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CONTINUOUS REPRESENTATION OF INTERVAL ORDERS BY MEANS OF DECREASING SCALES

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Abstract. We characterize the representability of an interval order on a topological space through a pair of continuous real-valued functions which in addition represent two total preorders associated to the given interval order. Such a continuous representation is obtained by using the notion of a decreasing scale.

1. Introduction

Interval orders are reflexive and total binary relations which are not transitive in general. Such a model may be viewed as the simplest one fulfilling these requirements, in the sense that interval orders may be fully represented by a pair of real-valued functions. The real representability of interval orders was first deeply studied by Fishburn (see e.g. Fishburn [18,19], and then considered by other authors (see e.g. Bridges [9–11], Bridges and Mehta [13], and Oloriz et al. [22]).

Some authors were concerned with the existence of a (semi)continuous representation of an interval order on a topological space (see e.g. Bridges [12], Candeal et al. [15], Chateauneuf [16], Bosi [3], Bosi and Isler [4], and Bosi et al. [5]). In particular, Chateauneuf [16] provided a characterization of the existence of a pair of continuous real-valued functions representing an interval order on a connected topological space. A characterization of the existence of a continuous representation of an interval order on a topological space has been recently obtained by Bosi et al. [6] by using a suitable notion of order separability, called *i.o.separability.*

In this paper we provide a characterization of the existence of a pair (U, V) of continuous real-valued functions representing an interval order $\precsim$ on a topological space (X, τ) (in the sense that, for all $x, y \in X$, $x \precsim y$ if and only if $U(x) \leq V(y)$). The functions U and V may be chosen so that they represent two total preorders associated to the interval order $\precsim$. In order to obtain such a characterization, we use the notion of a *decreasing scale* which was first introduced by Burgess and Fitzpatrick [14], and then considered by other authors (see e.g. Herden [20], Alcantud et al. [1], Bosi and Mehta [7] and Bosi and Zuanon [8]).

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2. Notation and preliminaries

An *interval order* $\precsim$ on an arbitrary nonempty set X is a binary relation on X which is *reflexive* and in addition verifies the following condition for all $x, y, z, w \in X$:

$$(x \precsim z) \quad \text{and} \quad (y \precsim w) \Rightarrow (x \precsim w) \quad \text{or} \quad (y \precsim z).$$

The *strict part* of a given interval order $\precsim$ will be denoted by $\prec$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if *not*$(y \precsim x)$). An interval order $\precsim$ on a set X is necessarily total, in the sense that, for any two elements $x, y \in X$, either $x \precsim y$ or $y \precsim x$ (see Oloriz et al. [22]).

If $\precsim$ is an interval order on a set X, then we may consider the binary relations $\precsim^*$ and $\precsim^{**}$ on X defined as follows:

$$x \precsim^* y \Leftrightarrow (z \precsim x \Rightarrow z \precsim y \text{ for every } z \in X) \quad (x, y \in X)$$
$$x \precsim^{**} y \Leftrightarrow (y \precsim z \Rightarrow x \precsim z \text{ for every } z \in X) \quad (x, y \in X)$$

Fishburn [19] proved that the binary relations $\precsim^*$ and $\precsim^{**}$ associated to any interval order $\precsim$ on a set X are total preorders on X (i.e., they are reflexive, transitive and total). It is clear that, for any two elements $x, y \in X$, if either $x \precsim^* y$ or $x \precsim^{**} y$, then we have that $x \precsim y$.

Obviously every total preorder $\precsim$ on a set X is an interval order on X. In this case, we have that $\precsim = \precsim^* = \precsim^{**}$. The importance of interval orders in economics lies on the fact that they are not transitive in general.

A total preorder $\precsim$ on a set X is representable by means of a real-valued function U on X if, for all $x, y \in X$:

$$x \precsim y \Leftrightarrow U(x) \leq U(y).$$

We also say that U is a *utility function* for the total preorder $\precsim$ on the set X.

An interval order $\precsim$ on a set X is said to be representable through a pair (U, V) of real-valued functions on X if, for all $x, y \in X$:

$$x \precsim y \Leftrightarrow U(x) \leq V(y).$$

If $\precsim$ is an interval order on a set X, then a subset G of X is said to be $\precsim$-*decreasing* if, for all $x, y \in X$, $x \precsim y$ and $y \in G$ imply $x \in G$.

An interval order $\precsim$ on a topological space (X, τ) is said to be *upper (lower) semicontinuous* if $L_{\prec}(x) = \{y \in X : y \prec x\}$ $(U_{\prec}(x) = \{y \in X : x \prec y\})$ is a τ-open subset of X for every $x \in X$. If $\precsim$ is both upper and lower semicontinuous, then it is said to be *continuous* .

3. Continuous representability

We present a characterization of the existence of a pair of continuous real-valued functions representing an interval order on a topological space, where the two functions are utilities for two total preorders naturally associated with the given interval order.

THEOREM 3.1. *The following conditions are equivalent for an interval order* $\precsim$ *on a topological space* (X, τ)*:*

i) *The interval order* $\precsim$ *is representable through a pair of continuous real-valued functions* (U, V) *with values in* $[0,1]$*, where* U *is a representation for the total preorder* $\precsim^{**}$ *and* V *is a representation for the total preorder* $\precsim^{*}$*;*

ii) *There exist two families* $\{G_r^*\}_{r\in\mathbf{Q}\cap]0,1]}$ *and* $\{G_r^{**}\}_{r\in\mathbf{Q}\cap]0,1]}$ *of open subsets of* (X, τ) *with* $G_1^* = G_1^{**} = X$ *satisfying the following conditions:*

   (a) $x \precsim y$ *and* $y \in G_r^*$ *imply* $x \in G_r^{**}$ *for every* $x, y \in X$ *and* $r \in \mathbf{Q}\cap]0,1]$*;*

   (b) G_r^* *is* $\precsim^*$*-decreasing and* G_r^{**} *is* $\precsim^{**}$*-decreasing for every* $r \in \mathbf{Q}\cap]0,1]$*;*

   (c) $\overline{G_{r_1}^*} \subseteq G_{r_2}^*$ *and* $\overline{G_{r_1}^{**}} \subseteq G_{r_2}^{**}$ *for every* $r_1, r_2 \in \mathbf{Q}\cap]0,1]$ *such that* $r_1 < r_2$*;*

   (d) *for every* $x, y \in X$ *such that* $x \prec y$ *there exist* $r_1, r_2 \in \mathbf{Q}\cap]0,1[$ *such that* $r_1 < r_2$, $x \in G_{r_1}^*$, $y \notin G_{r_2}^{**}$.

Proof. $i) \Rightarrow ii)$. If (U, V) is a representation of the interval order $\precsim$ with the indicated properties, then just define $G_r^* = V^{-1}([0, r[)$, $G_r^{**} = U^{-1}([0, r[)$ for every $r \in \mathbf{Q}\cap]0,1[$, and $G_1^* = G_1^{**} = X$ in order to immediately verify that $\{G_r^*\}_{r\in\mathbf{Q}\cap]0,1]}$ and $\{G_r^{**}\}_{r\in\mathbf{Q}\cap]0,1]}$ are two families of open subsets of (X, τ) satisfying conditions (a) through (d).

$ii) \Rightarrow i)$. Assume that the above condition $ii)$ holds. Define two functions $U, V : X \to [0,1]$ as follows:

$$U(x) = \inf\{r \in \mathbf{Q}\cap]0,1] : x \in G_r^{**}\} \quad (x \in X),$$
$$V(x) = \inf\{r \in \mathbf{Q}\cap]0,1] : x \in G_r^{*}\} \quad (x \in X).$$

We claim that (U, V) is a pair of continuous functions on (X, τ) with values in $[0,1]$ representing the interval order $\precsim$ where U is a representation for the total preorder $\precsim^{**}$ and V is a representation for the total preorder $\precsim^*$.

From the definition of the functions U and V, it is clear that they both take values in $[0,1]$. Let us first show that the pair (U, V) represents the interval order $\precsim$. Consider any two elements $x, y \in X$ such that $x \precsim y$, and observe that, for every $r \in \mathbf{Q}\cap]0,1]$, if $y \in G_r^*$ then it must be that $x \in G_r^{**}$ by the above condition (a). Hence it must be that $U(x) \leq V(y)$ from the definition of U and V. Now consider any two elements $x, y \in X$ such that $y \prec x$. Then by condition (d), there exist $r_1, r_2 \in \mathbf{Q}\cap]0,1]$ such that $r_1 < r_2$, $y \in G_{r_1}^*$, $x \notin G_{r_2}^{**}$. Hence we have that $V(y) \leq r_1 < r_2 \leq U(x)$, which obviously implies that $V(y) < U(x)$.

Let us now prove that V is a representation for the total preorder $\precsim^*$. From the first part of condition (b) we have that G_r^* is a $\precsim^*$ decreasing subset of X for every $r \in \mathbf{Q}\cap]0,1]$. Hence if x, y are any two elements of X such that $x \precsim^* y$, then it must be that $V(x) \leq V(y)$ from the definition of V. Now consider any two elements $x, y \in X$ such that $y \prec^* x$. Hence there exists another element $z \in X$ such that $y \prec z \precsim x$. So, by condition (d), there exist $r_1, r_2 \in \mathbf{Q}\cap]0,1]$ such that $r_1 < r_2$,

$y \in G^*_{r_1}$, $z \notin G^{**}_{r_2}$. By condition (a), we have that $x \notin G^*_{r_2}$ since $z \precsim x$. Finally, we may guarantee the existence of $r_1, r_2 \in \mathbf{Q}\cap]0,1]$ such that $r_1 < r_2$, $y \in G^*_{r_1}$, $x \notin G^*_{r_2}$. Hence from the definition of V, we have that $V(y) \leq r_1 < r_2 \leq V(x)$ which obviously implies that $V(y) < V(x)$.

Analogously it may be shown that U is a representation for the total preorder $\precsim^{**}$.

To conclude the proof, let us show that U and V are both continuous functions by condition (c). We only prove that U is continuous. Then analogous arguments will show that also V is continuous. Let us first prove that U is upper semicontinuous. Consider any $x \in X$, and $\alpha \in \mathbf{R}\cap]0,1]$ such that $U(x) < \alpha$. Then from the definition of U, there exists $r \in \mathbf{Q}\cap]0,1]$ such that $U(x) \leq r < \alpha$, $x \in G^{**}_r$. Observe that $U(z) \geq \alpha$ $(z \in X)$ implies that $U(z) > r$ which in turn implies that $z \notin G^{**}_r$. Hence G^{**}_r is an open subset of X containing x such that $U(z) < \alpha$ for every $z \in G^{**}_r$. In order to show that U is lower semicontinuous, let us first prove that

$$U(x) = \inf\{r \in \mathbf{Q}\cap]0,1] : x \in \overline{G^{**}_r}\} \text{ for every } x \in X.$$

Since $G^{**}_r \subseteq \overline{G^{**}_r}$ for every $r \in \mathbf{Q}\cap]0,1]$, it is clear that, for every $x \in X$,

$$\inf\{r \in \mathbf{Q}\cap]0,1] : x \in \overline{G^{**}_r}\} \leq \inf\{r \in \mathbf{Q}\cap]0,1] : x \in G_r\}.$$

Now assume that there exists $x \in X$ with

$$\inf\{r \in \mathbf{Q}\cap]0,1] : x \in \overline{G^{**}_r}\} < \inf\{r \in \mathbf{Q}\cap]0,1] : x \in G^{**}_r\}.$$

Consider r_1, $r_2 \in \mathbf{Q}\cap]0,1]$ such that

$$\inf\{r \in \mathbf{Q}\cap]0,1] : x \in \overline{G^{**}_r}\} < r_1 < r_2 < \inf\{r \in \mathbf{Q}\cap]0,1] : x \in G^{**}_r\}.$$

Then we have that $x \in \overline{G^{**}_{r_1}}$, $x \notin G^{**}_{r_2}$, and this is contradictory, since $\overline{G^{**}_{r_1}} \subseteq G^{**}_{r_2}$. So it must be that, for every $x \in X$,

$$\inf\{r \in \mathbf{Q}\cap]0,1] : x \in \overline{G^{**}_r}\} = \inf\{r \in \mathbf{Q}\cap]0,1] : x \in G^{**}_r\}.$$

Now consider any $x \in X$, and any $\alpha \in \mathbf{R}\cap]0,1]$ such that $\alpha < U(x)$. Further let $r_1, r_2 \in \mathbf{Q}\cap]0,1]$ be such that $\alpha < r_1 < r_2 < U(x)$. Then we have that $x \notin \overline{G^{**}_{r_1}}$ because otherwise $x \in \overline{G^{**}_{r_1}}$ implies that $x \in G^{**}_{r_2}$ and this contradicts the fact that $U(x) > r_2$. Observe that $U(z) \leq \alpha$ $(z \in X)$ implies that $U(z) < r_1$ which in turn implies that $z \in \overline{G^{**}_{r_1}}$ since $U(x) = \inf\{r \in \mathbf{Q}\cap]0,1] : x \in \overline{G^{**}_r}\}$ for every $x \in X$. Hence $X \setminus \overline{G^{**}_{r_1}}$ is an open subset of X containing x such that $\alpha < U(z)$ for every $z \in X \setminus \overline{G^{**}_{r_1}}$. This consideration completes the proof. ∎

Remark 3.2. The family $\{G^*_r\}_{r \in \mathbf{Q}\cap]0,1]}$ is a $\precsim^*$-*decreasing scale* according to the definition introduced by Burgess and Fitzpatrick [14].

As an application of the previous characterization, in the following proposition we present a generalization of the Theorem in Chateauneuf [16]. Chateauneuf showed that a *strongly separable* interval order $\precsim$ on a connected topological space

(X, τ) is representable through a pair of continuous real-valued functions (U, V), where U is a representation for the total preorder $\precsim^{**}$ and V is a representation for the total preorder $\precsim^{*}$, provided that the total preorders $\precsim^{*}$ and $\precsim^{**}$ are both continuous.

We recall that an interval order $\precsim$ on a set X is said to be *strongly separable* if there exists a countable set $D \subseteq X$ such that, for every $x, y \in X$ with $x \prec y$, there exists $d_1, d_2 \in D$ with $x \prec d_1 \precsim d_2 \prec y$. D is said to be an *order dense subset of* X (see Chateauneuf [16]).

Observe that, in contrast to the *Chateauneuf Representation Theorem*, ours does not need any connectedness assumption on the topological space. The following proposition was already proved by Bosi [3] by using the proof of the existence of a continuous utility function provided by Jaffray [21].

Proposition 3.3. *Let $\precsim$ be a strongly separable interval order on a topological space (X, τ), and assume that the total preorders $\precsim^{*}$ and $\precsim^{**}$ are both continuous. Then the interval order $\precsim$ is representable through a pair of continuous real-valued functions (U, V) with values in $[0, 1]$, where U is a representation for the total preorder $\precsim^{**}$ and V is a representation for the total preorder $\precsim^{*}$.*

Proof. Let $\precsim$ be a strongly separable interval order on a topological space (X, τ), and assume that the associated total preorders $\precsim^{*}$ and $\precsim^{**}$ are both continuous. Then from the Proposition in Chateauneuf [16], we have that $\precsim$ is continuous. Further strong separability of $\precsim$ implies *order separability* of $\precsim^{*}$ and $\precsim^{**}$. In particular, if D is an order dense subset of X, then for all $x, y \in X$ with $x \prec^{**} y$ there exists $d \in D$ with $x \prec^{**} d \prec^{**} y$. Without loss of generality, we may assume that $(D, \precsim^{**})$ is actually a totally ordered set (or a *chain*) without extreme points. Therefore by using considerations in Birkhoff [2] and following a construction analogous to Construction 3.3 in Alcantud et al. [1], we may conclude that there exists an order-preserving function $f\colon (D, \precsim^{**}) \to (\mathbf{Q} \cap]0, 1[, \leq)$. Further we may assume that the mapping f is onto.

For reader's convenience we recall that any countable chain $(C, \preceq)$ is order isomorphic with a subchain of $(\mathbf{Q}, \leq)$ (see Theorem 22 on page 200 in Birkhoff [2]) and in particular order isomorphic with $(\mathbf{Q}, \leq)$ if $(C, \preceq)$ is *dense in itself* and has neither a minimal nor a maximal element (see Theorem 23 on page 200 in Birkhoff [2]). Therefore we may conclude that any countable chain $(C, \preceq)$ with the above properties is also order isomorphic with $(\mathbf{Q} \cap]0, 1[, \leq)$.

Let us now go back to our case and consider an order-preserving function $f\colon (D, \precsim^{**}) \to (\mathbf{Q} \cap]0, 1[, \leq)$ which is also onto. If $f^{-1}(r) = d$ ($r \in \mathbf{Q} \cap]0, 1[$), then define

$$G_r^{*} = L_{\prec}(d), \;\; G_r^{**} = L_{\prec^{**}}(d) \quad (r \in \mathbf{Q} \cap]0, 1[),$$

and set $G_1^{*} = G_1^{**} = X$. We claim that $\{G_r^{*}\}_{r \in \mathbf{Q} \cap]0,1]}$ and $\{G_r^{**}\}_{r \in \mathbf{Q} \cap]0,1]}$ are two families of subsets of X satisfying condition *ii)* of Theorem 3.1. It is clear that G_r^{*} is open and $\precsim^{*}$-decreasing, and G_r^{**} is open and $\precsim^{**}$-decreasing for every $r \in \mathbf{Q} \cap]0, 1]$, so that condition *(b)* holds. In order to show that condition *(a)* is verified, just observe that, for every $d \in D$, if $x \precsim y$ and $y \prec d$, then $x \prec^{**} d$. In order to prove that condition *(c)* holds, observe that for all $d_1, d_2 \in D$ such that $d_1 \prec^{**} d_2$ there exists $z \in X$ such that $d_1 \precsim z \prec d_2$, and therefore we have that

$$L_{\prec}(d_1) \subseteq \overline{L_{\prec}(d_1)} \subseteq L_{\precsim^*}(z) \subseteq L_{\prec}(d_2),$$
$$L_{\prec^{**}}(d_1) \subseteq \overline{L_{\prec^{**}}(d_1)} \subseteq L_{\precsim^{**}}(d_1) \subseteq L_{\prec^{**}}(d_2),$$

where $L_{\precsim^*}(z) = \{w \in X : w \precsim^* z\}$ and $L_{\precsim^{**}}(d_1) = \{w \in X : w \precsim^{**} d_1\}$ are closed subsets of X. Finally, condition *(d)* of Theorem 3.1 holds, since strong separability of the interval order $\precsim$ implies that for all $x, y \in X$ such that $x \prec y$ there exist $d_1, d_2 \in D$ such that $x \prec d_1 \prec^{**} d_2 \prec^{**} y$, and therefore $x \in L_{\prec}(d_1)$ and $y \notin L_{\prec^{**}}(d_2)$. This consideration completes the proof. ∎

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