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SOME GENERALIZATIONS OF LITTLEWOOD-PALEY INEQUALITY IN THE POLYDISC

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Abstract. The paper generalizes the well-known inequality of Littlewood-Paley in the polydisc. We establish a family of inequalities which are analogues and extensions of Littlewood-Paley type inequalities proved by Sh. Yamashita and D. Luecking in the unit disk. Some other generalizations of the Littlewood-Paley inequality are stated in terms of anisotropic Triebel-Lizorkin spaces. With the help of an extension of Hardy-Stein identity, we also obtain area inequalities and representations for quasi-norms in weighted spaces of holomorphic functions in the polydisc.

1. Introduction

Let $\mathbf{D}^n = \{z = (z_1, \dots, z_n) \in \mathbf{C}^n : |z_j| < 1, 1 \le j \le n\}$ be the unit polydisc in $\mathbf{C}^n$, and $\mathbf{T}^n = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n : |\xi_j| = 1, 1 \le j \le n\}$ be the n-dimensional torus, the distinguished boundary of $\mathbf{D}^n$. Denote by $H(\mathbf{D}^n)$ the set of all holomorphic functions in $\mathbf{D}^n$. If $f(z) = f(r\xi)$ is a measurable function in $\mathbf{D}^n$, then

$$M_p(f,r) = \left[\frac{1}{(2\pi)^n}\int_{\mathbf{T}^n} |f(r\xi)|^p dm_n(\xi)\right]^{1/p}, \quad r = (r_1, \dots, r_n) \in I^n,$$

where $0 < p < \infty$, $I^n = (0,1)^n$, m_n is the n-dimensional Lebesgue measure on $\mathbf{T}^n$. The collection of holomorphic functions $f(z)$, for which $\|f\|_{H^p} = \sup\limits_{r\in I^n} M_p(f,r) < +\infty$, is the usual Hardy space H^p. For a radial weight function $\omega(r) = \prod_{j=1}^n \omega_j(r_j)$ the quasi-normed space L^p_ω $(0 < p < \infty)$ is the set of those functions $f(z)$ measurable in the polydisc $\mathbf{D}^n$, for which the quasi-norm

$$\|f\|_{L^p_\omega} = \left(C_\omega \int_{\mathbf{D}^n} |f(z)|^p \prod_{j=1}^n \omega_j(|z_j|)\, dm_{2n}(z)\right)^{1/p}$$

is finite. Here $dm_{2n}(z) = r\,dr\,dm_n(\xi)$ is the Lebesgue measure on $\mathbf{D}^n$, and the constant C_ω is chosen so that $\|1\|_{L^p_\omega} = 1$. For the subspace of L^p_ω consisting of

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holomorphic functions let $A^p_\omega = H(\mathbf{D}^n) \cap L^p_\omega$. We will write L^p_α, A^p_α instead of L^p_ω, A^p_ω if $\omega_j(r_j) = (1-r_j)^{\alpha_j}$ $(\alpha_j > -1, 1 \le j \le n)$.

The classical inequality of Littlewood and Paley for functions holomorphic in the unit disk $\mathbf{D} = \mathbf{D}^1$ (see, e.g., [23]) is well known.

Theorem A. (Littlewood-Paley) *If* $2 \le p < \infty$, *then for any* $f \in H^p(\mathbf{D})$

$$\int_{\mathbf{D}} |f'(z)|^p (1-|z|)^{p-1}\, dm_2(z) \le C\|f\|^p_{H^p}. \tag{1.1}$$

Many generalizations and extensions of Theorem A are known, see, for example, [1–2, 8–13, 17–22]. The next theorem is Luecking's [9] generalization of (1.1).

Theorem B. (Luecking) *Let* $0 < p, s < \infty$. *Then*

$$\int_{\mathbf{D}} |f(z)|^{p-s} |f'(z)|^s (1-|z|)^{s-1}\, dm_2(z) \le C\|f\|^p_{H^p} \tag{1.2}$$

for any $f \in H^p(\mathbf{D})$ *if and only if* $2 \le s < p+2$.

We see that the case $0 < s < 2$ is omitted. So, it would be of interest to obtain analogues of (1.2) for $0 < s < 2$.

The present paper is organized as follows. Theorem 1 deals with Luecking's integral (1.2) in the polydisc for $0 < s < 2$. We obtain a family of inequalities reducing to the Littlewood-Paley inequality in the limiting case $s, p \to 2$. Note that the proof of D. Luecking [9] essentially uses some one variable methods which are not extendible to the polydisc case by a direct iteration. We exploit function spaces introduced by R. Coifman, Y. Meyer and E. Stein [3] and apply methods for estimating of Luecking's integral, which are quite different from those of [9]. In Theorem 2 we prove another extension of the Littlewood-Paley inequality in terms of anisotropic Triebel-Lizorkin spaces. Then we consider in Theorem 3 fractional derivatives of arbitrary order and estimate more general integrals for all $0 < s \le p < \infty$. We establish in Theorem 4 other analogues of (1.2) by means of general weight functions $\omega(r)$. To this end, we extend to the polydisc the well-known Hardy-Stein identity. Finally, in Theorem 5 we give a characterization of weighted spaces A^p_ω on the polydisc with the use of (1.2) type integrals.

2. Notation and main theorems

We will use the conventional multi-index notations: $r\zeta = (r_1\zeta_1, \dots, r_n\zeta_n)$, $dr = dr_1 \cdots dr_n$, $(1-|\zeta|)^\alpha = \prod_{j=1}^n (1-|\zeta_j|)^{\alpha_j}$, $\zeta^\alpha = \prod_{j=1}^n \zeta_j^{\alpha_j}$, $\alpha q + 1 = (\alpha_1 q + 1, \dots, \alpha_n q + 1)$ for $\zeta \in \mathbf{C}^n$, $r \in I^n$, $q \in \mathbf{R}$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. Let $\mathbf{Z}^n_+$ denote the set of all multi-indices $k = (k_1, \dots, k_n)$ with nonnegative integers $k_j \in \mathbf{Z}_+$. Any inequality (or equality) $A \le B$ quoted or proved is to be interpreted as meaning 'if B is finite, then A is finite, and $A \le B$'. Throughout the paper, the letters $C(\alpha,\beta,\dots), C_\alpha$ etc. stand for positive constants possibly different at different places and depending only on the parameters indicated. For $A, B > 0$ we will write $A \lesssim B$, if there exists an inessential constant $c > 0$ independent of variables involved such that $A \le cB$. The symbol $A \asymp B$ means $A \lesssim B$ and $B \lesssim A$. For any $p, 1 \le p \le \infty$, we define the conjugate index p' as $p' = p/(p-1)$ (we interpret $1/\infty = 0$ and $1/0 = +\infty$).

For every function $f \in H(\mathbf{D}^n)$ having a series expansion $f(z) = \sum_{k\in\mathbf{Z}^n_+} a_k r^k \xi^k$, where $z = r\xi, r \in I^n, \xi \in \mathbf{T}^n$, we define the radial fractional integro-differentiation of arbitrary order $\alpha = (\alpha_1,\dots,\alpha_n),\ \alpha_j \in \mathbf{R}$ by

$$\mathcal{D}^\alpha f(z) \equiv \mathcal{D}^\alpha_r f(z) = \sum_{k\in\mathbf{Z}^n_+} \prod_{j=1}^n (1+k_j)^{\alpha_j} a_k r^k \xi^k.$$

It is easily seen that $\mathcal{D}^\alpha_r f(z) = \mathcal{D}^{\alpha_1}_{r_1}\mathcal{D}^{\alpha_2}_{r_2}\dots\mathcal{D}^{\alpha_n}_{r_n} f$, where $\mathcal{D}^{\alpha_j}_{r_j}$ means the same operator acting in the variable r_j only.

We now formulate the main theorems of the paper. First we establish a family of inequalities which are analogues of Littlewood-Paley type inequalities (1.2) proved by Sh. Yamashita [22] and D. Luecking [9] in the unit disk.

THEOREM 1. *Let $0 < \alpha < s < 2$, $s < p$. Then for any $\lambda > (p-s)/\alpha$*

$$\int_{\mathbf{D}^n} |f(z)|^{p-s} |\mathcal{D}^1 f(z)|^s (1-|z|)^{s-1} dm_{2n}(z) \lesssim \|f\|^{p-s}_{H^\lambda} \|\mathcal{D}^{\alpha/s} f\|^s_{H^s}. \tag{2.1}$$

REMARK 1. Taking $p = 2$ in (2.1) and formally passing to the limit as $s \to 2-$ and $\alpha \to +0$, we get the classical Littlewood-Paley inequality (1.1) for $p = 2$ in the polydisc.

Recall now anisotropic Triebel-Lizorkin spaces on the polydisc, see [5], [11], [12], [15], [16]]. The function $f(z)$ holomorphic in $\mathbf{D}^n$, is said to belong to the space F^{pq}_α $\big(0 < p < \infty, 0 < q \le \infty, \alpha = (\alpha_1,\dots,\alpha_n), \alpha_j \ge 0\big)$, if for some multi-index $\beta = (\beta_1,\dots,\beta_n)$, $\beta_j > \alpha_j$ the (quasi-)norm

$$\|f\|_{F^{pq}_\alpha} = \begin{cases} \left[\displaystyle\int_{\mathbf{T}^n}\left(\int_{I^n}(1-r)^{(\beta-\alpha)q-1}|\mathcal{D}^\beta f(r\xi)|^q dr\right)^{p/q} dm_n(\xi)\right]^{1/p}, & 0 < q < \infty, \\ \left[\displaystyle\int_{\mathbf{T}^n}\left(\sup_{r\in I^n}(1-r)^{\beta-\alpha}|\mathcal{D}^\beta f(r\xi)|\right)^p dm_n(\xi)\right]^{1/p}, & q = \infty, \end{cases}$$

is finite. For different $\beta\,(\beta_j > \alpha_j)$ equivalent norms appear. Many well-studied function spaces are included in the Triebel–Lizorkin spaces. For $p = q$ the space F^{pp}_α coincides with the holomorphic Besov space; for $q = 2$ Hardy-Sobolev spaces arise, and for $q = 2, \alpha_j = 0$ the space F^{p2}_0 coincides with H^p.

THEOREM 2. *For any $0 < p < \infty$, $0 < q \le q_1 \le \infty$, $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_j \ge 0$ the following inclusion is continuous*

$$F^{pq}_\alpha \subset F^{pq_1}_\alpha. \tag{2.2}$$

Remark 2. The inclusion (2.2) is proved in [11] in the setting of the unit ball of $\mathbf{C}^n$. For the polydisc, (2.2) is a generalization of the inclusion $F_0^{p2} \subset F_0^{p\infty}$ proved in [6] as well as of that in [1], where $\alpha_j = 0$ and n-harmonic functions are considered. In particular, for $\alpha_j = 0, q = 2, p = q_1$ the inclusion (2.2) reduces to (1.1).

In the next theorem the fractional derivative of the first order is replaced by the same operator $\mathcal{D}^\alpha$ of arbitrary order $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$ and more general integrals are studied.

Theorem 3. *Let $0 < s \le p < \infty$, $\alpha_j > 0$ $(1 \le j \le n)$, and $f(z)$ is a function of Hardy space $H^p(\mathbf{D}^n)$, and a function $g(z)$ belongs to the mixed norm space $H(p, s, \alpha)$, that is*

$$\|g\|^s_{H(p,s,\alpha)} = \int_{I^n} M_p^s(g, r)(1-r)^{\alpha s - 1}\, dr < +\infty.$$

Then

$$\frac{1}{(2\pi)^n} \int_{\mathbf{D}^n} |f(z)|^{p-s} |g(z)|^s (1-|z|)^{\alpha s-1}\, dm_{2n}(z) \le \|f\|^{p-s}_{H^p} \|g\|^s_{H(p,s,\alpha)}.$$

In particular, if $\mathcal{D}^\alpha f \in H(p, s, \alpha)$, then

$$\frac{1}{(2\pi)^n} \int_{\mathbf{D}^n} |f(z)|^{p-s} |\mathcal{D}^\alpha f(z)|^s (1-|z|)^{\alpha s-1}\, dm_{2n}(z) \le \|f\|^{p-s}_{H^p} \|\mathcal{D}^\alpha f\|^s_{H(p,s,\alpha)}. \tag{2.3}$$

Theorem 4. (i) *Let $f(z)$ be a holomorphic function in $\mathbf{D}^n$, $0 < p < \infty$, $\omega_j(r_j)$, $j = 1, \ldots, n$ be weight functions positive and continuously differentiable in $[0, 1)$ such that*

$$\omega_j(r_j) \frac{\partial}{\partial r_j} M_p^p(f, r) = o(1) \qquad \text{as} \qquad r_j \to 1- . \tag{2.4}$$

Then the following identity holds:

$$\int_{\mathbf{D}^n} \prod_{j=1}^n \omega_j(r_j) \cdot f^\#(z)\, dm_{2n}(z) = (-1)^n \int_{\mathbf{D}^n} \prod_{j=1}^n \omega_j'(r_j) \frac{\partial^n}{\partial r_1 \cdots \partial r_n} |f(z)|^p\, dm_{2n}(z), \tag{2.5}$$

where $f^\#(z) = \Delta_{z_1} \Delta_{z_2} \ldots \Delta_{z_n} |f(z)|^p$, and Δ_{z_j} is the usual Laplacian in the variable z_j. For the standard weight functions $\omega_j(r_j) = (1 - r_j)^{\alpha_j}$ $(\alpha_j > 0)$ the assumptions (2.4) can be dropped.

(ii) *For $n = 1$ the following improvements of (2.5) are valid: The identity*

$$\int_{\mathbf{D}} (1-|z|)^\alpha f^\#(z)\, dm_2(z) = \alpha \int_{\mathbf{D}} (1-|z|)^{\alpha-1} \frac{\partial}{\partial r} |f(z)|^p\, dm_2(z), \qquad p > 0,\ \alpha > 0, \tag{2.6}$$

holds if one of the integrals in (2.6) exists. Here

$$f^\#(z) = \Delta |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2. \tag{2.7}$$

(iii) *The integrals*

$$A(f;p,\alpha) = \int_{\mathbf{D}} |f(z)|^{p-2}|f'(z)|^2(1-|z|)^{\alpha}\, dm_2(z),$$
$$B(f;p,\alpha) = \int_{\mathbf{D}} |f(z)|^{p-1}|f'(z)|(1-|z|)^{\alpha-1}\, dm_2(z)$$

are comparable. More precisely,

– *If $p > 0$, $\alpha > 0$, then*

$$A(f;p,\alpha) \le \frac{\alpha}{p}\, B(f;p,\alpha), \tag{2.8}$$

where the constant α/p is sharp.

– *If $p > 0, \alpha > 1$, then there exists a constant $C_{\alpha,p} > 0$ such that*

$$B(f;p,\alpha) \le C_{\alpha,p}\, A(f;p,\alpha). \tag{2.9}$$

Remark 3. The inequalities (2.8) and (2.9) for $p = 2$ are proved in [21]. Their analogues for integers p ($p \ge 2$) in the unit disk and in the unit ball of $\mathbf{C}^n$ are proved in [17], [18] in another way.

The next theorem gives a characterization of weighted Bergman spaces A^p_ω on the bidisc and a representation for (quasi-)norms in A^p_ω with the use of (1.2) type integrals.

Theorem 5. *Let $0 < p < \infty$, $f(z) \in H(\mathbf{D}^2)$, $\omega_j(r_j) \in L^1(0,1)$, $\omega_j(r_j) > 0$, $j = 1,2$. Then the following representations are valid:*

$$\|f\|^p_{A^p_\omega(\mathbf{D}^2)} \asymp |f(0,0)|^p + \int_{\mathbf{D}^2}\Big(\Delta_{z_1}\Delta_{z_2}|f(z_1,z_2)|^p + \Delta_{z_1}|f(z_1,0)|^p + \\ + \Delta_{z_2}|f(0,z_2)|^p\Big)\prod_{j=1}^{2} h_{\omega_j}(|z_j|)\, dm_4(z), \tag{2.10}$$

$$\|f\|^p_{A^p_\omega(\mathbf{D}^2)} + |f(0,0)|^p = \big\|f(\cdot,0)\big\|^p_{A^p_{\omega_1}} + \big\|f(0,\cdot)\big\|^p_{A^p_{\omega_2}} + \\ + C_\omega \int_{\mathbf{D}^2} f^{\#}(z_1,z_2)\prod_{j=1}^{2} h_{\omega_j}(|z_j|)\, dm_4(z), \tag{2.11}$$

where $A^p_{\omega_j}$ is the weighted Bergman space in the variable z_j, and h_{ω_j} is the weight function

$$h_{\omega_j}(|z_j|) = \int_{|z_j|}^1 \left(\int_{\rho_j}^1 \omega_j(x)x\, dx\right)\frac{d\rho_j}{\rho_j}.$$

In particular, $f \in A^p_\alpha(\mathbf{D}^2)$ if and only if $f^{\#} \in L^1_{\alpha+2}(\mathbf{D}^2)$ ($\alpha_j > -1$).

Remark 4. For $n = 1$ and $\omega(r) = (1-r)^\alpha$ ($\alpha > -1$) and by virtue of the formula (2.7), the relation (2.10) in the limiting case $\alpha \to -1$ coincides with Yamashita's [22] characterization of Hardy spaces $H^p(\mathbf{D})$, while some analogues of (2.10) and (2.11) for the unit ball of $\mathbf{C}^n$ are established in [2], [10], [20].

Without loss of generality and to simplify notation, we may assume that $n = 2$ everywhere below in the proofs.

3. Preliminaries and proof of Theorem 1

Let us introduce some more notation i order to formulate several auxiliary lemmas. In what follows, for a fixed $\delta > 1$ let $\Gamma_\delta(\xi) = \{z \in \mathbf{D} : |1-\overline{\xi}z| \le \delta(1-|z|)\}$ be the admissible approach region whose vertex is at $\xi \in \mathbf{T}$. For any arc $I \subset \mathbf{T}$ of the length $|I|$ define the Carleson square over I to be $\Box I = \left\{z \in \mathbf{D};\, \frac{z}{|z|} \in I,\, 1-|z| \le \frac{1}{2\pi}|I|\right\}$. Following [3], consider the functions

$$A_p(f)(\xi) = \left(\int_{\Gamma_\delta(\xi)} \frac{|f(z)|^p}{(1-|z|)^2}\, dm_2(z)\right)^{1/p}, \qquad p < \infty,$$

$$A_\infty(f)(\xi) = \sup\{|f(z)|; z \in \Gamma_\delta(\xi)\},$$

$$C_p(f)(\xi) = \sup_{I \supset \xi} \left(\frac{1}{|I|}\int_{\Box I} \frac{|f(z)|^p}{1-|z|} dm_2(z)\right)^{1/p}, \qquad p < \infty, \quad \xi \in \mathbf{T}.$$

Lemma C. ([3], [12]) *For any functions $f(z)$ and $g(z)$ measurable in the unit disk*

$$\int_{\mathbf{D}} \frac{|f(z)|}{1-|z|}\, dm_2(z) \lesssim \int_{\mathbf{T}} \left(\int_{\Gamma_\delta(\xi)} \frac{|f(z)|}{(1-|z|)^2}\, dm_2(z)\right) dm(\xi), \tag{3.1}$$

$$\int_{\mathbf{D}} \frac{|f(z)||g(z)|}{1-|z|}\, dm_2(z) \lesssim \int_{\mathbf{T}} A_p(f)(\xi)\, C_{p'}(g)(\xi)\, dm(\xi), \quad 1 < p \le \infty, \tag{3.2}$$

where $dm(\xi) = dm_1(\xi)$ is the Lebesgue measure on the circle $\mathbf{T}$.

For a proof of Lemma C see [3, pp. 313, 316, 326], [12, Th. 2.1].

Lemma D. ([3], [12]) *For $0 < q < \infty, \alpha > 0, \beta > 0$ and a function $f(z)$ measurable in the unit disk*

$$\left\| C_q\big(|f(z)|(1-|z|)^\alpha\big) \right\|^q_{L^\infty} \asymp \sup_{w \in \mathbf{D}} (1-|w|)^\beta \int_{\mathbf{D}} \frac{|f(z)|^q (1-|z|)^{\alpha q - 1}}{|1-\overline{w}z|^{\beta+1}}\, dm_2(z). \tag{3.3}$$

For a proof of Lemma D including estimates of Carleson measures see [12, pp. 736–737], and also [4, Ch. VI, Sec. 3].

Define a version of Lusin's area integral (see, e.g., [23])

$$S(f)(\xi) = \left(\int_{\Gamma_\delta(\xi)} |\mathcal{D}^1 f(z)|^2\, dm_2(z)\right)^{1/2}, \qquad \xi \in \mathbf{T}, \quad \delta > 1.$$

Lemma E. (Lusin [23]) *If $f \in H(\mathbf{D})$, $0 < p < \infty$, then $\|S(f)\|_{L^p(\mathbf{T})} \asymp \|f\|_{H^p}$.*

We now turn to the *proof of Theorem 1.* Denote by L the integral on the left-hand side of (2.1) and write

$$L = \int_{\mathbf{D}} (1-|z_2|)^{s-1} \left[\int_{\mathbf{D}} |f(z)|^{p-s} |\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-1}\, dm_2(z_1) \right] dm_2(z_2). \quad (3.4)$$

Denote also the inner integral in (3.4) by J. Choosing any α, $0 < \alpha < s$, we estimate J by Lemma C:

$$\begin{aligned} J &= \int_{\mathbf{D}} |\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-\alpha} \cdot |f(z)|^{p-s} (1-|z_1|)^{\alpha} \frac{dm_2(z_1)}{1-|z_1|} \\ &\lesssim \int_{\mathbf{T}} A_{2/s}\Big(|\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-\alpha}\Big)(\xi_1) \cdot C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^{\alpha}\Big)(\xi_1)\, dm(\xi_1) \\ &\le \Big\| C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^{\alpha}\Big)\Big\|_{L^\infty} \int_{\mathbf{T}} A_{2/s}\Big(|\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-\alpha}\Big)(\xi_1)\, dm(\xi_1). \end{aligned} \quad (3.5)$$

Estimate the last integral separately:

$$\begin{aligned} J_1 &\equiv \int_{\mathbf{T}} A_{2/s}\left(|\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-\alpha}\right)(\xi_1)\, dm(\xi_1) \\ &= \int_{\mathbf{T}} \left[\int_{\Gamma_\delta(\xi_1)} |\mathcal{D}^1 f(z)|^2 (1-|z_1|)^{-2\alpha/s} dm_2(z_1) \right]^{s/2} dm(\xi_1). \end{aligned}$$

According to a result of [11, pp. 179, 186] on fractional differentiation and then by Lemma E

$$J_1 \lesssim \int_{\mathbf{T}} \left[\int_{\Gamma_\delta(\xi_1)} |\mathcal{D}_{r_1}^{\alpha/s} \mathcal{D}^1 f(z)|^2\, dm_2(z_1) \right]^{s/2} dm(\xi_1) \lesssim \left\| \mathcal{D}_{r_2}^1 \mathcal{D}_{r_1}^{\alpha/s} f \right\|_{H^s_{z_1}}^s, \quad (3.6)$$

where $H^s_{z_1}$ means the Hardy space in the variable z_1. Combining the inequalities (3.4)–(3.6), we conclude that

$$L \lesssim \int_{\mathbf{D}} (1-|z_2|)^{s-1} \Big\| C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^{\alpha}\Big)(\xi_1)\Big\|_{L^\infty} \left\| \mathcal{D}_{r_2}^1 \mathcal{D}_{r_1}^{\alpha/s} f \right\|_{H^s_{z_1}}^s dm_2(z_2).$$

By Fatou's lemma and Lemma C

$$\begin{aligned} L &\lesssim \liminf_{r_1 \to 1} \int_{\mathbf{T}}\int_{\mathbf{D}} (1-|z_2|)^{s-1} \big\| C_{(\frac{2}{s})'}\big(|f(z)|^{p-s}(1-|z_1|)^{\alpha}\big)\big\|_{L^\infty} \times \\ &\quad \times \left| \mathcal{D}_{r_2}^1 \mathcal{D}_{r_1}^{\alpha/s} f \right|^s dm(\xi_1)\, dm_2(z_2) \\ &\lesssim \Big\| C_{(2/s)'}\Big(\Big\| C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^{\alpha}\Big)\Big\|_{L^\infty} (1-|z_2|)^{\alpha}\Big)(\xi_2)\Big\|_{L^\infty} \times \\ &\quad \times \liminf_{r_1 \to 1} \int_{\mathbf{T}}\int_{\mathbf{T}} A_{2/s}\Big(\left|\mathcal{D}_{r_2}^1 \mathcal{D}_{r_1}^{\alpha/s} f\right|^s (1-|z_2|)^{s-\alpha}\Big)(\xi_2)\, dm(\xi_2)\, dm(\xi_1) \equiv J_2 \cdot J_3. \end{aligned}$$

Let us now evaluate each factor J_2 and J_3 separately. Applying again the rule of fractional differentiation [11, pp. 179, 186], Lemma E, Fatou's lemma and using the equality $\mathcal{D}_r^{\gamma_1}\mathcal{D}_r^{\gamma_2}=\mathcal{D}_r^{\gamma_2}\mathcal{D}_r^{\gamma_1}$, we get

$$
\begin{aligned}
J_3 &= \liminf_{r_1\to 1}\int_{\mathbf{T}}\int_{\mathbf{T}}\left[\int_{\Gamma_\delta(\xi_2)}\left|\mathcal{D}_{r_2}^1\mathcal{D}_{r_1}^{\alpha/s}f\right|^2(1-|z_2|)^{-2\alpha/s}dm_2(z_2)\right]^{s/2}dm(\xi_2)\,dm(\xi_1)\\
&\lesssim \liminf_{r_1\to 1}\int_{\mathbf{T}}\int_{\mathbf{T}}\left[\int_{\Gamma_\delta(\xi_2)}\left|\mathcal{D}_{r_2}^{\alpha/s}\mathcal{D}_{r_2}^1\mathcal{D}_{r_1}^{\alpha/s}f\right|^2dm_2(z_2)\right]^{s/2}dm(\xi_2)\,dm(\xi_1)\\
&\lesssim \liminf_{r_1\to 1}\int_{\mathbf{T}}\left\|\mathcal{D}^{\alpha/s}f\right\|_{H^s_{z_2}}^s dm(\xi_1)=\left\|\mathcal{D}^{\alpha/s}f\right\|_{H^s}^s.
\end{aligned}
$$

Estimate now J_2 choosing $\beta>0$ large enough:

$$
J_2=\left\|C_{(2/s)'}\left(\left\|C_{(2/s)'}\left(|f(z)|^{p-s}(1-|z_1|)^\alpha\right)\right\|_{L^\infty}(1-|z_2|)^\alpha\right)(\xi_2)\right\|_{L^\infty}.
$$

By Lemma D, the inner norm can be estimated as follows

$$
\begin{aligned}
&\left\|C_{2/(2-s)}\left(|f(z)|^{p-s}(1-|z_1|)^\alpha\right)\right\|_{L^\infty}^{2/(2-s)}\\
&\lesssim \sup_{w\in\mathbf{D}}(1-|w|)^\beta\int_{\mathbf{D}}|f(z_1,z_2)|^{2(p-s)/(2-s)}\frac{(1-|z_1|)^{2\alpha/(2-s)-1}}{|1-\overline{w}z_1|^{\beta+1}}\,dm_2(z_1)\\
&\le \|f\|_{H^\lambda_{z_1}}^{2(p-s)/(2-s)}\sup_{w\in\mathbf{D}}(1-|w|)^\beta\int_{\mathbf{D}}\frac{(1-|z_1|)^{2\alpha/(2-s)-(2/\lambda)(p-s)/(2-s)-1}}{|1-\overline{w}z_1|^{\beta+1}}\,dm_2(z_1)\\
&\lesssim \|f\|_{H^\lambda_{z_1}}^{2(p-s)/(2-s)},
\end{aligned}
$$

where the inequality $|f(\zeta)|\lesssim\|f\|_{H^q}(1-|\zeta|)^{-1/q},\ \zeta\in\mathbf{D},$ and another well-known inequality ([14, Sec. 1.4.10]) are used. Hence

$$
\begin{aligned}
J_2 &\lesssim \left\|C_{2/(2-s)}\left(\|f\|_{H^\lambda_{z_1}}^{p-s}(1-|z_2|)^\alpha\right)(\xi_2)\right\|_{L^\infty}\\
&\lesssim \left[\sup_{w\in\mathbf{D}}(1-|w|)^\beta\int\limits_{\mathbf{D}}\|f(z_1,z_2)\|_{H^\lambda_{z_1}}^{2(p-s)/(2-s)}\frac{(1-|z_2|)^{2\alpha/(2-s)-1}}{|1-\overline{w}z_2|^{\beta+1}}\,dm_2(z_2)\right]^{\frac{2-s}{2}}\\
&\lesssim \|f\|_{H^\lambda(\mathbf{D}^2)}^{p-s}\left[\sup_{w\in\mathbf{D}}(1-|w|)^\beta\int\limits_{\mathbf{D}}\frac{(1-|z_2|)^{2\alpha/(2-s)-(2/\lambda)(p-s)/(2-s)-1}}{|1-\overline{w}z_2|^{\beta+1}}\,dm_2(z_2)\right]^{\frac{2-s}{2}}\\
&\lesssim \|f\|_{H^\lambda}^{p-s}.
\end{aligned}
$$

Thus, for any $\lambda>(p-s)/\alpha$

$$
L\lesssim\|f\|_{H^\lambda}^{p-s}\left\|\mathcal{D}^{\alpha/s}f\right\|_{H^s}^s.
$$

This completes the proof of Theorem 1. ■

4. Proof of Theorems 2 and 3

We begin by proving the inclusion (2.2) for $q_1 = \infty$, i.e.

$$\|f\|_{F_\alpha^{p\infty}} \lesssim \|f\|_{F_\alpha^{pq}}. \tag{4.1}$$

Throughout the proof, J_ξ denotes the arc on $\mathbf{T}$ centered at $\xi \in \mathbf{T}$

$$J_\xi(t) = \{\eta \in \mathbf{T}; |1 - \bar{\xi}\eta| < t\}.$$

On the torus $\mathbf{T}^n$ the symbol $J_\xi(t)$ means $J_\xi(t) = J_{\xi_1}(t_1) \times \cdots \times J_{\xi_n}(t_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{T}^n$, $t = (t_1, \ldots, t_n)$. Consider a version of Hardy-Littlewood maximal function on the circle:

$$M(\psi)(\xi) = \sup_{t>0} \frac{1}{|J_\xi(t)|} \int_{J_\xi(t)} |\psi(\eta)|\, dm(\eta), \qquad \xi \in \mathbf{T}.$$

It is well known (see, e.g., [23]) that the operator M is bounded in L^p for $p > 1$.

Let $f(r\xi)$ be a function of the space F_α^{pq} on the bidisc. For ε, $0 < \varepsilon < \min\{p, q\}$, in view of 2-subharmonicity, we can find small numbers c, $c' \in (0, 1)$ such that

$$|\mathcal{D}^\beta f(r\xi)|^\varepsilon \lesssim \frac{1}{(1-r)^2} \int\limits_{J_\xi(c(1-r))} \int\limits_{r-c(1-r)}^{r+c'(1-r)} |\mathcal{D}^\beta f(t\eta)|^\varepsilon \, dt\, dm_2(\eta), \quad r \in I^2, \quad \xi \in \mathbf{T}^2.$$

A similar argument in the setting of the unit ball of $\mathbf{C}^n$ can be found in [11, p. 189].

Then an application of Hölder's inequality with indices q/ε and $q/(q-\varepsilon)$ leads to

$$\begin{aligned}
&(1-r)^{\varepsilon(\beta-\alpha)} |\mathcal{D}^\beta f(r\xi)|^\varepsilon \\
&\quad \lesssim \frac{1}{(1-r)^2} \int_{J_\xi(c(1-r))} \int\limits_{r-c(1-r)}^{r+c'(1-r)} (1-t)^{\varepsilon(\beta-\alpha)} |\mathcal{D}^\beta f(t\eta)|^\varepsilon dt\, dm_2(\eta) \\
&\quad \lesssim \frac{1}{1-r} \int_{J_\xi(c(1-r))} \Bigg(\int\limits_{r-c(1-r)}^{r+c'(1-r)} (1-t)^{q(\beta-\alpha)-1} |\mathcal{D}^\beta f(t\eta)|^q dt \Bigg)^{\varepsilon/q} dm_2(\eta).
\end{aligned}$$

Denoting

$$\psi(\eta_1, \eta_2) = \left(\int_{I^2} (1-t)^{q(\beta-\alpha)-1} |\mathcal{D}^\beta f(t\eta)|^q dt \right)^{\varepsilon/q},$$

we get

$$\begin{aligned}
&(1-r)^{p(\beta-\alpha)} |\mathcal{D}^\beta f(r\xi)|^p \lesssim \left[\frac{1}{1-r} \int_{J_\xi(c(1-r))} \psi(\eta_1, \eta_2)\, dm_2(\eta) \right]^{p/\varepsilon} \\
&\lesssim \left[\frac{1}{|J_{\xi_1}(c(1-r_1))|} \int\limits_{J_{\xi_1}(c(1-r_1))} \left(\frac{1}{|J_{\xi_2}(c(1-r_2))|} \int\limits_{J_{\xi_2}(c(1-r_2))} \psi(\eta_1, \eta_2) dm(\eta_2) \right) dm(\eta_1) \right]^{p/\varepsilon}.
\end{aligned}$$

Taking supremum over all $r \in I^2$, and then integrating the inequality in ξ_1, ξ_2, and twice applying the boundedness of the Hardy-Littlewood operator M in $L^{p/\varepsilon}$, we obtain

$$\int\limits_{\mathbf{T}^2} \sup_{r\in I^n} (1-r)^{p(\beta-\alpha)} |\mathcal{D}^\beta f(r\xi)|^p \, dm_2(\xi) \lesssim \int\limits_{\mathbf{T}^2} \psi^{p/\varepsilon}(\eta_1,\eta_2)\, dm(\eta_1)\, dm(\eta_2) = \|f\|^p_{F^{pq}_\alpha}.$$

The inclusion (4.1) is proved. The general case $0 < q \le q_1 < \infty$ follows easily from (4.1). Indeed, an application of Hölder's inequality with indices q_1/q and $q_1/(q_1-q)$ gives

$$\begin{aligned} &\|f\|^p_{F^{pq_1}_\alpha} \\ &= \int\limits_{\mathbf{T}^2} \left(\int\limits_{I^2} (1-r)^{(\beta-\alpha)(q_1-q)} (1-r)^{(\beta-\alpha)q-1} |\mathcal{D}^\beta f(r\xi)|^{q_1-q} |\mathcal{D}^\beta f(r\xi)|^q dr \right)^{p/q_1} dm_2(\xi) \\ &\lesssim \|f\|^{pq/q_1}_{F^{pq}_\alpha} \left(\int\limits_{\mathbf{T}^2} \sup_{r\in I^2} (1-r)^{p(\beta-\alpha)} |\mathcal{D}^\beta f(r\xi)|^p dm_2(\xi) \right)^{(q_1-q)/q_1}. \end{aligned}$$

Thus,

$$\|f\|_{F^{pq_1}_\alpha} \lesssim \|f\|^{q/q_1}_{F^{pq}_\alpha} \|f\|^{(q_1-q)/q_1}_{F^{p\infty}_\alpha} \lesssim \|f\|^{q/q_1}_{F^{pq}_\alpha} \|f\|^{(q_1-q)/q_1}_{F^{pq}_\alpha} = \|f\|_{F^{pq}_\alpha},$$

and this completes the proof of Theorem 2. ■

Proof of Theorem 3. Assuming that $\|f\|_{H^p} \ne 0$, we can apply Jensen's inequality to the integral

$$\begin{aligned} &\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(r\xi)|^{p-s} |g(r\xi)|^s dm_n(\xi) \\ &\qquad = M_p^p(f,r) \left[\frac{1}{M_p^p(f,r)} \int_{\mathbf{T}^n} \left| \frac{g(r\xi)}{f(r\xi)} \right|^s |f(r\xi)|^p \frac{dm_n(\xi)}{(2\pi)^n} \right]^{\frac{p}{s}\frac{s}{p}} \\ &\qquad \le M_p^p(f,r) \left[\frac{1}{M_p^p(f,r)} \int_{\mathbf{T}^n} \left| \frac{g(r\xi)}{f(r\xi)} \right|^p |f(r\xi)|^p \frac{dm_n(\xi)}{(2\pi)^n} \right]^{s/p} \\ &\qquad = M_p^{p-s}(f,r) \left[\int_{\mathbf{T}^n} |g(r\xi)|^p \frac{dm_n(\xi)}{(2\pi)^n} \right]^{s/p} = M_p^{p-s}(f,r) M_p^s(g,r). \end{aligned}$$

A similar method is applied in the proof of Theorem 4 of [19]. Further, a weighted integration leads now to

$$\begin{aligned} &\frac{1}{(2\pi)^n} \int_{\mathbf{D}^n} |f(z)|^{p-s} |g(z)|^s (1-|z|)^{\alpha s-1} dm_{2n}(z) \\ &\qquad \le \int_{I^n} M_p^{p-s}(f,r) M_p^s(g,r) (1-r)^{\alpha s-1} dr \\ &\qquad \le \|f\|^{p-s}_{H^p} \int_{I^n} M_p^s(g,r)(1-r)^{\alpha s-1} dr, \end{aligned}$$

and the proof is complete. ■

5. Proof of Theorems 4 and 5

We need the next lemma which extends the well-known Hardy-Stein identity (see, e.g., [7]) to the polydisc.

Lemma 1. *Suppose that $f(z) \in H(\mathbf{D}^n)$, $0 < p < \infty$. Then for any $r = (r_1, \dots, r_n) \in I^n$*

$$\prod_{j=1}^{n} r_j \cdot \frac{\partial^n}{\partial r_1 \dots \partial r_n} M_p^p(f,r) = \frac{1}{(2\pi)^n} \int\limits_{|z_1|<r_1} \cdots \int\limits_{|z_n|<r_n} f^{\#}(z) dm_{2n}(z), \tag{5.1}$$

where $f^{\#}(z) = \Delta_{z_1}\Delta_{z_2}\dots\Delta_{z_n}|f(z)|^p$, and Δ_{z_j} is the usual Laplacian in the variable z_j.

Proof. Fix z_2 for a moment and apply Green's formula (see, e.g., [4], [23]) to the function $|f(z_1,z_2)|^p$ in $|z_1| < r_1$:

$$\int_{|z_1|=r_1} \frac{\partial}{\partial r_1} |f(z_1,z_2)|^p d\ell = \int_{|z_1|<r_1} \Delta_{z_1} |f(z_1,z_2)|^p dm_2(z_1),$$

where $d\ell$ means arc length integration. With respect to the function

$$\psi(z_2) = r_1 \frac{\partial}{\partial r_1} \int_{\mathbf{T}} |f(r_1\xi_1, z_2)|^p dm(\xi_1) = \int_{|z_1|<r_1} \Delta_{z_1} |f(z_1,z_2)|^p dm_2(z_1),$$

we can again apply Green's formula in $|z_2| < r_2$:

$$\int_{|z_2|=r_2} \frac{\partial}{\partial r_2} \psi(z_2) d\ell = \int_{|z_2|<r_2} \Delta_{z_2} \psi(z_2)\, dm_2(z_2).$$

Combining these equalities, we obtain

$$r_1 r_2 \frac{\partial^2}{\partial r_1 \partial r_2} \int_{\mathbf{T}} \int_{\mathbf{T}} |f(r_1\xi_1, r_2\xi_2)|^p \, dm(\xi_1)\, dm(\xi_2)$$
$$= \int_{|z_1|<r_1} \int_{|z_2|<r_2} \Delta_{z_1}\Delta_{z_2} |f(z_1,z_2)|^p \, dm_4(z),$$

which finishes the proof. ∎

Remark 5. For $n = 1$ (5.1) coincides with the well-known Hardy-Stein identity [7] in view of formula (2.7).

Proof of Theorem 4. Lemma 1 enables us to establish another identity

$$\begin{aligned} & r_1 r_2 \int_{\mathbf{T}^2} \Delta_{z_1}\Delta_{z_2} |f(z_1,z_2)|^p dm(\xi_1)\, dm(\xi_2) \\ & = \frac{\partial^2}{\partial r_1 \partial r_2} \int_0^{r_1} \int_0^{r_2} \int_{\mathbf{T}^2} \Delta_{z_1}\Delta_{z_2} |f(z_1,z_2)|^p \rho_1\rho_2 dm(\xi_1)\, dm(\xi_2)\, d\rho_1\, d\rho_2 \\ & = (2\pi)^2 \frac{\partial^2}{\partial r_1 \partial r_2} \left[r_1 r_2 \frac{\partial^2}{\partial r_1 \partial r_2} M_p^p(f, r_1, r_2) \right]. \end{aligned} \tag{5.2}$$

First we prove the identity (2.6), i.e. the one variable version. We transform the left integral of (2.6), integrating by parts and using the identity (5.2):

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbf{D}}&(1-|z|)^{\alpha}f^{\#}(z)dm_2(z) \\ &= \frac{1}{2\pi}\int_0^1(1-r)^{\alpha}\left[\int_{-\pi}^{\pi}\Delta|f(re^{i\theta})|^p d\theta\right] r\,dr \\ &= \int_0^1(1-r)^{\alpha}\left[\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}M_p^p(f,r)\right)\right]dr \\ &= \lim_{r\to 1-}(1-r)^{\alpha}\,r\frac{\partial}{\partial r}M_p^p(f,r)+\alpha\int_0^1(1-r)^{\alpha-1}\,r\,\frac{\partial}{\partial r}M_p^p(f,r)\,dr. \end{aligned} \tag{5.3}$$

If the right-hand side integral in (2.6) or (5.3) exists, then the limit in (5.3) vanishes. Indeed, by the Hardy-Stein identity, the function $r\frac{\partial}{\partial r}M_p^p(f,r)$ is increasing in $r\in(0,1)$. Hence for any $\rho\in(0,1)$

$$\int_{\rho}^{(1+\rho)/2}(1-r)^{\alpha-1}\,r\,\frac{\partial}{\partial r}M_p^p(f,r)dr \ge C_{\alpha}\,\rho\,(1-\rho)^{\alpha}\,\frac{\partial}{\partial\rho}M_p^p(f,\rho).$$

By the Cauchy criterion for convergence

$$\lim_{\rho\to 1-}(1-\rho)^{\alpha}\,\frac{\partial}{\partial\rho}M_p^p(f,\rho)=0.$$

It follows from (5.3) that

$$\frac{1}{2\pi}\int_{\mathbf{D}}(1-|z|)^{\alpha}f^{\#}(z)dm_2(z)=\alpha\int_0^1(1-r)^{\alpha-1}\,r\,\frac{\partial}{\partial r}M_p^p(f,r)\,dr.$$

Part (ii) of the theorem is proved.

Proceeding to the proof of the inequality (2.8), note that the example $f(z)=z$ shows the sharpness of the constant α/p. Then the identity (2.6) can be written as follows

$$A(f;p,\alpha)=\frac{\alpha}{p}\int_0^1\int_{-\pi}^{\pi}|f(re^{i\theta})|^{p-1}\left(\frac{\partial}{\partial r}\big|f(re^{i\theta})\big|\right)(1-r)^{\alpha-1}r\,dr\,d\theta. \tag{5.4}$$

Since $\big||f(re^{i\theta})|-|f(\rho e^{i\theta})|\big|\le\big|f(re^{i\theta})-f(\rho e^{i\theta})\big|$, we have $\big|\frac{\partial}{\partial r}|f(re^{i\theta})|\big|\le\big|f'(re^{i\theta})\big|$. Hence, (2.8) follows.

We now turn to the proof of the inequality (2.9). By the Cauchy-Schwarz inequality,

$$B(f;p,\alpha)\le\sqrt{A(f;p,\alpha)}\left(\int_{\mathbf{D}}|f(z)|^p(1-|z|)^{\alpha-2}dm_2(z)\right)^{1/2}.$$

Therefore, it only remains to verify the inequality

$$\int_{\mathbf{D}}|f(z)|^p(1-|z|)^{\alpha-2}dm_2(z)\lesssim A(f;p,\alpha),\qquad p>0,\ \alpha>1. \tag{5.5}$$

To this end, we integrate by parts to get

$$\begin{aligned}\frac{p^2}{2\pi\alpha}A(f;p,\alpha) &= \int_0^1 (1-r)^{\alpha-1}\, r\, \frac{\partial}{\partial r} M_p^p(f,r)\, dr\\ &= \lim_{r\to 1-} (1-r)^{\alpha-1}\, r\, M_p^p(f,r) - \int_0^1 M_p^p(f,r)\, d\Big(r(1-r)^{\alpha-1}\Big)\\ &= \lim_{r\to 1-} (1-r)^{\alpha-1}\, r\, M_p^p(f,r) + \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi} |f(re^{i\theta})|^p (1-r)^{\alpha-2}(\alpha r-1)\, dr\, d\theta.\end{aligned}$$

This equality shows that if $A(f;p,\alpha)$ exists, then the function $f(z)$ is in the Bergman space $A^p_{\alpha-2}(\mathbf{D})$. Consequently $\lim_{r\to 1-}(1-r)^{\alpha-1}\, M_p^p(f,r) = 0$. So, we get

$$\begin{aligned}A(f;p,\alpha) &= \frac{\alpha}{p^2}\int_0^1\int_{-\pi}^{\pi} |f(re^{i\theta})|^p (1-r)^{\alpha-2}(\alpha r-1)\, dr\, d\theta\\ &\geq \frac{\alpha(\alpha-1)}{2p^2}\int_{(\alpha+1)/(2\alpha)}^1\int_{-\pi}^{\pi} |f(re^{i\theta})|^p (1-r)^{\alpha-2} dr\, d\theta\\ &\geq C(\alpha,p)\int_{\mathbf{D}} |f(z)|^p (1-|z|)^{\alpha-2} dm_2(z),\end{aligned}$$

and this gives the desired result. Part (iii) of the theorem is proved. Part (i) of the theorem can be proved from (5.2) similarly, so we omit the details. ■

Proof of Theorem 5. The integrated Hardy-Stein identity (see Lemma 1)

$$\begin{aligned}M_p^p(f,r_1,r_2) + |f(0,0)|^p &= M_p^p(f,0,r_2) + M_p^p(f,r_1,0) +\\ &+ \frac{1}{(2\pi)^2}\int_0^{r_1}\int_0^{r_2}\left(\int_{|z_1|<\rho_1}\int_{|z_2|<\rho_2} f^{\#}(z_1,z_2)dm_4(z)\right)\frac{d\rho_1 d\rho_2}{\rho_1\rho_2}\end{aligned}$$

can be integrated again with respect to the measure $(2\pi)^2 C_{\omega_1}C_{\omega_2}\omega_1(r_1)\omega_2(r_2)\times r_1 r_2\, dr_1\, dr_2$. We thus have

$$\|f\|^p_{A^p_\omega} + |f(0,0)|^p = J_1 + J_2 + J_3,$$

where

$$J_1 = \big\|f(z_1,0)\big\|^p_{A^p_{\omega_1}(\mathbf{D})} = |f(0,0)|^p + 2\pi C_{\omega_1}\int_0^1 M_1\Big(\Delta_{z_1}|f(z_1,0)|^p, r_1\Big)h_{\omega_1}(r_1)r_1 dr_1,$$

$$J_2 = \big\|f(0,z_2)\big\|^p_{A^p_{\omega_2}(\mathbf{D})} = |f(0,0)|^p + 2\pi C_{\omega_2}\int_0^1 M_1\Big(\Delta_{z_2}|f(0,z_2)|^p, r_2\Big)h_{\omega_2}(r_2)r_2 dr_2.$$

Besides, a further application of Fubini's theorem shows that

$$\begin{aligned}J_3 &= C_{\omega_1}C_{\omega_2}\int_{I^2}\left[\int_0^{r_1}\int_0^{r_2}\left(\int_{|z_1|<\rho_1}\int_{|z_2|<\rho_2} f^{\#}(z_1,z_2)dm_4(z)\right)\frac{d\rho_1 d\rho_2}{\rho_1\rho_2}\right]\omega(r)rdr\\ &= (2\pi)^2 C_{\omega_1}C_{\omega_2}\int_0^1\int_0^1 M_1\big(f^{\#}(z_1,z_2), r_1, r_2\big)h_{\omega_1}(r_1)h_{\omega_2}(r_2)r_1 r_2 dr_1 dr_2.\end{aligned}$$

This completes the proof of Theorem 5. ■

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