МАТЕМАТИЧКИ ВЕСНИК
60 (2008), 121–135

UDK 517.547
оригинални научни рад
research paper

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION WITH THEIR DERIVATIVES

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Abstract. Using the notion of weakly weighted sharing we prove two uniqueness theorems concerning meromorphic functions sharing a small function with their derivatives, the first of which will improve all the results recently obtained by Lin-Lin [7] and thus provide a better answer to the questions posed by Yu [11] in this regard. Also with the aid of a recently introduced sharing notion in [1] known as relaxed weighted sharing we supplement one recent result of Lin-Lin [7].

1. Introduction, definitions and results

Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbf{C}$. A meromorphic function a is said to be a small function of f provided that $T(r,a)=S(r,f)$, that is $T(r,a)=o(T(r,f))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure. We denote by $S(f)$ the set of all small functions of f.

If for some $a\in S(f)\cap S(g)$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g share a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share a IM (ignoring multiplicities).

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure.

Let $N_E(r,a;f,g)$ ($\overline{N}_E(r,a;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a;f,g)$ ($\overline{N}_0(r,a;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities.

If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a;f,g)=S(r,f)+S(r,g),$$

AMS Subject Classification: 30D35

Keywords and phrases: Meromorphic function; derivative; small function; weakly weighted sharing.

then we say that f and g share a "CM". On the other hand if

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share a "IM".

In 2003 Yu [11] considered the uniqueness problem of an entire or meromorphic function when it shares a small function with its derivative. Yu proved the following two theorems.

Theorem A. [11] *Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f-a$ and $f^{(k)}-a$ share 0 CM and $\delta(0;f) > \frac{3}{4}$ then $f \equiv f^{(k)}$.*

Theorem B. [11] *Let f be a non-constant non entire meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If*

i) *f and a have no common poles,*

ii) *$f-a$ and $f^{(k)}-a$ share the value 0 CM,*

iii) *$4\delta(0;f)+2(8+k)\Theta(\infty;f) > 19+2k$,*

then $f \equiv f^{(k)}$ where k is a positive integer.

In the same paper Yu [11] posed the following open questions.

(i) Can a CM shared be replaced by an IM shared value in Theorem A?

(ii) Can the condition $\delta(0;f) > \frac{3}{4}$ of Theorem A be further relaxed?

(iii) Can the condition (iii) in Theorem B be further relaxed?

(iv) Can in general the condition (i) of Theorem B be dropped?

In 2004, P.Liu and Y.X.Gu [8] provided affirmative answers to the last three questions of Yu [11] and obtained the following results.

Theorem C. [8] *Let $k \geq 1$ and let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $f^{(k)}$ and a do not have any common poles of same multiplicity and $2\delta(0;f)+4\Theta(\infty;f) > 5$ then $f \equiv f^{(k)}$.*

Theorem D. [8] *Let $k \geq 1$ and let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $\delta(0;f) > \frac{1}{2}$ then $f \equiv f^{(k)}$.*

To state the next results we require the following definition known as weighted sharing of values which measure how close a shared value is to be shared IM or to be shared CM.

Definition 1.1. [3,4] Let k be a nonnegative integer or infinity. For $a \in \mathbf{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f,g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If a is a small function we define that f and g share (a, l) which means f and g share a with weight l if $f - a$ and $g - a$ share $(0, l)$.

Though we use the standard notations and definitions of the value distribution theory available in [2], we explain some definitions and notations which are used in the paper.

Definition 1.2. [10] For $a \in \mathbf{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.3. [6] For a positive integer p and $a \in \mathbf{C} \cup \{\infty\}$ we put

$$\delta_p(a; f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, a; f)}{T(r, f)}$$

Clearly $0 \leq \delta(a; f) \leq \delta_p(a; f) \leq \delta_{p-1}(a; f) \leq \cdots \leq \delta_2(a; f) \leq \delta_1(a; f) = \Theta(a; f)$.

In 2004 Lahiri and Sarkar [6] gave some affirmative answers to the first three questions imposing some restrictions on the zeros and poles of a. But they did not provide any definite answer corresponding to the question (i) of Yu as mentioned above. Rather they confined their investigations of sharing of small function up to weight 2.

Recently Lin and Lin [7] introduced the notion of weakly weighted sharing which we shall define next.

Definition 1.4. [7] Let f, g share a "IM" for $a \in S(f) \cap S(g)$ and k be a positive integer or ∞.

(i) $\overline{N}^E(r, a; f, g \mid\leq k)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k.

(ii) $\overline{N}^0(r, a; f, g \mid> k)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k.

Definition 1.5. [7] For $a \in S(f) \cap S(g)$, if k be a positive integer or ∞ and $\overline{N}(r, a; f \mid\leq k) - \overline{N}^E(r, a; f, g \mid\leq k) = S(r, f)$, $\overline{N}(r, a; g \mid\leq k) - \overline{N}^E(r, a; f, g \mid\leq k) = S(r, g)$, $\overline{N}(r, a; f \mid\geq k + 1) - \overline{N}^0(r, a; f, g \mid\geq k + 1) = S(r, f)$, $\overline{N}(r, a; g \mid\geq k + 1) - \overline{N}^0(r, a; f, g \mid\geq k + 1) = S(r, g)$ or if $k = 0$ and $\overline{N}(r, a; f) - \overline{N}_0(r, a; f, g) = S(r, f)$,

$\overline{N}(r,a;g) - \overline{N}_0(r,a;f,g) = S(r,g)$, then we say f, g weakly share a with weight k. Here we write f, g share "(a,k)" to mean that f, g weakly share a with weight k.

Obviously if f, g share "(a,k)", then f, g share "(a,p)" for any integer p, $0 \le p < k$. Also we note that f, g share a "IM" or "CM" if and only if f, g share "$(a,0)$" or "(a,∞)" respectively.

With the notion of weakly weighted sharing improving the results of Yu [11] and Liu-Gu [8] recently Lin and Lin [7] proved the following results.

Theorem E. [7] *Let f be a non-constant meromorphic function and $k(\ge 1)$, $l(\ge 0)$ be integers. Also let $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose that $f-a$ and $f^{(k)}-a$ share "$(0,l)$". If $2 \le l \le \infty$ and*

$$4\,\Theta(\infty,f) + 2\,\delta_{2+k}(0;f) > 5 \tag{1.1}$$

or l=1 and

$$\frac{9+k}{2}\,\Theta(\infty,f) + \frac{5}{2}\,\delta_{2+k}(0;f) > 6 + \frac{k}{2} \tag{1.2}$$

or $l=0$ and

$$(7+2k)\,\Theta(\infty,f) + 5\,\delta_{2+k}(0;f) > 11 + 2k \tag{1.3}$$

then $f \equiv f^{(k)}$.

In the present paper we shall improve Theorem E by replacing all the conditions (1.1)–(1.3) by three weaker ones and thus provide a better answer to the first question of Yu than that of Lin and Lin.

Following theorem is one of the main results of the paper.

Theorem 1.1. *Let f be a non-constant meromorphic function and $k(\ge 1)$, $l(\ge 0)$ be integers $a \in S(f)$ and $a(z) \not\equiv 0, \infty$. Suppose that $f-a$ and $f^{(k)}-a$ share "$(0,l)$". If $2 \le l \le \infty$ and*

$$3\Theta(\infty;f) + \delta_2(0;f) + \delta_{2+k}(0;f) > 4 \tag{1.4}$$

or $l=1$ and

$$\left(\frac{7}{2}+\frac{k}{2}\right)\Theta(\infty;f) + \delta_2(0;f) + \frac{1}{2}\delta_{1+k}(0;f) + \delta_{2+k}(0;f) > 5 + \frac{k}{2} \tag{1.5}$$

or $l=0$ and

$$(6+2k)\,\Theta(\infty;f) + 2\,\Theta(0;f) + \delta_2(0;f) + \delta_{1+k}(0;f) + \delta_{2+k}(0;f) > 10 + 2k \tag{1.6}$$

then $f \equiv f^{(k)}$.

From Theorem 1.1 we immediately have the following corollary.

Corollary 1.1. *Let f be a non-constant entire function and $k(\ge 1)$, $l(\ge 0)$ be integers. Also let $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose that $f-a$ and $f^{(k)}-a$ share "$(0,l)$". If $2 \le l \le \infty$ and*

$$\delta_{2+k}(0;f) > \frac{1}{2} - \frac{1}{2}[\delta_2(0;f) - \delta_{2+k}(0;f)] \tag{1.7}$$

or $l = 1$ *and*

$$\delta_{2+k}(0;f) > \frac{3}{5} - \frac{1}{5}\left[2\delta_2(0;f) + \delta_{1+k}(0;f) - 3\delta_{2+k}(0;f)\right] \tag{1.8}$$

or $l = 0$ *and*

$$\delta_{2+k}(0;f) > \frac{4}{5} - \frac{1}{5}\left[2\Theta(0;f) + \delta_2(0;f) + \delta_{1+k}(0;f) - 4\delta_{2+k}(0;f)\right] \tag{1.9}$$

then $f \equiv f^{(k)}$.

Conditions (1.7)–(1.9) weaken all the conditions of Theorem E corresponding to entire functions. Also conditions (1.7) and (1.9) provides respectively the better answers corresponding to the second and first question of Yu [11] than that given by Lin and Lin [7].

Now it is clear from Definition 1.1 and Definition 1.5 that weighted sharing and weakly weighted sharing are respectively scalings between IM, CM and "IM", "CM". Also weakly weighted sharing includes the definition of weighted sharing.

Recently in [1] another sharing notion known as relaxed weighted sharing has been introduced which is also a scaling between "IM" and "CM" but weaker than weakly weighted sharing and hence include the same definition. We first require the following notation.

Definition 1.6. [1] For $a \in S(f) \cap S(g)$ we denote by $\overline{N}(r,a;f \mid= p; g \mid= q)$ the reduced counting function of common zeros of $f-a$ and $g-a$ with multiplicities p and q respectively.

We are now at a stage to discuss about the definition of relaxed weighted sharing.

Definition 1.7. [1] For $a \in S(f) \cap S(g)$ let f, g share a "IM". Also let k be a positive integer or ∞. If

$$\sum_{\substack{p,q \le k \\ p \ne q}} \overline{N}(r,a;f \mid= p; g \mid= q) = S(r)$$

then we say f, g share a with weight k in a relaxed manner. Here we write f, g share $(a,k)^*$ to mean that f, g share a with weight k in a relaxed manner.

Obviously if f, g share $(a,k)^*$, then f, g share $(a,p)^*$ for any integer p, $1 \le p < k$. Also we note that f, g share "$(a,0)$" or "(a,∞)" if and only if f, g share $(a,1)^*$ or $(a,\infty)^*$ respectively. We note that f, g share "(a,k)" means they share $(a,k)^*$ for $k \ge 1$ but not conversely. Also from the definition of relaxed weighted sharing it is clear that for finite k f, g share $(a,k)^*$ actually means they share a "IM" with some restrictions imposed on the common zeros of $f-a$ and $g-a$ up to multiplicity k. In particular if $k=2$ the restrictions are minimum. In the next theorem we will show that if in Theorem 1.1 f and g share $(a,2)^*$ instead of "$(a,0)$" the condition (1.6) can further be weakened. Following theorem is another main result of the paper.

Theorem 1.2. *Let f be a non-constant meromorphic function and $k(\geq 1)$ be an integer, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, 2)^*$. If*

$$(4 + k)\,\Theta(\infty; f) + \Theta(0; f) + \delta_2(0; f) + \delta_{2+k}(0; f) > 6 + k \tag{1.10}$$

then $f \equiv f^{(k)}$.

We now give some more definitions.

Definition 1.8. [6] Let p be a positive integer and $a \in \mathbf{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ ($\overline{N}(r, a; f \mid\geq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r, a; f \mid\leq p)$ ($\overline{N}(r, a; f \mid\leq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.9. [5] Let $a, b \in \mathbf{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid\ g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.10. [5] Let $a, b \in \mathbf{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid\ g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

Definition 1.11. Let $a \in \mathbf{C} \cup \{\infty\}$ and m, n be two positive integers. We denote by $N(r, a; f \mid\ m \leq f \leq n)$ ($\overline{N}(r, a; f \mid\ m \leq f \leq n)$) the counting function (reduced counting function) of those a-points of f whose multiplicities are between m and n.

Definition 1.12. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 "IM". Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$ and by $\overline{N}_E^{(2}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.

Definition 1.13. Let k be a positive integer and f, g be two non-constant meromorphic functions such that f and g share the value 1 "IM". Let z_0 be a zero of $f(z) - 1$ of multiplicity p and a zero of $g(z) - 1$ of multiplicity q. We denote by $\overline{N}_{f \geq k+1}(r, 1; f \mid\ g = m)$ the reduced counting functions of those 1-points of f and g for which $p \geq k + 1$ and $q = m$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right). \tag{2.1}$$

LEMMA 2.1. *If F, G be share $(1,1)^*$ and $H \not\equiv 0$. Then*

$$N^E(r,1;F,G \mid\leq 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Proof. Since F, G share $(1,1)^*$ it follows that if z_0 be a common simple 1-point of F and G, then in some neighborhoods of z_0 we have $H = (z - z_0)\alpha(z)$, where $\alpha(z)$ is analytic at z_0. Hence by the first fundamental theorem and Milloux theorem [2, p. 55] we get

$$N^E(r,1;F,G \mid\leq 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G). \quad \blacksquare$$

LEMMA 2.2. *If for two positive integers p, and k, $N_p(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ with multiplicity m is counted m times if $m \leq p$ and p times if $m > p$ then*

$$N_p(r,0;f^{(k)} \mid f \neq 0) \leq N_k(r,0;f) + k\overline{N}(r,\infty;f) - \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;\frac{f^{(k)}}{f} \mid\geq m\right) + S(r,f).$$

Proof. By the first fundamental theorem and Milloux theorem [2, p. 55] we get

$$\begin{aligned} N(r,0;f^{(k)} \mid f \neq 0) &= N(r,0;\frac{f^{(k)}}{f}) \leq N(r,\infty;\frac{f^{(k)}}{f}) + m(r,\frac{f^{(k)}}{f}) + O(1) \\ &= N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + k\overline{N}(r,\infty;f) + S(r,f) \\ &= N_k(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

Now

$$\begin{aligned} N_p(r,0;\frac{f^{(k)}}{f}) + \sum_{m=p+1}^{\infty} \overline{N}(r,0;\frac{f^{(k)}}{f} \mid\geq m) &= N(r,0;f^{(k)} \mid f \neq 0) \\ &\leq N_k(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

The lemma follows from above. $\blacksquare$

LEMMA 2.3. *For two positive integers p and k*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) - \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;\frac{f^{(k)}}{f} \mid\geq m\right) + S(r,f).$$

Proof. We note that

$$\begin{aligned} N_p(r,0;f^{(k)} \mid f = 0) = N(r,0;f \mid k \leq f \leq k+p-1) \\ - k\overline{N}(r,0;f \mid k \leq f \leq k+p-1) + p\overline{N}(r,0;f \mid\geq k+p). \end{aligned}$$

Since using Lemma 2.2 we get

$$\begin{aligned}N_p(r,0;f^{(k)}) &= N_p(r,0;\frac{f^{(k)}}{f}) + N_p(r,0;f^{(k)} \mid f=0)\\ &= N_k(r,0;f) + k\overline{N}(r,\infty;f) - \sum_{m=p+1}^{\infty}\overline{N}(r,0;\frac{f^{(k)}}{f}\mid\geq m)\\ &\quad + N_p(r,0;f^{(k)}\mid f=0) + S(r,f),\end{aligned}$$

the lemma follows. ■

Lemma 2.4. *For two positive integers p and k*

$$N_p(r,0;f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) - \sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f}\mid\geq m\right).$$

Proof. Since

$$N(r,0;f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N(r,0;f) + S(r,f),$$

it follows that

$$N_p(r,0;f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N(r,0;f) - \sum_{m=p+1}^{\infty}\overline{N}\left(r,0;f^{(k)}\mid\geq m\right) + S(r,f).$$

But

$$\begin{aligned}\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;f^{(k)}\mid\geq m\right) &= \sum_{m=p+1}^{\infty}\overline{N}\left(r,0;f^{(k)}\mid f=0\mid\geq m\right)\\ &\quad + \sum_{m=p+1}^{\infty}\overline{N}(r,0;\frac{f^{(k)}}{f}\mid\geq m),\end{aligned}$$

where by $\overline{N}(r,0;f^{(k)}\mid f=0\mid\geq m)$ we mean the reduced counting function of those zeros of $f^{(k)}$ with multiplicities not less than m which are also the zeros of f. Also

$$N(r,0;f) - \sum_{m=p+1}^{\infty}\overline{N}(r,0;f^{(k)}\mid f=0\mid\geq m) = N_{p+k}(r,0;f)$$

the lemma follows. ■

Lemma 2.5. *Let f, g share $(1,2)^*$. Then*

$$\begin{aligned}\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f\geq 3}(r,1;g\mid=1)&\\ \leq N(r,1;g) - \overline{N}(r,1;g) + S(r).&\end{aligned}$$

Proof. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. For each possible value of q, possible values of p is always ≥ 0. Since f, g share $(1,2)^*$ implies f, g share "$(1,0)$", the sum of the reduced counting functions corresponding to the 1 points of g for which $p=0$ are $S(r)$. Also from the definition of relaxed weighted sharing we note that the sum of the reduced counting functions corresponding to the common 1 points of f and g for which (i) $q=1, p=2$ and (iii) $q=2, p=1$ are $S(r)$, the lemma follows. ■

Lemma 2.6. *Let f, g share $(1,2)^*$. Then*

$$\overline{N}_L(r,1;f)+\overline{N}_{f\geq 3}(r,1;g\mid=1)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\sum_{p=3}^{\infty}\overline{N}\Big(r,0;\frac{f'}{f}\mid\geq p\Big)+S(r).$$

Proof. Using Lemma 2.2 we get

$$\begin{aligned}\overline{N}_L(r,1;f)+\overline{N}_{f\geq 3}(r,1;g\mid=1)&=\overline{N}(r,1;f\mid=2;g=1)+2\overline{N}(r,1;f\mid\geq 3)+S(r)\\&=2\overline{N}(r,1;f\mid\geq 3)+S(r)\\&\leq N_2(r,0;f'\mid f=1)+S(r)\leq N_2(r,0;f'\mid\ f\neq 0)+S(r)\\&\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{f'}{f}\mid\geq p\right)+S(r).\quad ■\end{aligned}$$

Lemma 2.7. [9] *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n\neq 0$ and $b_m\neq 0$ Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.8. [2, p. 68] *Suppose that f is meromorphic and transcendental in the plane and that*

$$f^nP=Q$$

where P and Q are differential polynomials in f and the degree of Q is at most n. Then

$$m\{r,P\}=S(r,f)\ \ as\ \ r\to+\infty.$$

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F=\frac{f}{a}$ and $G=\frac{f^{(k)}}{a}$. Then $F-1=\frac{f-a}{a}$ and $G-1=\frac{f^{(k)}-a}{a}$. Since $f-a$ and $f^{(k)}-a$ share "$(0,l)$" it follows that F, G share "$(1,l)$" except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $H\not\equiv 0$.

Subcase 1.1. $l\geq 1$. Since H has only simple poles from (2.1) it can be easily calculated that

$$\begin{aligned}N(r,\infty;H)\leq{}&\overline{N}(r,\infty;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,0;F\mid\geq 2)+\\&+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+\overline{N}(r,0;a)+\overline{N}(r,\infty;a),\quad(3.1)\end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined. Let z_0 be a simple zero of $F-1$ and $G-1$ but $a(z_0)\neq 0,\infty$. Then z_0 is a zero of H. So

$$\begin{aligned} N(r,1;F\mid\leq 1) &= N(r,1;G\mid\leq 1)+S(r)\\ &\leq N(r,0;H)+N(r,\infty;a)+N(r,0;a)+S(r)\\ &\leq N(r,\infty;H)+S(r). \end{aligned}\tag{3.2}$$

From (3.1) and (3.2) we get

$$\begin{aligned} \overline{N}(r,1;G) &\leq N(r,1;G\mid\leq 1)+\overline{N}(r,1;F\mid\geq 2)+S(r)\\ &\leq \overline{N}(r,\infty;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r.0;F\mid\geq 2)\\ &\quad+\overline{N}(r.0;G\mid\geq 2)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}(r,0;F^{'})+\overline{N}(r,0;G^{'})+S(r). \end{aligned}\tag{3.3}$$

By the second fundamental theorem, (3.3) and noting that $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)$ $+\,S(r)$ we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,1;G)-N_0(r,0;G^{'})+S(r,G)\\ &\leq 2\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,0;F\mid\geq 2)\\ &\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})+S(r). \end{aligned}\tag{3.4}$$

While $l\geq 2$ using Lemma 2.2 we obtain

$$\begin{aligned} &\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})\\ &\qquad\leq \overline{N}(r,1;F\mid\geq 3)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})\\ &\qquad\leq N_2(r,0;F^{'}\mid F\neq 0)\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+S(r). \end{aligned}\tag{3.5}$$

Using (3.5) and Lemma 2.4 we get from (3.4)

$$\begin{aligned} T\left(r,f^{(k)}\right) &\leq 3\,\overline{N}(r,\infty;f)+N_2(r,0;f)+N_2(r,0;f^{(k)})+S(r)\\ &\leq 3\,\overline{N}(r,\infty;f)+N_2(r,0;f)+T(r,f^{(k)})-T(r,f)+N_{2+k}(r,0;f)+S(r) \end{aligned}$$

that is

$$T(r,f)\leq 3\,\overline{N}(r,\infty;f)+N_2(r,0;f)+N_{2+k}(r,0;f)+S(r)$$

and hence it follows that

$$3\Theta(\infty;f)+\delta_2(0;f)+\delta_{2+k}(0;f)\leq 4,$$

which contradicts (1.4).

While $l = 1$ (3.5) changes to

$$\begin{aligned}
\overline{N}_L(r,1;F) &+ \overline{N}_L(r,1;G) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F^{'}) \\
&\leq \overline{N}(r,1;F \mid\geq 3) + \overline{N}(r,1;G \mid\geq 3) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F^{'}) \\
&\leq N_2(r,0;F^{'} \mid F \neq 0) + \frac{1}{2}N_2(r,0;G^{'} \mid G \neq 0) \\
&\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \frac{1}{2}\overline{N}(r,0;G) + \frac{1}{2}\overline{N}(r,\infty;G) + S(r).
\end{aligned} \tag{3.6}$$

Using (3.6) and Lemma 2.4 we get from (3.4)

$$\begin{aligned}
T\left(r, f^{(k)}\right) &\leq \frac{7}{2}\ \overline{N}(r,\infty;f) + N_2(r,0;f) + N_2(r,0;f^{(k)}) + \frac{1}{2}\ \overline{N}(r,0;G) + S(r) \\
&\leq \frac{7}{2}\ \overline{N}(r,\infty;f) + N_2(r,0;f) + T(r,f^{(k)}) - T(r,f) + N_{2+k}(r,0;f) \\
&\quad + \frac{1}{2}\ \overline{N}(r,0;f^{(k)}) + S(r).
\end{aligned} \tag{3.7}$$

Now from (3.7) using Lemma 2.3 with $p = 1$ we get

$$T(r,f) \leq \left(\frac{7}{2} + \frac{k}{2}\right)\ \overline{N}(r,\infty;f) + N_2(r,0;f) + N_{2+k}(r,0;f) + \frac{1}{2}\ N_{1+k}(r,0;f) + S(r),$$

from which it follows that

$$\left(\frac{7}{2} + \frac{k}{2}\right)\ \Theta(\infty;f) + \delta_2(0;f) + \frac{1}{2}\ \delta_{1+k}(0;f) + \delta_{2+k}(0;f) \leq 5 + \frac{k}{2},$$

which contradicts (1.5).

Subcase 1.2. $l = 0$. In this case F and G share "$(1,0)$" except the zeros and poles of $a(z)$. Let z_0 be a common simple zero of $F-1$ and $G-1$. By a simple calculation we see that z_0 is a zero of H and hence (3.2) is replaced by

$$N^E(r,1;F,G \mid\leq 1) \leq N(r,\infty;H) + S(r). \tag{3.8}$$

So from (3.1), (3.8) we note that

$$\begin{aligned}
\overline{N}(r,1;F) + \overline{N}(r,1;G) &\leq N^E(r,1;F,G \mid\leq 1) + \overline{N}_E^{(2}(r,1;F) \\
&\quad + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_E^{(2}(r,1;F) \\
&\quad + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + \overline{N}_0(r,0;F^{'}) \\
&\quad + \overline{N}_0(r,0;G^{'}) + S(r) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + 2\overline{N}_L(r,1;F) \\
&\quad + \overline{N}_L(r,1;G) + N(r,1;G) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r).
\end{aligned} \tag{3.9}$$

Using Lemma 2.2 for $p = 1$ we obtain

$$
\begin{aligned}
2\,\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) &\le 2\overline{N}(r,1;F \mid\ge 2) + \overline{N}(r,1;G \mid\ge 2) + S(r)\\
&\le 2\,\overline{N}(r,1;F^{'} \mid F \ne 0) + \overline{N}(r,1;G^{'} \mid G \ne 0) + S(r)\\
&\le 2\overline{N}(r,0;F) + 3\,\overline{N}(r,\infty;f) + \overline{N}(r,0;G) + S(r). \qquad (3.10)
\end{aligned}
$$

By the second fundamental theorem (3.9) and (3.10) we obtain

$$
\begin{aligned}
T(r,F) + T(r,G) &\le \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F) + \overline{N}(r,0;G)\\
&\quad + \overline{N}(r,1;F) + \overline{N}(r,1;G) - N_0(r,0;F^{'}) - N_0(r,0;G^{'}) + S(r)\\
&\le N_2(r,0;F) + N_2(r,0;G) + 6\,\overline{N}(r,\infty;f)\\
&\quad + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + T(r,G) + S(r). \qquad (3.11)
\end{aligned}
$$

Hence using Lemma 2.3 for $p = 2$ and also for $p = 1$ we get from (3.11)

$$
\begin{aligned}
T(r,f) &\le N_2(r,0;f) + 2\,\overline{N}(r,0;f) + N_{2+k}(r,0;f) + N_{1+k}(r,0;f)\\
&\quad + (6+2k)\,\overline{N}(r,\infty;f) + S(r),
\end{aligned}
$$

that is

$$
(6+2k)\,\Theta(\infty;f) + \delta_2(0;f) + 2\,\Theta(0;f) + \delta_{2+k}(0;f) + \delta_{1+k}(0;f) \le 10 + 2k,
$$

which contradicts with (1.6).

Case 2. Let $H \equiv 0$.

On integration we get from (2.1)

$$
\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \qquad (3.12)
$$

where C, D are constants and $C \ne 0$. If there exist a pole z_0 of f with multiplicity p which is not a pole and zero of $a(z)$, then z_0 is the pole of F with multiplicity p and the pole of G with multiplicity $p + k$. This contradicts (3.12). So

$$
N(r,\infty;f) \le N(r,0;a) + N(r,\infty;a) = S(r,f),
$$

and hence $N\left(r,\infty;f^{(k)}\right) = S(r,f)$. So

$$
\Theta(\infty;f) = 1.
$$

From (1.4), (1.5) and (1.6) we know respectively

$$
\delta_2(0;f) + \delta_{2+k}(0;f) > 1, \qquad (3.13)
$$

$$
\delta_2(0;f) + \frac{1}{2}\delta_{1+k}(0;f) + \delta_{2+k}(0;f) > \frac{3}{2} \qquad (3.14)
$$

and

$$
\delta_2(0;f) + 2\,\Theta(0;f) + \delta_{1+k}(0;f) + \delta_{2+k}(0;f) > 4. \qquad (3.15)
$$

Suppose $D \ne 0$.

From (3.12) we can deduce

$$\frac{f^{(k)}}{a}=\frac{(C-D)\dfrac{f}{a}+D+1-C}{-D\dfrac{f}{a}+D+1}. \tag{3.16}$$

From (3.16) we have

$$-Dff^{(k)}=a((C-D)f+a(D+1-C))-a(D+1)f^{(k)}. \tag{3.17}$$

Hence using Lemma 2.8 we obtain from (3.17)

$$T\left(r,f^{(k)}\right)=m\left(r,f^{(k)}\right)+N\left(r,\infty;f^{(k)}\right)=S(r,f).$$

Using Lemma 2.7 from (3.12) we get

$$T(r,F)=T(r,G)+O(1).$$

Since a is a small function it follows that

$$T(r,f)=T\left(r,f^{(k)}\right)+S(r,f)=S(r,f),$$

which is absurd. Hence $D=0$ and so from (3.12) we get $G-1\equiv C\ (F-1)$. If $C\neq 1$, then $G\equiv C\ \left(F-1+\dfrac{1}{C}\right)$ and

$$\overline{N}(r,0;G)=\overline{N}\left(r,1-\frac{1}{C};F\right).$$

By the second fundamental theorem and noting that $\overline{N}(r,\infty;F)=S(r,f)$ we get

$$\begin{aligned}T(r,F)&\le\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}\left(r,1-\frac{1}{C};F\right)+S(r,G)\\&\le\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,f).\end{aligned}$$

By Lemma 2.3 for $p=1$ we have

$$\begin{aligned}T(r,f)&\le\overline{N}(r,0;f)+\overline{N}\left(r,0;f^{(k)}\right)+S(r,f)\\&\le\overline{N}(r,0;f)+N_{1+k}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\\&\le\overline{N}(r,0;f)+N_{1+k}(r,0;f)+S(r,f).\end{aligned}$$

Hence

$$\Theta(0;f)+\delta_{1+k}(0;f)\le 1. \tag{3.18}$$

So we have

$$\delta_2(0;f)+\delta_{2+k}(0;f)\le\Theta(0;f)+\delta_{1+k}(0;f)\le 1,$$

$$\delta_2(0;f)+\frac{1}{2}\delta_{1+k}(0;f)+\delta_{2+k}(0;f)\le\Theta(0;f)+\delta_{1+k}(0;f)+\frac{1}{2}\ \delta_{1+k}(0;f)\le\frac{3}{2}$$

and

$$\begin{aligned}&\Theta(0;f)+\delta_{2+k}(0;f)+\Theta(0;f)+\delta_{1+k}(0;f)+\delta_2(0;f)\\&\qquad\le 2\{\Theta(0;f)+\delta_{1+k}(0;f)\}+\delta_2(0;f)\le 3.\end{aligned}$$

These contradict (3.13), (3.14) and (3.15). Hence $C=1$ and so $F\equiv G$, that is $f\equiv f^{(k)}$. This completes the proof of the theorem. ∎

Proof of Theorem 1.2. Let $F = \frac{f}{a}$ and $G = \frac{f^{(k)}}{a}$. Then $F - 1 = \frac{f-a}{a}$ and $G - 1 = \frac{f^{(k)}-a}{a}$. Since f and $f^{(k)}$ share $(a,2)^*$ it follows that $f - a$ and $f^{(k)} - a$ share $(0,2)^*$ and hence F, G share $(1,2)^*$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $H \not\equiv 0$.

Using Lemmas 2.1, 2.5, 2.6 and (3.1) we note that

$$\begin{aligned}
\overline{N}(r,1;F) + \overline{N}(r,1;G) &\le N^E(r,1;F,G \mid\le 1) + \overline{N}_E^{(2}(r,1;F) \\
&\quad + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r) \\
&\le \overline{N}(r,\infty;f) + \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + \overline{N}_E^{(2}(r,1;F) \\
&\quad + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + \overline{N}_0(r,0;F^{'}) \\
&\quad + \overline{N}_0(r,0;G^{'}) + S(r) \\
&\le \overline{N}(r,\infty;f) + \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + \overline{N}_L(r,1;F) \\
&\quad + \overline{N}_{F\ge 3}(r,1;G \mid= 1) + N(r,1;G) + \overline{N}_0(r,0;F^{'}) \\
&\quad + \overline{N}_0(r,0;G^{'}) + S(r) \\
&\le 2\,\overline{N}(r,\infty;f) + \overline{N}(r,0;F) + \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) \\
&\quad + T(r,G) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r). \qquad (3.19)
\end{aligned}$$

By the second fundamental theorem and (3.19) we obtain

$$\begin{aligned}
T(r,F) + T(r,G) &\le \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F) + \overline{N}(r,0;G) \\
&\quad + \overline{N}(r,1;F) + \overline{N}(r,1;G) - N_0(r,0;F^{'}) - N_0(r,0;G^{'}) + S(r) \\
&\le 4\,\overline{N}(r,\infty;f) + N_2(r,0;F) + N_2(r,0;G) \\
&\quad + \overline{N}(r,0;F) + T(r,G) + S(r). \qquad (3.20)
\end{aligned}$$

Hence using Lemma 2.3 for $p = 2$ we get

$$T(r,f) \le N_2(r,0;f) + \overline{N}(r,0;f) + N_{2+k}(r,0;f) + (4+k)\,\overline{N}(r,\infty;f) + S(r),$$

that is

$$(4+k)\,\Theta(\infty;f) + \delta_2(0;f) + \Theta(0;f) + \delta_{2+k}(0;f) \le 6 + k,$$

which contradicts with (1.10).

Case 2. Let $H \equiv 0$.

Integrating (2.1) we get (3.12). In this case proceeding in the same way as done in Theorem 1.1 we can obtain

$$\Theta(\infty;f) = 1.$$

So from (1.10) we get

$$\delta_2(0;f) + \Theta(0;f) + \delta_{2+k}(0;f) > 2. \qquad (3.21)$$

Now again proceeding in the same way as done in Theorem 1.1, from (3.12) we can obtain $D = 0$. Again supposing $C \neq 1$ we get (3.18), which together with (3.21) leads to a contradiction. Hence $C = 1$. So from (3.12) we have $f = f^{(k)}$. This completes the proof of the theorem. ∎

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(received 15.06.2007, in revised form 25.03.2008)

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