МАТЕМАТИЧКИ ВЕСНИК
60 (2008), 9–22

UDK 515.124
оригинални научни рад
research paper

LOCAL LIPSCHITZ PROPERTY FOR THE CHEBYSHEV CENTER MAPPING OVER N-NETS

Pyotr N. Ivanshin and Evgenii N. Sosov

Abstract. We prove local Lipschitz property of the map which puts in correspondence to each exact N-net its Chebyshev center. If dimension of Euclidean or Lobachevsky space is greater than 1 and the net consists of more than 2 points we show that this map is not Lipschitz in a neighbourhood of the space of all 2-nets embedded into the space of N-nets endowed with Hausdorff metric.

1. Definitions and notation

We assume the following notation.

$\mathbb{R}_+$ – the set of all non-negative real numbers.

(X, ρ) – metric space with metric ρ.

$xy = \rho(x, y)$, $xZ = \inf\{xu : u \in Z\}$ for x, $y \in X$, $Z \subset X$.

$\omega(x, y)$ – the set such that if non-empty (for example in case (X, ρ) is a convex metric space) then for any $z \in \omega(x, y)$, $2xz = 2yz = xy$ for x, $y \in X$ (in the Euclidean space it is the middle-point of the interval $[x, y]$).

$B[x, r]$ ($B(x, r)$, $S(x, r)$) – a closed ball (open ball, sphere) centered at $x \in (X, \rho)$ with radius $r \geq 0$.

For each positive integer N, denote by $\Sigma_N(X)$ the class of all nonempty subsets Y of X with at most N points; each element of it will be called an N-net [1]. Let also $\Sigma_N^*(X)$ stand for the class of all $Y \in \Sigma_N(X)$ with exactly N points. Clearly, $\Sigma_N^*(X) = \Sigma_N(X) \setminus \Sigma_{N-1}(X)$; each element of it will be referred to as an exact N-net.

Let $B(X)$ stand for the set of all non-empty bounded closed subsets of the space X. $\alpha\colon B(X) \times B(X) \to \mathbb{R}_+$, $\alpha(M, T) = \max\{\max\{xT : x \in M\}, \max\{tM : t \in T\}\}$ is the Hausdorff metric on the set $B(X)$ (cf. [2], p. 223). It is also a metric on the set $\Sigma_N(X) \subset B(X)$. Let us denote for brevity by $\Sigma_N(X)$ the metric space $(\Sigma_N(X), \alpha)$.

AMS Subject Classification: 54E40, 52C35.

Keywords and phrases: Chebyshev center, local Lipschitz property, N-net, Hausdorff metric.

$B_\alpha(M,r)$ – open ball centered at the point $M \in (\Sigma_N(X),\alpha)$, with radius $r>0$.

$D[M]$ – diameter of the set $M \in \Sigma_N(X)$.

Consider $M \in \Sigma_N(X)$. Assume that $R(M)=\inf\{\sup[yx : y \in M] : x \in X\}$ is Chebyshev radius of the N-net M. The point $\mathrm{cheb}(M) \in X$ is called a Chebyshev center if $\sup[x\,\mathrm{cheb}(M) : x \in M] = R(M)$ [1].

Let (X,ρ) and (X_1,ρ_1) be metric spaces. The mapping $f : X \to X_1$ will be called (ρ,ρ_1)-Lipschitz (or, simply, Lipschitz when ρ and ρ_1 are understood) if there is a constant $L \geq 0$ (called the Lipschitz constant of f) such that $\rho_1(f(x),f(y)) \leq L\rho(x,y)$, for all $x,y \in X$. When $L=1$, we say that f is (ρ,ρ_1)-nonexpansive (or, simply, nonexpansive when ρ and ρ_1 are understood)(cf. [3], p. 10).

The notation given below will be used in Euclidean space as well as in Lobachevsky one.

$co(M)$ – convex hull of the set M.

$[x,y]$ ((x,y), $(x,y]$) – closed (open, half-closed) interval of the end-points x,y.

$\Pi(x_1,\dots,x_{m+1})$ – m-plane defined by points $x_1,\dots,x_{m+1}$.

$\Lambda(x,y)$ – ray with the vertex in the point x comprising $y \neq x$.

2. Statement of the problem

The notion of Chebyshev center is of a fundamental importance in many areas of Analysis and Geometry (fixed point and approximation theory, metrical geometry, etc.). A natural problem to be addressed here is the study of local and global Lipschitz properties of the mapping cheb: $B(X) \to X$ over different classes K of $B(X)$. Despite its obvious importance, this problem was not completely solved until now even in the easiest case of Euclidean space.

It is well known that the Chebyshev center mapping cheb: $B(X) \to X$ is continuous, whenever X is Euclidean or Lobachevsky; cf. [4], [5]. Moreover, restriction of the map cheb to the set of all balls of the space with inner metric is a nonexpansive map [7]. Nevertheless the restriction of the map cheb to the set $(B(\mathbb{R}^2),\alpha)$ is not Lipschitz even in a neighbourhood of a closed disk [8].

Thus we arrive to the following (rather complicated in general case) task: Find classes of subsets (different from one consisting of the points or balls) of the given metric space such that for any class K the map cheb: $(K,\alpha) \to X$, $M \mapsto \mathrm{cheb}(M)$ is locally Lipschitz.

It seems natural to solve this problem first for Euclidean space (X,ρ) (X may be infinite-dimensional and ρ is the standard metric) and $K=\Sigma_N(X)$ or $K=\Sigma^*_N(X)$. The brief formulation of the result achieved by the authors is as follows:

i) if X is Euclidean the Chebyshev center mapping cheb: $\Sigma^*_N(X) \to X$ is locally Lipschitz;

ii) if the dimension of Euclidean or Lobachevsky space is greater than 1 and $N>2$ the mapping in question is not Lipschitz in any neighbourhood of the class $\Sigma_2(X) \subset \Sigma_N(X)$.

3. Statements of the main results

Here we present the detailed formulation of our results. For $N = 2$ lemma 1 gives the answer to the problem stated in the previous paragraph. This lemma is a part of lemma 2 from [8].

LEMMA 1. *Let the metric space (X, ρ) be such that a) for each $x, y \in X$, $\omega(x, y)$ is a singleton, and b) $2\omega(p, x)\omega(p, y) \le xy$, for all $x, y, p \in X$. Then, the inequalities*

$$\operatorname{cheb}(M)\operatorname{cheb}(Z) \le \alpha(M, Z) \le \operatorname{cheb}(M)\operatorname{cheb}(Z) + (D[M] + D[Z])/2$$

hold true for any M, $Z \in \Sigma_2(X)$.

We give the answer to our question for Euclidean line $\mathbb{R}$ in lemma 2.

LEMMA 2. *Let X be the Euclidean line. Then the map* $\operatorname{cheb}\colon (\Sigma_N(X), \alpha) \to X$ *is nonexpansive.*

The global Lipschitz property of the map cheb vanishes if the dimension of the space is greater than 1 and $N > 2$.

LEMMA 3. *The following properties hold:*

i) *if the dimension of the Euclidean or Lobachevsky space X is greater than* 1 *and $N > 2$, the Chebyshev center mapping* $\operatorname{cheb}\colon U \to X$ *is not Lipschitz for any neighborhood U of the class $\Sigma_2(X) \subset \Sigma_N(X)$;*

ii) *if X is Euclidean, the Chebyshev center mapping* $\operatorname{cheb}\colon \Sigma_N(X) \to X$ *is not uniformly continuous.*

The following lemmas help us to prove local Lipschitz property of the map cheb in case $N = 3$.

LEMMA 4. *Let X be an Euclidean space. Then for any pair $M = \{u, v, w\}$ and $Z = \{u, v, z\} \in \Sigma_3^*$ such that $w \in (u, z)$ we get inequality*

$$\operatorname{cheb}(M)\operatorname{cheb}(Z) \le L\alpha(M, Z).$$

Here $L = 1/(2\sin(\varphi))$ if $\varphi = \angle vuw \ne 0$ and $\angle uvw$ are acute angles; and $L = 1/2$ in other cases.

LEMMA 5. *Let X be an Euclidean space, $M = \{u, v, w\} \in \Sigma_3^*$ and $\varepsilon = \min[ab : a \ne b, a\,b \in M]/8$. Then there exists a constant $L > 0$ such that for all z_1, $z_2 \in B(w, \varepsilon)$ the following inequality holds true*

$$\operatorname{cheb}(W_1)\operatorname{cheb}(W_2) \le L\alpha(W_1, W_2).$$

Here $W_1 = \{u, v, z_1\}$, $W_2 = \{u, v, z_2\}$.

Now we are ready to prove a local Lipschitz property of the map cheb for exact 3-nets.

Corollary 1. *Let X be an Euclidean space, $M \in \Sigma_3^*$ and $\varepsilon = \min[ab : a \neq b, a\, b \in M]/8$. Then* $\mathrm{cheb}\colon (B_\alpha(M,\varepsilon),\alpha) \to X$ *is a Lipschitz map.*

Theorem 1 is an extension of this result to the case $N > 3$ in Euclidean plane.

Theorem 1. *Let X be the Euclidean plane, $N > 3$, $M \in \Sigma_N^*$ and $\varepsilon = \min[ab : a \neq b, a\, b \in M]/8$. Then* $\mathrm{cheb}\colon (B_\alpha(M,\varepsilon),\alpha) \to X$ *is a Lipschitz map.*

Then we turn to the case of Euclidean space of dimension greater than 2.

Lemma 6. *Assume that X is an Euclidean space of dimension greater than 1 and a N-net $M = \{x_1, \dots, x_N\} \in \Sigma_N^*$ defines an $(N-1)$-dimensional simplex. Then there exists a constant $L > 0$ such that for each N-net $Z = \{x_1, \dots, x_{N-1}, y_N\}$ meeting the condition $x_N \in (x_1, y_N)$ the inequality*

$$\mathrm{cheb}(M)\,\mathrm{cheb}(Z) \leq L\alpha(M,W)$$

holds true.

Lemma 7. *Let X be an Euclidean space of dimension greater than 1, $M = \{x_1, \dots, x_N\} \in \Sigma_N^*(X)$ be an $(N-1)$-dimensional simplex and $W = M \bigcup \{x_{N+1}\} \in \Sigma_{N+1}^*(X)$ be such that $x_{N+1} \in \Pi(x_1, \dots, x_N) \backslash M$ and*

$$\varepsilon = \min[\min[ab : a \neq b, a,\, b \in \{x_1, \dots, x_{N-1}\}], x_N\Pi(x_1, \dots, x_{N-1})]/8.$$

Then

(i) there exists a constant $L > 0$ such that for all z_1, $z_2 \in B(x_N, \varepsilon)$ the inequality

$$\mathrm{cheb}(Z_1)\,\mathrm{cheb}(Z_2) \leq L\alpha(Z_1, Z_2)$$

holds true, where $Z_1 = \{x_1, \dots, x_{N-1}, z_1\}$, $Z_2 = \{x_1, \dots, x_{N-1}, z_2\}$.

(ii) there exist constants $\delta > 0$ and $L > 0$ such that for all y_1, $y_2 \in B(x_{N+1}, \delta)$ the inequality

$$\mathrm{cheb}(Y_1)\,\mathrm{cheb}(Y_2) \leq L\alpha(Y_1, Y_2),$$

holds, where $Y_1 = \{x_1, \dots, x_N, y_1\}$, $Y_2 = \{x_1, \dots, x_N, y_2\}$.

Theorem 2. *Let (X, ρ) be an Euclidean space and $N \geq 2$ be arbitrary fixed. Then, for each $M \in \Sigma_N^*(X)$ there exists $\varepsilon = \varepsilon(M) > 0$ such that* $\mathrm{cheb}\colon B_\alpha(M,\varepsilon) \to X$ *is Lipschitz.*

It turns out that Lipschitz property of the map cheb holds for two N-nets which lying sufficiently far from each other.

Proposition 1. *Let X be the Euclidean plane and $N > 2$. Then for any M, $Z \in \Sigma_N(X)$ such that $B(\mathrm{cheb}(M), R(M)) \cap B(\mathrm{cheb}(Z), R(Z)) = \emptyset$ the inequality*

$$\mathrm{cheb}(M)\,\mathrm{cheb}(Z) \leq L\alpha(M,Z)$$

holds true, where $L = (1+\sqrt{5})/2$ for $N > 3$ and $L = \sqrt{2}$ for $N = 3$.

Proposition 2. (i) *Consider an Euclidean space* X *and a* 3*-net* $M = \{u, v, w\}$. *Let the* 3*-net* $Z = \{u, v, z\} \in \Sigma_3^*(X)$ *be such that:*

1. $co(\{u, v, w\}) \cap co(\{u, v, z\}) = [u, v]$;

2. *if angles* $\angle(uwv)$, $\angle(uzv)$ *are acute then* $\alpha(M, Z) < wz$.

Then the inequality

$$\mathrm{cheb}(M)\,\mathrm{cheb}(Z) \le \alpha(M, Z)$$

holds true.

(ii) *Let* X *be the Euclidean plane. Then the inequality*

$$\mathrm{cheb}(M)\,\mathrm{cheb}(Z) \le 2\alpha(M, Z)$$

holds for all $M = \{u, v, w\}$ *and* $Z = \{u, q, z\} \in \Sigma_3(X)\backslash\Sigma_2(X)$ *such that*

$$co(\{u, v, w\}) \cap co(\{u, v, z\}) = \{u\}.$$

4. Proofs of the results

Proof of Lemma 2. Consider two arbitrary N-nets $M = \{x_1, \dots, x_N\}$, $Z = \{y_1, \dots, y_N\}$. Let us assume that $x_1 \le \dots \le x_N$, $y_1 \le \dots \le y_N$ and $x_1 \le y_1$. Then Lemma 1 together with the definitions of the Hausdorff metric and of the Chebyshev center gives us the inequality

$$\begin{aligned}\mathrm{cheb}(M)\,\mathrm{cheb}(Z) &= \mathrm{cheb}(\{x_1, x_N\})\,\mathrm{cheb}(\{y_1, y_N\})\\ &\le \alpha(\{x_1, x_N\}, \{y_1, y_N\}) \le \alpha(M, Z). \quad \blacksquare\end{aligned}$$

Proof of Lemma 3. (i) It suffices to verify the statement in case $N = 3$. Let x, y be two different points in Euclidean plane (Lobachevsky plane) and S_+ be the semicircle constructed on the interval $[x, y]$ as diameter of the circle. Consider a non-fixed point $z \in S_+$ such that $yz \in (0, xy/2)$. Now find a point u meeting the following conditions: a) $z \in [x, u]$; b) the point y lies on a circle based on $[x, u]$ as diameter. Consider 3-nets $M = \{x, y, z\}$, $W = \{x, y, u\} \in \Sigma_3^*$ in Euclidean space (or in Lobachevsky space). Then by construction $v = \mathrm{cheb}(M) = \omega(x, y)$, $w = \mathrm{cheb}(W) = \omega(x, u)$ and $uz = \alpha(M, W)$. In Euclidean plane for any constant $L > 0$ we can choose a point $z \in S_+$ so that $2Lyz < xy$. Then $\mathrm{cheb}(M)\,\mathrm{cheb}(W) = vw = (xy)(uz)/(2zy) > Luz = L\alpha(M, W)$. This proves lemma 3 for Euclidean space.

Now let p be the base of a perpendicular to the interval $[x, w]$ passing through point v and ψ be the angle with vertex w of the triangle $\triangle xvw$ in Lobachevsky plane. The triangles $\triangle xvw$, $\triangle wpv$ are right-angled by construction. Hence, we get formulae (cf. (4a), (4b) from [8], p. 58) $\tanh(pv) = \sinh(pw)\tan(\psi)$, $\tanh(vx) = \sinh(vw)\tan(\psi)$. Then the fraction $\sinh(vw)/\sinh(pw) = \tanh(vx)/\tanh(pv)$ tends to ∞ as $(pv \to 0)$. But then $\mathrm{cheb}(M)\,\mathrm{cheb}(W)/\alpha(M, W) = (vw)/(2pw)$ also infinitely increases as $(pv \to 0)$. This completes the proof for Lobachevsky space.

(ii) Let us use the notation of the previous part of the proof. At the same time let us introduce new points $x_n = y+n(x-y)$, $z_n = \Lambda(u, x_n) \cap S(\omega(x_n, y), x_n y/2)$ and triples $M_n = \{x_n, y, z_n\}$, $Z_n = \{x_n, y, u\}$ where we identify points with their radius-vectors and $n = 1, 2, \ldots$. Then $\alpha(M_n, Z_n) \to 0$ but $\operatorname{cheb}(M_n)\operatorname{cheb}(Z_n) \to yu$ as $(n \to \infty)$. This completes the proof of lemma 3. ■

Proof of Lemma 4. Let φ be a nonzero acute angle. Then

$$\operatorname{cheb}(M)\operatorname{cheb}(W) \leq wz/2 = \alpha(M, W)/2.$$

Let $\varphi \neq 0$ be acute angle, p – base of the perpendicular to the ray $\Lambda(u, w)$ passing through v, q – point on the ray $\Lambda(u, w)$ such that the interval $[q, v]$ is perpendicular to the interval $[u, v]$. Then the set of points of the ray $\Lambda(u, w)$ can be represented as follows: $\Lambda(u, w) = (u, p] \cup (p, q) \cup (\Lambda(u, w)\backslash(u, q))$. Let us consider position of $\operatorname{cheb}(M)$ depending on w. If $w \in (u, p]$ then $c = \operatorname{cheb}(M) = \omega(u, v)$. If $w \in (p, q)$ then $\operatorname{cheb}(M) \in (c, b)$, where the point $b \in (u, q]$ is such that the interval $[b, c]$ is perpendicular to the ray $\Lambda(u, v)$ and $\operatorname{cheb}(M) = pw/(2\sin(\varphi))$. If $w \in \Lambda(u, w)\backslash(u, q)$ then $\operatorname{cheb}(M) = \omega(u, w)$. Let us investigate all possibilities for location of the points w, z on the ray $\Lambda(u, w)$.

If either $w, z \in (u, p]$ or $w, z \in \Lambda(u, w)\backslash(u, q)$ then $\operatorname{cheb}(M)\operatorname{cheb}(W) = 0$.

In case $w \in (u, p)$, $z \in (p, q)$ we have $\operatorname{cheb}(M)\operatorname{cheb}(W) = pz/(2\sin(\varphi)) = \alpha(M, W)/(2\sin(\varphi))$.

If $w \in (u, p)$, $z \in \Lambda(u, w)\backslash(u, q)$ then $\operatorname{cheb}(M)\operatorname{cheb}(W) = c\operatorname{cheb}(W) \leq cb + b\operatorname{cheb}(W) = pq/(2\sin(\varphi)) + qz/2 \leq pz/(2\sin(\varphi)) \leq \alpha(M, W)/(2\sin(\varphi))$.

If $w, z \in (p, q)$ then $\operatorname{cheb}(M)\operatorname{cheb}(W) = (pz - pw)/(2\sin(\varphi)) \leq$
$\leq \alpha(M, W)/(2\sin(\varphi))$.

If $w \in (p, q)$, $z \in \Lambda(u, w)\backslash(u, q)$ then $\operatorname{cheb}(M)\operatorname{cheb}(W) \leq \operatorname{cheb}(M)b + b\operatorname{cheb}(W) = (pq - pw)/(2\sin(\varphi)) + qz/2 = \alpha(M, W)/(2\sin(\varphi))$.

Thus in all considered cases we can fix $1/(2\sin(\varphi))$ as Lipschitz constant L. This completes the proof of the lemma. ■

Proof of Lemma 5. Let triangles defined by 3-nets W_1, W_2 be not acute. Then with the help of lemma 1 we get the desired inequality with constant $L = 1/2$. Now let X be a Euclidean plane and 3-net W_1 define an acute triangle. If the angle $\angle uz_2v$ is blunt then there exists a point $z_3 \in (z_1, z_2)$ such that the angle $\angle uz_3v$ is right and $\operatorname{cheb}(\{u, v, z\}) = \operatorname{cheb}(W_2)$. Thus without loss of generality one can assume that the triangle given by the 3-net W_2 is not blunt. Let us consider all possible cases of the location of the point $z_2 \in B(w, \varepsilon)$.

1. Let a point $z_2 \neq z_1$ belong to the set $co\{u, v, z_1\}$ or to the closed angle with vertex z_1 and sides on the rays $\Lambda(v, z_1)$, $\Lambda(u, z_1)$. If $z_2 \in \Lambda(u, z_1) \cup \Lambda(v, z_1)$ then we get the desired inequality from Lemma 4. In the other case let us denote by p a point of intersection of the rays $\Lambda(v, z_2)$, $\Lambda(u, z_1)$ and $W_3 = \{u, v, p\}$. Note that the angle $\angle z_1pz_2$ is not acute. Now we use Lemma 4 and triangle inequality. Thus we get constants $L_1 = \sup\{1/(2\sin(\angle z_1uv)) : z_1 \in B(w, \varepsilon)\} < \infty$ and $L_2 =$

$\sup\{1/(2\sin(\angle z_2vu)) : z_2 \in B(w,\varepsilon)\} < \infty$ such that

$$\begin{aligned}\operatorname{cheb}(W_1)\operatorname{cheb}(W_2) &\le \operatorname{cheb}(W_1)\operatorname{cheb}(W_3) + \operatorname{cheb}(W_3)\operatorname{cheb}(W_2)\\ &\le L_1\alpha(W_1,W_3) + L_2\alpha(W_3,W_2) \le (L_1+L_2)\alpha(W_1,W_2).\end{aligned}$$

So, in this case $L = L_1 + L_2$.

2. Now let a point $z_2 \ne z_1$ belong to the open angle with vertex z_1 and sides on the rays $\Lambda(z_1,u)$ and $\Lambda(v,z_1)$ (the case in which $z_2 \ne z_1$ belongs to the open acute angle with vertex z_1 and sides on the rays $\Lambda(z_1,v)$ and $\Lambda(u,z_1)$ is similar to this one). Let us denote by p the intersection point of the rays $\Lambda(v,z_2)$, $\Lambda(u,z_1)$ and $W_3 = \{u,v,p\}$. If the angle $\angle z_1pz_2$ is not acute we can use the same considerations as in the first case. If the angle $\angle z_1pz_2$ is acute then we have inequalities $\alpha(W_1,W_3) = pz_1 = pv\sin(\angle pvz_1)/\sin(\angle uz_1v) \le vz_1\sin(\angle pvz_1)/\sin(\angle uz_1v) \le z_1z_2/\sin(\angle uz_1v) \le L_3\alpha(W_1,W_2)$, here $L_3 = \sup\{1/(\sin(\angle uz_1v)) : z_1 \in B(w,\varepsilon)\} < \infty$ by Lemma 5. Now we again use this inequality, triangle inequality and Lemma 4. Thus we get

$$\begin{aligned}\operatorname{cheb}(W_1)\operatorname{cheb}(W_2) &\le \operatorname{cheb}(W_1)\operatorname{cheb}(W_3) + \operatorname{cheb}(W_3)\operatorname{cheb}(W_2)\\ &\le L_1\alpha(W_1,W_3) + L_2\alpha(W_3,W_2)\\ &\le L_1\alpha(W_1,W_3) + L_2(\alpha(W_1,W_3)+\alpha(W_1,W_2)) \le L\alpha(W_1,W_2).\end{aligned}$$

Here $L = L_3(L_1+L_2)+L_2$, $L_1 = \sup\{1/(2\sin(\angle z_1uv)) : z_1 \in B(w,\varepsilon)\} < \infty$, $L_2 = \sup\{1/(2\sin(\angle z_2vu)) : z_2 \in B(w,\varepsilon)\} < \infty$.

Now let X be Euclidean space of dimension greater than 2 and W_4 be an orthogonal projection of W_2 on the plane containing W_1; note that if W_1 lies on the line then $W_4 = W_2$. Then the previous considerations provide us with the constant $L_4 > 0$ such that for all z_1, $z_2 \in B(w,\varepsilon)$ the inequality

$$\operatorname{cheb}(W_1)\operatorname{cheb}(W_4) \le L_4\alpha(W_1,W_4) \le L_4\alpha(W_1,W_2)$$

holds true. Note that the inequality

$$\operatorname{cheb}(W_4)\operatorname{cheb}(W_2) \le \alpha(W_1,W_2)$$

follows from purely geometric considerations. It suffices now to apply triangle inequality to complete the proof. ■

Proof of Corollary 1. Fix an arbitrary pair of nets $W_1 = \{x,y,z\}$, $W_2 = \{u,v,w\} \in B_\alpha(M,\varepsilon)$ and put $W_3 = \{x,y,w\}$, $W_4 = \{x,v,w\}$. Then Lemma 5, definition of the Hausdorff metric and triangle inequality provide us with the constants $L_1, L_2, L_3 > 0$ such that

$$\begin{aligned}\operatorname{cheb}(W_1)\operatorname{cheb}(W_2) &\le \operatorname{cheb}(W_1)\operatorname{cheb}(W_3) + \operatorname{cheb}(W_3)\operatorname{cheb}(W_4)+\\ &+ \operatorname{cheb}(W_4)\operatorname{cheb}(W_2) \le L_1\alpha(W_1,W_3) + L_2\alpha(W_3,W_4) + L_3\alpha(W_4,W_2)\\ &\le (L_1+L_2+L_3)\alpha(W_1,W_2).\end{aligned}$$

Thus $\operatorname{cheb}\colon (B_\alpha(M,\varepsilon),\alpha) \to X$ is a Lipschitz map. This completes the proof. ■

Proof of Theorem 1. (i) Let us consider two arbitrary N-nets of the special kind $Z_1 = \{x_1, x_2, \dots, x_N\}$, $Z_2 = \{y_1, x_2, \dots, x_N\} \in B_\alpha(M, \varepsilon)$ and introduce the parametrisation $x = x(s)$ by the length of the interval $[x_1, y_1]$ so that $x_1 = x(0)$, $y_1 = x(x_1y_1)$. If $z \in (x_1, y_1] \cap B[\operatorname{cheb}(Z_1), R(Z_1)]$ then $\operatorname{cheb}(Z_1) = \operatorname{cheb}(\{z, x_2, \dots, x_N\}$. Hence we may assume that $x_1 \in S(\operatorname{cheb}(Z_1), R(Z_1))$. Let us put in correspondence to any $s \in [0, x_1y_1]$ an N-net $Z(s) = \{x(s), x_2, \dots, x_N\}$ and a convex polygon $Q(s)$, given by vertices $Z(s) \cap S(\operatorname{cheb}(M(s)), R(M(s)))$. For any $s \in [x_1y_1]$ divide the polygon $Q(s)$ into triangles $\{\triangle_1(s), \dots, \triangle_{k(s)}(s)\}$ with common vertex $x(s)$ by connecting this vertex by intervals with all other vertices of $Q(s)$. Now for each $s \in [x_1y_1]$ $\operatorname{cheb}(Q(s)) = \operatorname{cheb}(M(s))$ belongs either to the interior of one of the acute triangles or to the side common to two adjacent triangles (it is possible that these triangles are one and the same or even they are just points on an interval) from the division $\{\triangle_1(s), \dots, \triangle_{k(s)}(s)\}$. Consider two cases.

1. Let $\operatorname{cheb}(Z_1)$ belong to the interior of the triangle $\triangle x_1ab \in \{\triangle_1(0), \dots, \triangle_{k(0)}(0)\}$. Then by continuity of the mapping cheb one can find a minimal number $s_1 \in (0, x_1y_1]$ such that either $\operatorname{cheb}(Z_2) = \operatorname{cheb}(\{y_1, a, b\})$ in case $s_1 = x_1y_1$, or $\operatorname{cheb}(Z(s_1)) = \operatorname{cheb}(\{x(s_1), a, b\})$ lies on the side of one or two of triangles of the division $\{\triangle_1(s_1), \dots, \triangle_{k(s_1)}(s_1)\}$ in case $s_1 \in (0, x_1y_1)$.

2. Let $\operatorname{cheb}(Z_1)$ belong to the common side of the triangles $\triangle x_1ab$, $\triangle x_1ac \in \{\triangle_1(0), \dots, \triangle_{k(0)}(0)\}$. Then continuity of the map cheb implies the existence of a minimal $s_1 \in (0, x_1y_1]$ such that either $\operatorname{cheb}(Z_2) = \operatorname{cheb}(\{y_1, a, b\})$ or $\operatorname{cheb}(Z_2) = \operatorname{cheb}(\{y_1, a, c\})$ if $s_1 = x_1y_1$, or $\operatorname{cheb}(Z(s_1)) = \operatorname{cheb}(\{x(s_1), a, b\})$ or $\operatorname{cheb}(Z(s_1)) = \operatorname{cheb}(\{x(s_1), a, c\})$ lies on the side of one or two of triangles of the division $\{\triangle_1(s_1), \dots, \triangle_{k(s_1)}(s_1)\}$ in case $s_1 \in (0, x_1y_1)$.

In either of the considered cases Lemma 5 and compactness of the interval $[x_1, y_1]$ imply the existence of the constant $L_1 > 0$ such that $\operatorname{cheb}(Z_1)\operatorname{cheb}(Z(s_1)) \le L_1\alpha(Z_1, Z(s_1))$. If $s_1 \ne x_1y_1$ then for $\operatorname{cheb}(Z(s_1))$ we can proceed in similar way and get s_2. Continuing this process we get from Lemma 5 constants $0 < s_1 < \dots < s_m = x_1y_1$, $L_1, \dots, L_m > 0$ such that

$$\begin{aligned}\operatorname{cheb}(Z_1)\operatorname{cheb}(Z_2) &\le \operatorname{cheb}(Z_1)\operatorname{cheb}(Z(s_1)) + \operatorname{cheb}(Z(s_1))\operatorname{cheb}(Z(s_2)) + \dots + \\ &+ \operatorname{cheb}(Z(s_{m-1}))\operatorname{cheb}(Z_2) \le L_1\alpha(Z_1, Z(s_1)) + \dots + \\ &+ L_m\alpha(Z(s_{m-1}), Z_2) \le (L_1 + \dots + L_m)\alpha(Z_1, Z_2).\end{aligned}$$

This completes the proof of the Theorem 1 in this special case.

(ii) Choose arbitrary nets $Z_1 = \{x_1, x_2, \dots, x_N\}$, $Z_2 = \{y_1, y_2, \dots, y_N\} \in B_\alpha(M, \varepsilon)$ and fix $Z_3 = \{y_1, x_2, \dots, x_N\}, \dots, Z_{N+1} = \{y_1, y_2, \dots, y_{N-1}, x_N\}$. Then triangle inequality, Lemma 5 and the definition of the Hausdorff metric give us constants $L_1 \dots, L_{N-1} > 0$ such that

$$\begin{aligned}\operatorname{cheb}(Z_1)\operatorname{cheb}(Z_2) &\le \operatorname{cheb}(Z_1)\operatorname{cheb}(Z_3) + \operatorname{cheb}(Z_3)\operatorname{cheb}(Z_4) + \dots + \\ &+ \operatorname{cheb}(Z_{N+1})\operatorname{cheb}(Z_2) \le L_1\alpha(Z_1, Z_3) + L_2\alpha(Z_3, Z_4) + \dots + \\ &+ L_{N-1}\alpha(Z_{N+1}, Z_2) \le (L_1 + \dots + L_{N-1})\alpha(Z_1, Z_2).\end{aligned}$$

Thus $\operatorname{cheb}\colon (B_\alpha(M, \varepsilon), \alpha) \to X$ is a Lipschitz map. ■

Proof of Lemma 6. The proof is by induction. Lemma 4 implies consistence of the statement in the case $N = 3$. Assume now that Lemma 6 holds true for all numbers less or equal than $N-1$. Let us show that the statement holds true for N. Let $p \neq x_1$ be an intersection point of the ray $\Lambda(x_1, x_N)$ with the sphere $S(\operatorname{cheb}(\{x_1, \ldots, x_{N-1}\}), R(\{x_1, \ldots, x_{N-1}\}))$. and q be a point on $\Lambda(x_1, x_N)$ such that there exists $k \in \{2, \ldots, N\}$ such that

$$x_k \in S(\operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, q\}), R(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, q\})),$$

(here cap on the symbol means that this symbol must be excluded) closest to x_1. Then the set of points of the ray $\Lambda(x_1, x_N)$ can be represented as follows: $\Lambda(x_1, x_N) = (x_1, p] \cup (p, q) \cup (\Lambda(x_1, x_N) \backslash (x_1, q))$. Consider all possible locations of the points x_N, y_N on $\Lambda(x_1, x_N)$.

If $x_N, y_N \in (x_1, p]$ then $\operatorname{cheb}(M) \operatorname{cheb}(Z) = 0$. If $x_N, y_N \in (\Lambda(x_1, x_N) \backslash (x_1, q))$ then there exist $k, j \in \{2, \ldots, N\}$ such that $\operatorname{cheb}(M) = \operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_N\})$, $\operatorname{cheb}(Z) = \operatorname{cheb}(\{x_1, \ldots, \hat{x}_j, \ldots, x_{N-1}, y_N\})$. Moreover by induction assumption there exists a constant $L > 0$ such that

$$\begin{aligned} \operatorname{cheb}(M) \operatorname{cheb}(Z) &\leq L\alpha(\{x_1, \ldots, \hat{x}_k, \ldots, x_N\}, \{x_1, \ldots, \hat{x}_j, \ldots, x_{N-1}, y_N\}) \\ &\leq L\alpha(M, Z). \end{aligned}$$

If $x_N \in (x_1, q)$, $y_N \in (p, q)$ then $\operatorname{cheb}(M) \operatorname{cheb}(Z) \leq \alpha(M, Z)/(2\cos(\psi))$, here ψ is an angle between the ray $\Lambda(x_1, x_N)$ and the normal to the scale $(x_1, \ldots, x_{N-1})$ of the simplex.

Let b be an intersection-point of the boundary of the simplex $(x_1, \ldots, x_N)$ with normal to the scale $(x_1, \ldots, x_{N-1})$ passing through the point $\operatorname{cheb}(\{x_1, \ldots, x_{N-1}\})$ which does not belong to this face. If $x_N \in (x_1, q)$, $y_N \in (\Lambda(x_1, x_N) \backslash (x_1, q))$ then there exist $k, j \in \{2, \ldots, N\}$ such that $b = \operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, q\})$, $\operatorname{cheb}(Z) = \operatorname{cheb}(\{x_1, \ldots, \hat{x}_j, \ldots, x_{N-1}, y_N\})$. Moreover by assumption there exist a constant $L_1 > 0$ such that

$$\begin{aligned} \operatorname{cheb}(M) \operatorname{cheb}(Z) &\leq \operatorname{cheb}(M) b + b \operatorname{cheb}(Z) \leq q\{p, x_N\}/(2\cos\psi) + \\ &\quad + \operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, q\}) \operatorname{cheb}(\{x_1, \ldots, \hat{x}_j, \ldots, x_{N-1}, y_N\}) \\ &\leq q\{p, x_N\}/(2\cos\psi) + L_1\alpha(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, q\}, \\ &\quad \{x_1, \ldots, \hat{x}_j, \ldots, x_{N-1}, y_N\}) \leq L\alpha(M, Z), \end{aligned}$$

where $L = (1 + L_1)/(2\cos\psi)$. This completes the proof. ∎

Proof of Lemma 7. The proof is by induction.

(i) Lemma 5 implies that statement (i) of Lemma 7 holds true for $N = 3$. Assume now that it holds for all numbers less or equal than $N-1$ and prove it for N. Consider two cases.

1. Let there exists $k \in \{1, \ldots, N-1\}$ such that $pz_1 \leq z_1 z_2$, where p is the intersection point of the ray $\Lambda(x_k, z_2)$ with the $(N-2)$-plane $\Pi(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, z_1\})$.

Let us denote by Z_3 the set $\{x_1, \ldots, x_{N-1}, p\}$. Now Lemma 6 implies the existence of a constant $L_2 > 0$ such that $\operatorname{cheb}(Z_3) \operatorname{cheb}(Z_2) \leq L_2\alpha(Z_3, Z_2)$.

By induction assumption there exists a constant $L_1 > 0$ such that

$$\operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, z_1\}) \operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, p\})$$
$$L_1\alpha(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, z_1\}, \{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, p\}) \leq L_1\alpha(Z_1, Z_3).$$

Let ψ stand for the angle between normal vector to the scale $(x_1, \ldots, x_{N-1})$ directed into the simplex and normal vector to the scale $(x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, z_1)$ directed outside of the simplex. Now purely geometrical considerations imply the inequality

$$\operatorname{cheb}(Z_1)\operatorname{cheb}(Z_3) \leq$$
$$\leq \max[1/\sin\psi, 1] \operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, z_1\}) \operatorname{cheb}(\{x_1, \ldots, \hat{x}_k, \ldots, x_{N-1}, p\}).$$

Thus

$$\operatorname{cheb}(Z_1)\operatorname{cheb}(Z_2) \leq \operatorname{cheb}(Z_1)\operatorname{cheb}(Z_3) + \operatorname{cheb}(Z_3)\operatorname{cheb}(Z_2) \leq$$
$$\leq L_1 \max[1/\sin\psi, 1]\alpha(Z_1, Z_3) + L_2\alpha(Z_3, Z_2) \leq (L_1 \max[1/\sin\psi, 1] + 2L_2)\alpha(Z_1, Z_2).$$

2. Let the assumption of case 1 be wrong, $j \in \{1, \ldots, N-1\}$ and denote by p_j the intersection point of the ray $\Lambda(x_j, z_2)$ and the $(N-2)$-plane $\Pi(x_1, \ldots, \hat{x}_j, \ldots, x_{N-1}, z_1)$. Now choose $k \in \{1, \ldots, N-1\}$ such that $p_k z_1 = \min[p_j z_1 : j \in \{1, \ldots, N-1\}]$ and consider $Z_3 = \{x_1, \ldots, x_{N-1}, p_k\}$.

Note that the angle $\angle z_1 p_k z_2$ is acute and introduce the inequality

$$\alpha(Z_1, Z_3) = p_k z_1 = p_k x_k \sin\angle p_k x_k z_1 / \sin\angle p_k z_1 x_k \leq$$
$$\leq x_k z_1 \sin\angle p_k x_k z_1 / \sin\angle p_k z_1 x_k \leq z_1 z_2 / \sin\angle p_k z_1 x_k \leq L_3\alpha(Z_1, Z_2),$$

where general assumptions imply that $L_3 = \sup\{1/\sin\angle p_k z_1 x_k : z_1 \in B(x_N, \varepsilon)\} < \infty$. This inequality, the triangle one, Lemma 6 and similar estimate for $\operatorname{cheb}(Z_1)\operatorname{cheb}(Z_3)$ of the first case put together give us constants L_1, $L_2 > 0$ such that

$$\operatorname{cheb}(Z_1)\operatorname{cheb}(Z_2) \leq \operatorname{cheb}(Z_1)\operatorname{cheb}(Z_3) + \operatorname{cheb}(Z_3)\operatorname{cheb}(Z_2) \leq$$
$$\leq L_1\alpha(Z_1, Z_3) + L_2\alpha(Z_3, Z_2)$$
$$\leq L_1\alpha(Z_1, Z_3) + L_2(\alpha(Z_1, Z_3) + \alpha(Z_1, Z_2)) \leq L\alpha(Z_1, Z_2),$$

where $L = L_3(L_1 + L_2) + L_2$. Thus statement (i) of Lemma 7 holds true.

(ii) Lemma 5 provides us with the proof of statement (ii) of Lemma 7 for $N = 2$. Let us assume that statement (ii) holds true also for all natural numbers less or equal than $N-1$ and prove it for N. Let $0 < \delta < \min[ab : a \neq b, a, b \in W]/8$. Consider three cases.

1. Let $[y_1, y_2] \cap \Pi(x_1, \ldots, x_N) = \emptyset$. Then our statement holds by compactness of the interval $[y_1, y_2]$ and previously proved statement (i) of Lemma 7.

2. Let $[y_1, y_2] \subset \Pi(x_1, \ldots, x_N)$. Introduce the natural parametrisation $y = y(s)$ of the interval $[y_1, y_2]$ by length so that $y_1 = y(0)$, $y_2 = y(y_1 y_2)$. If $y \in (y_1, y_2] \cap B[\operatorname{cheb}(Y_1), R(Y_1)]$ then $\operatorname{cheb}(Y_1) = \operatorname{cheb}(\{x_1, x_2, \ldots, x_N, y\})$.

Hence we can assume that $y_1 \in S(\mathrm{cheb}(Y_1), R(Y_1))$. Consider for any $s \in [0, y_1y_2]$ $(N+1)$-net $Y(s) = \{x_1, x_2, \dots, x_N, y(s)\}$ and convex polygon $Q(s)$ with vertices $Y(s) \cap S(\mathrm{cheb}(Y(s)), R(Y(s)))$. Now we divide polygon $Q(s)$ for each $s \in [x_1, y_1]$ into simplices $\{\triangle_1(s), \triangle_{k(s)}(s)\}$ with the common vertex $y(s)$, here $1 \le k(s) \le 2$. For any $s \in [y_1, y_2]$ $\mathrm{cheb}(Q(s)) = \mathrm{cheb}(Y(s))$ belongs either to the interior of one simplex or to the common face of two simplices of the division $\{\triangle_1(s), \triangle_{k(s)}(s)\}$ (the degenerate cases are also possible). Consider two cases.

A. Let $\mathrm{cheb}(Y_1)$ belong to the interior of the simplex $(a_1, \dots, a_{N-1}, y_1) \in \{\triangle_1(0), \triangle_{k(0)}(0)\}$.

Then—since the mapping cheb is continuous—one can find a minimal $s_1 \in (0, y_1y_2]$ such that either $\mathrm{cheb}(Y_2) = \mathrm{cheb}(\{a_1, a_2, \dots, a_{N-1}, y_2\})$ for $s_1 = y_1y_2$ or $\mathrm{cheb}(Y(s_1)) = \mathrm{cheb}(\{y(s_1), x_2, \dots, x_N\})$ belongs to the scale of one or two simplices of the division $\{\triangle_1(s_1), \triangle_{k(s_1)}(s_1)\}$ for $s_1 \in (0, y_1y_2)$.

B. Let $\mathrm{cheb}(Y_1)$ belong to the common face of the simplices $(a_1, \dots, a_{N-2}, b, y_1)$, $(a_1, \dots, a_{N-2}, c, y_1) \in \{\triangle_1(0), \triangle_{k(0)}(0)\}$.

Then again continuity of the mapping cheb implies the existence of the minimal $s_1 \in (0, y_1y_2]$, such that either $\mathrm{cheb}(Y_2) = \mathrm{cheb}(\{a_1, \dots, a_{N-2}, b, y_2\})$ or $\mathrm{cheb}(Y_2) = \mathrm{cheb}(\{a_1, \dots, a_{N-2}, c, y_2\})$ for $s_1 = y_1y_2$ or $\mathrm{cheb}(Y(s_1)) =$
$= \mathrm{cheb}(\{a_1, \dots, a_{N-2}, b, y(s_1)\})$ or $\mathrm{cheb}(Y(s_1)) = \mathrm{cheb}(\{a_1, \dots, a_{N-2}, c, y(s_1)\})$ belongs to the scale of one or two simplices of the division $\{\triangle_1(s_1), \triangle_{k(s_1)}(s_1)\}$ for $s_1 \in (0, y_1y_2)$.

In either case compactness of the interval $[y_1, y_2]$ and statement (i) of Lemma 7 or induction hypothesis provide us with the constant $L_1 > 0$ such that $\mathrm{cheb}(Y_1)\,\mathrm{cheb}(Y(s_1)) \le L_1\alpha(Y_1, Y(s_1))$. If $s_1 \ne y_1y_2$ then similar considerations applied to $\mathrm{cheb}(Y(s_1))$ give us s_2. Continuing this process we get the set of constants $0 < s_1 < \dots < s_i = y_1y_2$, $L_1, \dots, L_i > 0$ such that

$$\begin{aligned}\mathrm{cheb}(Y_1)\,\mathrm{cheb}(Y_2) &\le \mathrm{cheb}(Y_1)\,\mathrm{cheb}(Y(s_1)) + \mathrm{cheb}(Y(s_1))\,\mathrm{cheb}(Y(s_2)) + \dots + \\ &+ \mathrm{cheb}(Y(s_{i-1}))\,\mathrm{cheb}(Y_2) \le L_1\alpha(Y_1, Y(s_1)) + \dots + L_i\alpha(Y(s_{i-1}), Y_2) \\ &\le (L_1 + \dots + L_i)\alpha(Y_1, Y_2).\end{aligned}$$

Thus our statement holds true in this special case.

3. There exists a unique point u such that $u = [y_1, y_2] \cap \Pi(x_1, \dots, x_N)$.

Since triangle inequality holds true it suffices to prove the statement for the interval $[y_1, u]$. Let $\tilde{y}_1$ denote orthogonal projection of y_1 onto $\Pi(x_1, \dots, x_N)$. Then geometrical considerations give us the inequality

$$\mathrm{cheb}(Y_1)\,\mathrm{cheb}(\{x_1, x_2, \dots, x_N, \tilde{y}_1\}) \le \alpha(Y_1, \{x_1, x_2, \dots, x_N, \tilde{y}_1\}).$$

Now case 2 of the proof implies existence of the constant $L > 0$ such that

$$\begin{aligned}\mathrm{cheb}(Y_1)\,\mathrm{cheb}(Y_2) &\le \mathrm{cheb}(Y_1)\,\mathrm{cheb}(\{x_1, x_2, \dots, x_N, \tilde{y}_1\}) + \\ &+ \mathrm{cheb}(\{x_1, x_2, \dots, x_N, \tilde{y}_1\})\,\mathrm{cheb}(\{x_1, x_2, \dots, x_N, u\}) \\ &\le \alpha(Y_1, \{x_1, x_2, \dots, x_N, \tilde{y}_1\}) + L\alpha(\{x_1, x_2, \dots, x_N, \tilde{y}_1\}, \{x_1, x_2, \dots, x_N, u\}) \\ &\le (1 + L)\alpha(Y_1, \{x_1, x_2, \dots, x_N, u\}).\end{aligned}$$

This completes the proof of the Lemma. ■

Proof of Theorem 2. (*i*) Let us consider two arbitrary N-nets of the special kind: $Z_1 = \{x_1, x_2, \dots, x_N\}$, $Z_2 = \{y_1, x_2, \dots, x_N\} \in B_\alpha(M, \varepsilon)$ and introduce a parametrisation $x = x(s)$ by the length of the interval $[x_1, y_1]$ such that $x_1 = x(0)$, $y_1 = x(x_1y_1)$. If $z \in (x_1, y_1] \cap B[\mathrm{cheb}(Z_1), R(Z_1)]$ then $\mathrm{cheb}(Z_1) = \mathrm{cheb}(\{z, x_2, \dots, x_N\}$. Hence, we can assume that $x_1 \in S(\mathrm{cheb}(Z_1), R(Z_1))$. Let us put in correspondence the N-net $Z(s) = \{x(s), x_2, \dots, x_N\}$ and the convex polygon $Q(s)$ defined by its vertices $Z(s) \cap S(\mathrm{cheb}(Z(s)), R(Z(s)))$ to any $s \in [0, x_1y_1]$. Now for any $s \in [x_1y_1]$ divide the polygon $Q(s)$ into simplices $\{\triangle_1(s), \dots, \triangle_{k(s)}(s)\}$ with the common vertex $x(s)$. Now for any $s \in [x_1y_1]$ $\mathrm{cheb}(Q(s)) = \mathrm{cheb}(Z(s))$ either belongs to the interior of one simplex or to the common plane of the two of that of the division $\{\triangle_1(s), \dots, \triangle_{k(s)}(s)\}$ (the degenerate cases are also possible). Consider two cases.

1. Let $\mathrm{cheb}(Z_1)$ belong to the interior of the simplex

$$(x_1, a_1, \dots, a_{N-1}) \in \{\triangle_1(0), \dots, \triangle_{k(0)}(0)\}.$$

Then—since the Chebyshev center depends on the set continuously—one can find a minimal $s_1 \in (0, x_1y_1]$ such that either $\mathrm{cheb}(Z_2) = \mathrm{cheb}(\{y_1, a_1, \dots, a_{N-1}\})$ for $s_1 = x_1y_1$, or $\mathrm{cheb}(Z(s_1)) = \mathrm{cheb}(\{x(s_1), a_1, \dots, a_{N-1}\})$ belongs to the common face of one or two simplices of the division $\{\triangle_1(s_1), \dots, \triangle_{k(s_1)}(s_1)\}$ if $s_1 \in (0, x_1y_1)$.

2. Let $\mathrm{cheb}(Z_1)$ belong to the face of the simplex $(x_1, a_1, \dots, a_{N-2}, b)$,

$$(x_1, a_1, \dots, a_{N-2}, c) \in \{\triangle_1(0), \dots, \triangle_{k(0)}(0)\}.$$

Then again as in the first case there exists a minimal $s_1 \in (0, x_1y_1]$ such that either $\mathrm{cheb}(Z_2) = \mathrm{cheb}(\{y_1, a_1, \dots, a_{N-2}, b\})$ or $\mathrm{cheb}(Z_2) = \mathrm{cheb}(\{y_1, a_1, \dots, a_{N-2}, c\})$ if $s_1 = x_1y_1$, or $\mathrm{cheb}(Z(s_1)) = \mathrm{cheb}(\{x(s_1), a_1, \dots, a_{N-2}, b\})$ or $\mathrm{cheb}(Z(s_1)) = \mathrm{cheb}(\{x(s_1), a_1, \dots, a_{N-2}, c\})$ lies on the face of one or two simplices of the division $\{\triangle_1(s_1), \dots, \triangle_{k(s_1)}(s_1)\}$ if $s_1 \in (0, x_1y_1)$.

In either case Lemma 7 and compactness of the interval $[x_1, y_1]$ imply existence of the constant $L_1 > 0$ such that $\mathrm{cheb}(Z_1)\,\mathrm{cheb}(Z(s_1)) \le L_1\alpha(Z_1, Z(s_1))$. If $s_1 \ne x_1y_1$ then we consider a similar construction for $\mathrm{cheb}(Z(s_1))$ and get the number s_2. Continuing the process we get the constants $0 < s_1 < \dots < s_i = x_1y_1$, $L_1, \dots, L_i > 0$ such that

$$\begin{aligned}\mathrm{cheb}(Z_1)\,\mathrm{cheb}(Z_2) &\le \mathrm{cheb}(Z_1)\,\mathrm{cheb}(Z(s_1)) + \mathrm{cheb}(Z(s_1))\,\mathrm{cheb}(Z(s_2)) + \dots + \\ &+ \mathrm{cheb}(Z(s_{i-1}))\,\mathrm{cheb}(Z_2) \le L_1\alpha(Z_1, Z(s_1)) + \dots + L_i\alpha(Z(s_{i-1}), Z_2) \le \\ &\le (L_1 + \dots + L_i)\alpha(Z_1, Z_2).\end{aligned}$$

This completes the proof of the statement in this special case.

(*ii*) Consider arbitrary $Z_1 = \{x_1, x_2, \dots, x_N\}$, $Z_2 = \{y_1, y_2, \dots, y_N\} \in B_\alpha(M, \varepsilon)$ and put $Z_3 = \{y_1, x_2, \dots, x_N\}, \dots, Z_{N+1} = \{y_1, y_2, \dots, y_{N-1}, x_N\}$. Again by Lemma 6, definition of the Hausdorff metric and triangle inequality together imply existence of the constants $L_1 \dots, L_{N-1} > 0$ such that

$$\begin{aligned}\mathrm{cheb}(Z_1)\,\mathrm{cheb}(Z_2) &\le \mathrm{cheb}(Z_1)\,\mathrm{cheb}(Z_3) + \mathrm{cheb}(Z_3)\,\mathrm{cheb}(Z_4) + \dots + \\ &+ \mathrm{cheb}(Z_{N+1})\,\mathrm{cheb}(Z_2) \le L_1\alpha(Z_1, Z_3) + L_2\alpha(Z_3, Z_4) + \dots + \\ &+ L_{N-1}\alpha(Z_{N+1}, Z_2) \le (L_1 + \dots + L_{N-1})\alpha(Z_1, Z_2).\end{aligned}$$

Thus $\text{cheb}\colon (B_\alpha(M,\varepsilon),\alpha) \to X$ is a Lipschitz map. This completes the proof. ■

Proof of Proposition 1. Let us denote by $r = R(M)$, $R = R(Z)$, $t = \text{cheb}(M)\,\text{cheb}(Z) - R - r$ and assume that $r \le R$. Now the definition of the Hausdorff metric and the inclusion $\text{cheb}(Z) \in co(Z)$ imply that $\sqrt{(R+r+t)^2 + R^2} - r \le \alpha(M,Z)$ if $N > 3$ and $\sqrt{(R+t)^2 + R^2} \le \alpha(M,Z)$ if $N = 3$. On the other hand $\text{cheb}(M)\,\text{cheb}(Z) = R + r + t \le (1+\sqrt{5})(\sqrt{(R+r+t)^2 + R^2}) - r)/2$. These inequalities put together complete the proof. ■

Proof of Proposition 2. (i) In the case both angles $\angle(uwv)$, $\angle(uzv)$ are not acute $\text{cheb}(M)\,\text{cheb}(Z) = 0$. Assume then that the angle $\angle(uwv)$ is acute one. Consider a 3-net $Z_1 = \{u, v, z_1\}$, where the point z_1 is constructed rotating the point z over the line $\Pi(u,v)$ by angle equal to one adjacent to that between the halfplanes $\Pi_+(u,v,w)$ and $\Pi_+(u,v,z)$. Then the conditions of the propostion and geometrical considerations imply that $\alpha(M,Z) = \alpha(M,Z_1)$ and $\text{cheb}(M)\,\text{cheb}(Z) \le \text{cheb}(M)\,\text{cheb}(Z_1)$. Consider two angles $\angle(vuw)$ and $\angle(uvw)$ for the 3-net M. Assume now that the angle $\angle(vuw)$ is acute or non-zero and $\angle(vuw) \le \angle(uvw)$. Then $\text{cheb}(M)\omega(u,v) \le vw/2 \le \alpha(M,Z_1)/2$. Another case (the acute angle $\angle(uzv)$) is considered similarly to the stated one. In the rest of the cases $\omega(u,v)\,\text{cheb}(Z_1) = 0$. So using all the inequalities found in this proof we get

$$\begin{aligned}\text{cheb}(M)\,\text{cheb}(Z) &\le \text{cheb}(M)\,\text{cheb}(Z_1)\\ &\le \text{cheb}(M)\omega(u,v) + \omega(u,v)\,\text{cheb}(Z_1) \le \alpha(M,Z).\end{aligned}$$

Thus statement (i) of the proposition holds true.

(ii) Assume without loss of generality that $[v,q]$ is the interval of the minimal length of the intervals v,q, $[v,z]$, $[w,q]$, $[w,z]$ such that its intersection with the interior of the set $co(M) \cup co(Z)$ is empty. Then using the definition of Hausdorff metric and the first part of the statement we find the following inequality:

$$\begin{aligned}\text{cheb}(M)\,\text{cheb}(Z) &\le \text{cheb}(M)\,\text{cheb}(\{u,v,q\}) + \text{cheb}(\{u,v,q\})\,\text{cheb}(Z)\\ &\le \alpha(M,\{u,v,q\}) + \alpha(\{u,v,q\},Z) \le 2\alpha(M,Z).\end{aligned}$$

This completes the proof of Proposition 2. ■

REFERENCES

[**1**] Garkavi, A. L., *The best possible net and the best possible cross-section of a set in a normed space*, Am. Math. Soc., Transl., II. Ser. **39**, 111–132 (1964); translation from Izv. Akad. Nauk SSSR, Ser. Mat. **26**, 87–106 (1962).

[**2**] Kuratowski, C., *Topologie I. Espaces metrisables, espaces complets*, 2. ed. revue et augmentee, Monografie Matematyczne, T. XX, Warszawa, Seminarium Matematyczne Uniwersytetu, XI, (1948).

[**3**] Burago, D., Burago, Yu., Ivanov, S., *A Course in Metric Geometry*, Graduate Studies in Mathematics, 33, Providence, RI, American Mathematical Society (AMS), (2001).

[**4**] Belobrov, P. K., *On the Chebyshev point of a system of sets*, Izv. Vyssh. Uchebn. Zaved., Mat. 1966, **6** (55), 18–24 (1966).

[**5**] Sosov, E. N. *The best net, the best section, and the Chebyshev center of bounded set in infinite-dimensional Lobachevskij space*, Russ. Math. **43**, 9, 39–43 (1999); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 1999, 9, 42–47 (1999).

[**6**] Sosov E. N., *On Hausdorff intrinsic metric*, Lobachevskii J. Math. **8** (2001), 185–189.

[**7**] Lang, U., Pavlović, B., Schroeder, V., *Extensions of Lipschitz maps into Hadamard spaces*, Geom. Funct. Anal. **10**, 6, 1527–1553 (2000).

[**8**] Sosov, E. N., *On metric space of* 2*-nets in the nonpositively curved space*, Izv. Vyssh. Uchebn. Zaved., Mat., **10**, 57–60 (1999).

[**9**] Shirokov, P. A., *A sketch of the fundamentals of Lobachevskian geometry*, Prepared by I. N. Bronshtejn, Groningen, P. Noordhoff Ltd. (1964).

(received 12.02.2007, in revised form 01.11.2007)

Pyotr N. Ivanshin, N.G. Chebotarev's RIMM, Kazan State University, 420008, Kazan, Universitetskaya, 17, Russia

E-mail: Pyotr.Ivanshin@ksu.ru

Evgenii N. Sosov, N.G. Chebotarev's RIMM, Kazan State University, 420008, Kazan, Universitetskaya, 17, Russia

E-mail: Evgenii.Sosov@ksu.ru