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HARMONIC FUNCTIONS STARLIKE OF THE COMPLEX ORDER

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Abstract. The main purpose of this paper is to introduce a class $TS_H^*(\gamma)$ ($\gamma \in \mathbf{C} \setminus \{0\}$) of functions which are harmonic in the unit disc. We give necessary and sufficient conditions for the functions to be in $TS_H^*(\gamma)$.

1. Introduction

A continuous complex-valued function $f = u+iv$ defined in a simply connected domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$.

Denote by S_H the class of functions $f = h+\bar{g}$ that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

In 1984 Clunie and Sheil-Small [2] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

Let TS_H denote the family of functions $f = h+\bar{g}$ that are harmonic in U with the normalization

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad a_n, b_n \geqslant 0, \ \ b_1 < 1. \tag{2}$$

We let $TS_H^*(\gamma)$ denote the subclass of TS_H consisting of functions $f = h+\bar{g} \in TS_H$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right)\right\} > 0, \qquad \gamma \in \mathbf{C} \setminus \{0\}. \tag{3}$$

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We further let $OS_H^*(\gamma)$ denote the subclass of TS_H consisting of functions $f = h + \bar{g} \in TS_H$ that satisfy the condition

$$\sum_{n=1}^{\infty} [2(n-1+|\gamma|)a_n + (n+1+|n+1-2\gamma|)b_n] \leqslant 4|\gamma|. \tag{4}$$

Denote by $PS_H^*(\gamma)$ the subclass of TS_H consisting of functions $f = h + \bar{g} \in TS_H$ that satisfy the condition

$$\sum_{n=1}^{\infty} \left[(n-1)\frac{\operatorname{Re}(\gamma)}{|\gamma|} + |\gamma|\right] a_n + \left[(n+1)\frac{\operatorname{Re}(\gamma)}{|\gamma|} - |\gamma|\right] b_n \leqslant 2|\gamma|. \tag{5}$$

Recently, Avcı and Zlotkiewicz [1], Jahangiri [3], Silverman [4], and Silverman and Silvia [5] studied the harmonic starlike functions. Avcı and Slotkiewicz [1] proved that the coefficient condition

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leqslant 1, \qquad b_1 = 0$$

is sufficient for functions $f = h + \bar{g}$ to be harmonic starlike. Silverman 4 proved that this coefficient condition is also necessary if $b_1 = 0$ and if a_n and b_n in (1) are negative. Jahangiri [3] proved that if $f = h + \bar{g}$ is given by (1) and if

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha}|b_n| \leqslant 1, \qquad 0 \leqslant \alpha < 1,$$

then f is a harmonic, univalent and starlike function of order α in U. This condition proved to be also necessary if h and g are of the form (2). The case when $\alpha = 0$ is given in [5] and for $\alpha = b_1 = 0$ see [4].

In this paper, we give an answer to the conjecture that $TS_H^*(\gamma) = OS_H^*(\gamma)$.

2. Main results

THEOREM 2.1. $OS_H^*(\gamma) \subset TS_H^*(\gamma)$.

Proof. Let $f \in OS_H^*(\gamma)$. According to the condition (2) we only need to show that if (4) holds then

$$\operatorname{Re}\left\{\frac{(\gamma-1)(h(z)+\overline{g(z)}) + zh'(z) - \overline{zg'(z)}}{\gamma(h(z)+\overline{g(z)})}\right\} > 0,$$

where $\gamma \in \mathbf{C} \setminus \{0\}$. Using the fact that $\operatorname{Re} w > 0$ if and only if $|1+w| > |1-w|$, it suffices to show that

$$|(2\gamma-1)(h(z)+\overline{g(z)}) + zh'(z) - \overline{zg'(z)}| - |h(z)+\overline{g(z)} - zh'(z) + \overline{zg'(z)}| > 0. \tag{6}$$

Substituting for $h(z)$ and $g(z)$ in (6) yields

$$|(2\gamma-1)(h(z)+\overline{g(z)}) + zh'(z) - \overline{zg'(z)}| - |h(z)+\overline{g(z)} - zh'(z) + \overline{zg'(z)}|$$

$$= \left|2\gamma z - \sum_{n=2}^{\infty}(2\gamma - 1 + n)a_n z^n - \sum_{n=1}^{\infty}(n+1-2\gamma)b_n \bar{z}^n\right| -$$

$$- \left|\sum_{n=2}^{\infty}(n-1)a_n z^n + \sum_{n=1}^{\infty}(n+1)b_n \bar{z}^n\right|$$

$$\geqslant 2|\gamma||z| - \sum_{n=2}^{\infty} 2(n-1+|\gamma|)a_n|z|^n - \sum_{n=1}^{\infty}(n+1+|n+1-2\gamma|)b_n|z|^n$$

$$> 2|\gamma| - \left(\sum_{n=2}^{\infty} 2(n-1+|\gamma|)a_n + \sum_{n=1}^{\infty}(n+1+|n+1-2\gamma|)b_n\right) \geqslant 0.$$

The functions

$$f(z) = z - \sum_{n=2}^{\infty} \frac{|\gamma|}{n-1+|\gamma|} x_n z^n + \sum_{n=1}^{\infty} \frac{2|\gamma|}{n+1+|n+1-2\gamma|} y_n \bar{z}^n \tag{7}$$

where x_n, y_n are non-negative and

$$\sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = 1,$$

show that the coefficient bound given by (4) is sharp. The functions of the form (7) are in $TS_H^*(\gamma)$ because

$$\sum_{n=2}^{\infty} 2(n-1+|\gamma|)a_n + \sum_{n=1}^{\infty}(n+1+|n+1-2\gamma|)b_n$$

$$= 2|\gamma|\left(1 + \sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n\right) = 4|\gamma|. \quad \blacksquare$$

THEOREM 2.2. $TS_H^*(\gamma) \subset PS_H^*(\gamma)$.

Proof. Let $f \in TS_H^*(\gamma)$. From (3) we have

$$\operatorname{Re}\left\{\frac{1}{\gamma}\left(\frac{-\sum_{n=2}^{\infty}(n-1)a_n z^n - \sum_{n=1}^{\infty}(n+1)b_n \bar{z}^n}{1 - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n}\right)\right\} > -1.$$

If we choose z on the real axis and $z \to 1^-$ we get

$$\frac{\sum_{n=2}^{\infty}(n-1)a_n + \sum_{n=1}^{\infty}(n+1)b_n}{1 - \sum_{n=2}^{\infty} a_n + \sum_{n=1}^{\infty} b_n} \operatorname{Re}\left(\frac{1}{\gamma}\right) \leqslant 1,$$

whence

$$\frac{\sum_{n=2}^{\infty}(n-1)a_n + \sum_{n=1}^{\infty}(n+1)b_n}{1 - \sum_{n=2}^{\infty} a_n + \sum_{n=1}^{\infty} b_n} \frac{\operatorname{Re}(\gamma)}{|\gamma|^2} \leqslant 1,$$

and so

$$\sum_{n=2}^{\infty}(n-1)a_n + \sum_{n=1}^{\infty}(n+1)b_n \leqslant \frac{|\gamma|^2}{\operatorname{Re}(\gamma)}\left(1 - \sum_{n=2}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\right)$$

which is equivalent to (5). Thus $f \in PS_H^*(\gamma)$. $\blacksquare$

THEOREM 2.3. *If* $\gamma \in (0,1]$, *then* $OS_H^*(\gamma) = TS_H^*(\gamma) = PS_H^*(\gamma)$.

Proof. If $\gamma \in (0,1]$, then the inequalities (4) and (5) are equivalent; hence $OS_H^*(\gamma) = PS_H^*(\gamma)$. By using Theorem 2.1 and Theorem 2.2, from this assertion we obtain the conclusion of the present theorem. ∎

THEOREM 2.4. *If* $-1/2 > \mathrm{Re}(\gamma) \leqslant 0$ *or* $\gamma \in (3/2, +\infty)$, *then*

$$PS_H^*(\gamma) \nsubseteq TS_H^*(\gamma).$$

Proof. Let

$$f(z) = z - z^2. \tag{8}$$

Then $f \in PS_H^*(\gamma)$ when $\gamma \in \mathbf{C} \setminus \{0\}$ and $\mathrm{Re}(\gamma) < 0$, because

$$\sum_{n=1}^{\infty} \left[(n-1)\frac{\mathrm{Re}(\gamma)}{|\gamma|} + |\gamma|\right] a_n + \left[(n+1)\frac{\mathrm{Re}(\gamma)}{|\gamma|} - |\gamma|\right] b_n$$
$$= |\gamma| \cdot 1 + \frac{\mathrm{Re}(\gamma)}{|\gamma|} + |\gamma| = 2|\gamma| + \frac{\mathrm{Re}(\gamma)}{|\gamma|} \leqslant 2|\gamma|.$$

Now let $r = \mathrm{Re}(\gamma) < 0$ and let s be a negative real number such that $1+2r(1-s) > 0$. If we choose $z = \dfrac{\gamma(1-s)}{1+\gamma(1-s)}$, then $z \in U$ and for f given by (8) we have

$$1 + \frac{1}{\gamma}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right) = s < 0,$$

hence $f \notin TS_H^*(\gamma)$.

Similarly, let

$$f(z) = z + \bar{z}^2. \tag{9}$$

Then $f \in PS_H^*(\gamma)$ when $\gamma \in (3/2, +\infty)$, because

$$\sum_{n=1}^{\infty} \left[(n-1)\frac{\mathrm{Re}(\gamma)}{|\gamma|} + |\gamma|\right] a_n + \left[(n+1)\frac{\mathrm{Re}(\gamma)}{|\gamma|} - |\gamma|\right] b_n$$
$$= |\gamma| \cdot 1 + \left(3\frac{\mathrm{Re}(\gamma)}{|\gamma|} - |\gamma|\right) \cdot 1 = 3\frac{\mathrm{Re}(\gamma)}{|\gamma|} \leqslant 2|\gamma|.$$

Now let $\gamma \in (3/2, +\infty)$ and let s be a negative real number such that $\gamma + \gamma(s-1) < 0$. If we choose $z = -\dfrac{\gamma(s-1)}{3+\gamma(s-1)}$, then $z \in U$ and for f given by (9) we have

$$1 + \frac{1}{\gamma}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right) = s < 0,$$

hence $f \notin TS_H^*(\gamma)$. ∎

Theorem 2.5. *If* $\gamma \in (-\infty, -1) \cup (-1/2, 0)$, *then*

$$TS^*_H(\gamma) \nsubseteq OS^*_H(\gamma).$$

Proof. Let $\gamma \in (-\infty, -1)$; we verify that the functions

$$f_\lambda(z) = z - \lambda z^2 \tag{10}$$

belong to $TS^*_H(\gamma)$ for $\lambda > \frac{\gamma}{1+\gamma}$ and that $f \notin OS^*_H(\gamma)$.

Indeed we have

$$\sum_{n=1}^{\infty} [2(n-1+|\gamma|)a_n + (n+1+|n+1-2\gamma|)b_n] = 2|\gamma| + 2(1+|\gamma|)\lambda > 4|\gamma|,$$

because $\lambda > \frac{\gamma}{1+\gamma} > 1$.

We also have

$$\mathrm{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zh'_\lambda(z) - \overline{zg'_\lambda(z)}}{h_\lambda(z) + \overline{g_\lambda(z)}} - 1\right)\right\} = \mathrm{Re}\left\{1 + \frac{\lambda z}{\gamma(\lambda z - 1)}\right\} > 0, \quad z \in U, \tag{11}$$

for $\lambda > \frac{\gamma}{1+\gamma}$ and $\gamma < -1$, hence $f_\lambda \in TS^*_H(\gamma)$.

Let now $\gamma \in (-1/2, 0)$, and let f_λ be defined by (10), where

$$-\frac{\gamma}{1-\gamma} < \lambda < -\frac{\gamma}{1+\gamma}.$$

Then $\lambda > -\frac{\gamma}{1-\gamma}$ implies $f_\lambda \notin OS^*_H(\gamma)$ and for $\lambda < -\dfrac{\gamma}{1+\gamma}$ the inequality (11) is also verified, hence $f_\lambda \in TS^*_H(\gamma)$. ∎

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