable subset) of X for every $y \in Y$. It is clear that every compact mapping from a metric space is an s- and π-mapping.

Definition 2.3. Let $f \colon X \longrightarrow Y$ be a mapping.

(1) f is called a pseudo-sequence-covering mapping [6] if for every convergent sequence S in Y, there exists a compact subset K of X such that $f(K) = S \cup \{y\}$, where y is the limit point of S;

(2) f is called a sequentially-quotient mapping [1] if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S;

(3) f is called a compact-covering mapping [12] if for every compact subset C of Y, there exists a compact subset K of X such that $f(K) = C$;

(4) f is called a sequence-covering mapping [14] if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L) = S$.

Remark 2.4. Every pseudo-sequence-covering mapping from a metric space is sequentially-quotient [10].

Definition 2.5. (1) Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X. $\mathcal{P}$ is called a network of X, if for every $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.

(2) Let $\{\mathcal{P}_n : n \in \mathbf{N}\}$ be a sequence of covers of a space X. $\{\mathcal{P}_n : n \in \mathbf{N}\}$ is called a point-star network of X [10], if $\{st(x, \mathcal{P}_n) : n \in \mathbf{N}\}$ is a network at x in X for every $x \in X$.

Definition 2.6. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a *wcs*-cover if for every convergent sequence S converging to x in X, there exists a finite subfamily $\mathcal{P}'$ of $(\mathcal{P})_x$ such that S is eventually in $\bigcup\mathcal{P}'$;

(2) $\mathcal{P}$ is called a cs^*-cover [10] if for every convergent sequence S in X, there exist $P \in \mathcal{P}$ and a subsequence S' of S such that S' is eventually in P;

(3) $\mathcal{P}$ is called a *cs*-cover of X [18] if for every convergent sequence S in X, there exists $P \in \mathcal{P}$ such that S is eventually in P.

(4) $\mathcal{P}$ is called a (point-)countable *wcs*-cover (resp. (point-)countable cs^*-cover, (point-)countable *cs*-cover) if $\mathcal{P}$ is a *wcs*-cover (resp. cs^*-cover, *cs*-cover) and is also (point-)countable.

Theorem 2.7. *For a space X, the following are equivalent:*

(1) *X is a pseudo-sequence-covering, π-image of a metric space;*

(2) *X has a point-star network consisting of wcs-covers.*

Proof. (1)$\Longrightarrow$(2): Let M be a metric space with a metric d, and let $f\colon M \longrightarrow X$ be a pseudo-sequence-covering, π-mapping. We write $B(a,\varepsilon) = \{b \in M : d(a,b) < \varepsilon\}$ for every $a \in M$, where $\varepsilon > 0$. For every $n \in \mathbf{N}$, put $\mathcal{B}_n = \{B(a,1/n) : a \in M\}$, and $\mathcal{P}_n = f(\mathcal{B}_n)$. Then $\mathcal{P}_n$ is a cover of X.

Claim 1. $\{\mathcal{P}_n\}$ is a point-star network of X. That is, $\{st(x,\mathcal{P}_n) : n \in \mathbf{N}\}$ is a network at x in X for every $x \in X$.

Let $x \in U$ with U open in X. Since f is a π-mapping, there exists $n \in \mathbf{N}$ such that $d(f^{-1}(x), M - f^{-1}(U)) > 1/n$. Pick $m \in \mathbf{N}$ such that $m > 2n$. It suffices to prove that $st(x,\mathcal{P}_m) \subset U$. Let $a \in M$ and let $x \in f(B(a,1/m)) \in \mathcal{P}_m$. We claim that $B(a,1/m) \subset f^{-1}(U)$. If fact, if $B(a,1/m) \not\subset f^{-1}(U)$, then pick $b \in B(a,1/m) - f^{-1}(U)$. Note that $f^{-1}(x) \cap B(a,1/m) \neq \emptyset$, pick $c \in f^{-1}(x) \cap B(a,1/m)$. Then $d(f^{-1}(x), M - f^{-1}(U)) \le d(c,b) \le d(c,a) + d(a,b) < 2/m < 1/n$. This is a contradiction. So $B(a,1/m) \subset f^{-1}(U)$, thus $f(B(a,1/m)) \subset ff^{-1}(U) = U$. This proves that $st(x,\mathcal{P}_m) \subset U$.

Claim 2. $\mathcal{P}_n$ is a *wcs*-cover of X for every $n \in \mathbf{N}$.

Let $n \in \mathbf{N}$. Suppose S is a sequence in X converging to $x \in X$. Since f is pseudo-sequence-covering, there exists a compact subset K in M such that $f(K) = S \cup \{x\}$. Note that $f^{-1}(x) \cap K$ is compact in M. There exists a finite subset M' of M such that $f^{-1}(x) \cap K \subset \bigcup_{a \in M'} B(a,1/n)$. We can assume that $f^{-1}(x) \cap B(a,1/n) \neq \emptyset$ for every $a \in M'$. Put $\mathcal{B} = \{B(a,1/n) : a \in M'\}$ and $B = \bigcup\mathcal{B}$, then $K - B$ is compact in M. Put $\mathcal{P}' = \{f(B(a,1/n)) : a \in M'\}$. Then $\mathcal{P}'$ is a finite subfamily of $(\mathcal{P}_n)_x$. We prove that S is eventually in $\bigcup\mathcal{P}'$ as follows.

If not, there exists a subsequence $\{x_k : k \in \mathbf{N}\} \cup \{x\}$ of S converging to x such that $x_k \notin \bigcup \mathcal{P}'$ for every $k \in \mathbf{N}$. Note that $f(K) = S$. Thus there exists $a_k \in K - B$ such that $f(a_k) = x_k$ for every $k \in \mathbf{N}$. Since $K - B$ is compact in M, there exists a subsequence $\{a_{k_i} : i \in \mathbf{N}\}$ of $\{a_k : k \in \mathbf{N}\}$ such that the sequence $\{a_{k_i} : i \in \mathbf{N}\}$ converges to a point $a \in K - B$. Thus $f(a) \neq x$. This contradicts the continuity of f. So S is eventually in $\bigcup \mathcal{P}'$. This proves that $\mathcal{P}_n$ is a *wcs*-cover of X.

By the above, X has a point-star network $\{\mathcal{P}_n : n \in \mathbf{N}\}$ consisting of *wcs*-covers.

(2)$\Longrightarrow$(1): Let X have a point-star network $\{\mathcal{P}_n : n \in \mathbf{N}\}$ consisting of *wcs*-covers. For every $n \in \mathbf{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$, and Λ_n is endowed with the discrete topology. Put

$$M = \{a = (\alpha_n) \in \prod_{n\in\mathbf{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ is a network at some } x_a \text{ in } X\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbf{N}} \Lambda_n$, is a metric space with metric d described as follows.

Let $a = (\alpha_n), b = (\beta_n) \in M$. If $a = b$, then $d(a, b) = 0$. If $a \neq b$, then $d(a, b) = 1/\min\{n \in \mathbf{N} : \alpha_n \neq \beta_n\}$.

Define $f \colon M \longrightarrow X$ by choosing $f(a) = x_a$ for every $a = (\alpha_n) \in M$, where $\{P_{\alpha_n}\}$ is a network at x_a in X. It is not difficult to check that f is continuous and onto.

Claim 1. f is a π-mapping.

Let $x \in U$ with U open in X. Since $\mathcal{P}_n$ is a point-star network of X, there exists $n \in \mathbf{N}$ such that $st(x, \mathcal{P}_n) \subset U$. Then $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$. In fact, let $a = (\alpha_n) \in M$ such that $d(f^{-1}(x), a) < 1/2n$. Then there is $b = (\beta_n) \in f^{-1}(x)$ such that $d(a, b) < 1/n$, so $\alpha_k = \beta_k$ if $k \leq n$. Notice that $x \in P_{\beta_n} \in \mathcal{P}_n$, $P_{\alpha_n} = P_{\beta_n}$, so $f(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_n) \subset U$, hence $a \in f^{-1}(U)$. Thus $d(f^{-1}(x), a) \geq 1/2n$ if $a \in M - f^{-1}(U)$, so $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$. This proves that f is a π-mapping.

Claim 2. f is a pseudo-sequence-covering mapping.

Let $S = \{x_n : n \in \mathbf{N}\} \cup \{x\}$ be a sequence in X converging to $x \in X$. Since every $\mathcal{P}_n$ is a *wcs*-cover, for every $n \in \mathbf{N}$, there exists a finite subfamily $\mathcal{F}_n$ of $(\mathcal{P}_n)_x$ such that S is eventually in $\bigcup \mathcal{F}_n$. Note that $S - \bigcup \mathcal{F}_n$ is finite. There exists a finite subfamily $\mathcal{G}_n$ of $\mathcal{P}_n$ such that $S - \bigcup \mathcal{F}_n \subset \bigcup \mathcal{G}_n$. Put $\mathcal{F}_n \cup \mathcal{G}_n = \{P_{\alpha_n} : \alpha_n \in \Gamma_n\}$, where Γ_n is a finite subset of Λ_n. For every $\alpha_n \in \Gamma_n$, if $P_{\alpha_n} \in \mathcal{F}_n$, put $S_{\alpha_n} = S \bigcap P_{\alpha_n}$, otherwise, put $S_{\alpha_n} = (S - \bigcup \mathcal{F}_n) \cap P_{\alpha_n}$. It is easy to see that $S = \bigcup_{\alpha_n \in \Gamma_n} S_{\alpha_n}$ and $\{S_{\alpha_n} : \alpha_n \in \Gamma_n\}$ is a family of compact subsets of X. Put $K = \{(\alpha_n) \in \prod_{n\in\mathbf{N}} \Gamma_n : \bigcap_{n\in\mathbf{N}} S_{\alpha_n} \neq \emptyset\}$. Then

(i) $K \subset M$ and $f(K) \subset S$: Let $a = (\alpha_n) \in K$, then $\bigcap_{n\in\mathbf{N}} S_{\alpha_n} \neq \emptyset$. Pick $y \in \bigcap_{n\in\mathbf{N}} S_{\alpha_n}$; then $y \in \bigcap_{n\in\mathbf{N}} P_{\alpha_n}$. Note that $\{P_{\alpha_n} : n \in \mathbf{N}\}$ is a network at y in X if and only if $y \in \bigcap_{n\in\mathbf{N}} P_{\alpha_n}$. So $a \in M$ and $f(a) = y \in S$. This proves that $K \subset M$ and $f(K) \subset S$.

(ii) $S \subset f(K)$: Let $y \in S$. For every $n \in \mathbf{N}$, pick $\alpha_n \in \Gamma_n$ such that $y \in S_{\alpha_n}$. Put $a = (\alpha_n)$, then $a \in K$ and $f(a) = y$. This proves that $S \subset f(K)$.

(iii) K is a compact subset of M: Since $K \subset M$ and $\prod_{n\in\mathbf{N}} \Gamma_n$ is a compact subset of $\prod_{n\in\mathbf{N}} \Lambda_n$, we only need to prove that K is a closed subset of $\prod_{n\in\mathbf{N}} \Gamma_n$. It is clear that $K \subset \prod_{n\in\mathbf{N}} \Gamma_n$. Let $a = (\alpha_n) \in \prod_{n\in\mathbf{N}} \Gamma_n - K$. Then $\bigcap_{n\in\mathbf{N}} S_{\alpha_n} = \emptyset$. There exists $n_0 \in \mathbf{N}$ such that $\bigcap_{n\le n_0} S_{\alpha_n} = \emptyset$. Put $W = \{(\beta_n) \in \prod_{n\in\mathbf{N}} \Gamma_n : \beta_n = \alpha_n \text{ for } n \le n_0\}$. Then W is open in $\prod_{n\in\mathbf{N}} \Gamma_n$ and $a \in \prod_{n\in\mathbf{N}} \Gamma_n$. It is easy to see $W \cap K = \emptyset$. So K is a closed subset of $\prod_{n\in\mathbf{N}} \Gamma_n$.

By (i), (ii) and (iii), f is a pseudo-sequence-covering mapping.

By the above, X is a pseudo-sequence-covering, π-image of a metric space. ∎

Are the conditions in Theorem 2.7 equivalent for a space to be a sequentially-quotient, π-image of a metric space? This question is still open (see [10, Question 3.1.14], for example). Recently, Lin gave a sequentially-quotient, π-mapping f from a metric space such that f is not pseudo-sequence-covering. This also shows that a point-star network consisting of cs^*-covers need not be a point-star network consisting of *wcs*-covers. But, we have the following result.

Lemma 2.8. *Let $\mathcal{P}$ be a cover of a space X.*

(1) *If $\mathcal{P}$ is a wcs-cover, then $\mathcal{P}$ is a cs^*-cover.*

(2) *If $\mathcal{P}$ is a point-countable cs^*-cover, then $\mathcal{P}$ is a wcs-cover.*

Proof. (1) holds by Definition 2.6. We only need to prove (2).

Let $\mathcal{P}$ be a point-countable cs^*-cover of X. Let $S = \{x_n : n \in \mathbf{N}\} \cup \{x\}$ be a sequence converging to $x \in X$. Since $\mathcal{P}$ is point-countable, put $(\mathcal{P})_x = \{P_n : n \in \mathbf{N}\}$. Then S is eventually in $\bigcup_{n\le k} P_n$ for some $k \in \mathbf{N}$. If not, then for any $k \in \mathbf{N}$, S is not eventually in $\bigcup_{n\le k} P_n$. So, for every $k \in \mathbf{N}$, there exists $x_{n_k} \in S - \bigcup_{n\le k} P_n$. We may assume $n_1 < n_2 < \cdots < n_{k-1} < n_k < n_{k+1} < \cdots$. Put $S' = \{x_{n_k} : k \in \mathbf{N}\} \cup \{x\}$, then S' is a sequence converging to x. Since $\mathcal{P}$ is a cs^*-cover, there exist $m \in \mathbf{N}$ and a subsequence S'' of S' such that S'' is eventually in P_m. Note that $P_m \in (\mathcal{P})_x$. This contradicts the construction of S'. ∎

Corollary 2.9. *Let X be a space. Then the following are equivalent:*

(1) *X is a pseudo-sequence-covering, s- and π-image of a metric space;*

(2) *X is a sequentially-quotient, s- and π-image of a metric space;*

(3) *X has a point-star network consisting of point-countable wcs-covers;*

(4) *X has a point-star network consisting of point-countable cs^*-covers.*

Proof. (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(4) from Theorem 1.3. (3)$\Longleftrightarrow$(4) from Lemma 2.8. ∎

3. π-images of separable metric spaces

Recall that a space is called a *cosmic*-space [12] if it has a countable network. It is known that a space is a cosmic-space if and only if it is an image of a separable metric space [12]. We characterize cosmic-spaces by π-images of separable metric spaces.

PROPOSITION 3.1. *For a space X, the following are equivalent:*

(1) *X is a cosmic-space;*

(2) *X has a point-star network consisting of countable covers;*

(3) *X is a π-image of a separable metric space;*

(4) *X is an image of a separable metric space.*

Proof. We only need to prove that (1)$\Longrightarrow$(2)$\Longrightarrow$(3).

(1)$\Longrightarrow$(2). Let $\mathcal{P} = \{P_n : n \in \mathbf{N}\}$ be a countable network of a cosmic-space X. For every $n \in \mathbf{N}$, put $\mathcal{P}_n = \{P_n\} \cup \{P_k - P_n : k \in N\}$. Then $\{st(x, \mathcal{P}_n) : n \in \mathbf{N}\}$ is a network at x in X for every $x \in X$. In fact, for every $x \in U$ with U is open in X, since $\mathcal{P}$ is a network of X, there exists $n \in \mathbf{N}$ such that $x \in P_n \subset U$. Note that $st(x, \mathcal{P}_n) = P_n$, thus $st(x, \mathcal{P}_n) \subset U$. So $\{\mathcal{P}_n : n \in \mathbf{N}\}$ is a point-star network consisting of countable covers.

(2)$\Longrightarrow$(3). Let X have a point-star network $\{\mathcal{P}_n : n \in \mathbf{N}\}$ consisting of countable covers. For every $n \in \mathbf{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ with Λ_n endowed with the discrete topology. Put

$$M = \{a = (\alpha_n) \in \prod_{n \in \mathbf{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ is a network at some } x_a \text{ in } X\}.$$

Note that Λ_n is countable for every $n \in N$. Then M, which is a subspace of the product space $\prod_{n \in N} \Lambda_n$, is a separable metric space with the metric d defined as follows.

Let $a = (\alpha_n), b = (\beta_n) \in M$. If $a = b$, then $d(a, b) = 0$. If $a \neq b$, then $d(a, b) = 1/\min\{n \in \mathbf{N} : \alpha_n \neq \beta_n\}$.

Define $f : M \longrightarrow X$ by choosing $f(a) = x_a$ for every $a = (\alpha_n) \in M$, where $\{P_{\alpha_n}\}$ is a network at x_a in X. As in the proof of (2)$\Longrightarrow$(1) in Theorem 2.7, it is easy to prove that f is a π-mapping. So X is a π-image of a separable metric space. ∎

Now we discuss pseudo-sequence-covering, π-images of separable metric spaces.

DEFINITION 3.2. [3] Let X be a space, and let $x \in X$. A subset P of X is called a sequential neighborhood of x if every sequence $S = \{x_n : n \in \mathbf{N}\} \cup \{x\}$ converging to x is eventually in P.

DEFINITION 3.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X. $\mathcal{P}$ is called an sn-network of X [11], if $\mathcal{P}_x$ satisfies the following (a), (b) and (c) for every $x \in X$, where $\mathcal{P}_x$ is called an sn-network at x in X.

(a) $\mathcal{P}_x$ is a network at x in X;

(b) if $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$;

(c) every element of $\mathcal{P}_x$ is a sequential neighborhood of x.

REMARK 3.4. In [9], a sequential neighborhood of x and an sn-network are called a sequence barrier at x and a universal cs-network respectively.

Lin and Yan proved that a regular space is a sequentially-quotient (or compact-covering), compact image of a separable metric space if and only if it has a countable sn-network, and where "regular" cannot be omitted [11]. But, it is possible to relax "compact" to "π" [4]. Thus we have the following result.

THEOREM 3.5. *For a regular space X, the following are equivalent:*

(1) *X is a pseudo-sequence-covering, π-image of a separable metric space;*

(2) *X is a sequentially-quotient, π-image of a separable metric space;*

(3) *X has a countable sn-network.*

Taking Theorem 3.5 into account, we ask whether "regular" can be omitted. Furthermore, without requiring the regularity of the spaces involved, are (1) and (2) equivalent? We answer these questions as follows.

EXAMPLE 3.6. A space with a countable base is not necessarily a sequentially-quotient, π-image of a metric space. So "regular" in Theorem 3.5 cannot be omitted.

Proof. Let $\mathbf{R}$ be the set of all real numbers, and let τ be the Euclidean topology on $\mathbf{R}$. Put $X = \mathbf{R}$ with the topology $\tau^* = \{\{x\} \cup (D \cap U) : x \in U \in \tau\}$, where D is the set of all irrational numbers. That is, X is the point irrational extension of R [15, Example 69]. It is easy to check that X is Hausdorff, non-regular [15].

Claim 1. X has a countable base [10, Example 3.13(5)].

Claim 2. X is not a sequentially quotient, π-image of a metric space. In fact, Lin proved that X is not a symmetric space [10, Example 3.13(5)]. So X is not a quotient, π-image of a metric space [16]. Note that every sequentially-quotient mapping onto a first countable space is quotient [1]. Thus X is not a sequentially-quotient, π-image of a metric space. ■

The proof of the following proposition is as the proof of Theorem 2.7. We omit it.

PROPOSITION 3.7. *A space is a pseudo-sequence-covering (resp. sequentially-quotient), π-image of a separable metric space if and only if it has a point-star network consisting of countable wcs-covers (resp. countable cs^*-covers).*

By Lemma 2.8, we have the following corollary.

COROLLARY 3.8. *A space is a pseudo-sequence-covering, π-image of a separable metric space if and only if it is a sequentially-quotient, π-image of a separable metric space.*

From the proofs of [11, Theorem 4.6, (3)$\Longrightarrow$(2)] and [4, Theorem 2.7, (3)$\Longrightarrow$(1)], we have the following results without requiring the regularity of the spaces involved.

Proposition 3.9. *For a space X, the following are true:*

(1) *If X is a sequentially-quotient, π-image of a separable metric space, then X has a countable sn-network.*

(2) *If X has a countable closed sn-network, then X is a compact-covering, compact image of a separable metric space.*

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Department of Mathematics, Suzhou University, Suzhou 215006, P.R.China
E-mail: geying@pub.sz.jsinfo.net