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λ-FRACTIONAL PROPERTIES OF GENERALIZED JANOWSKI FUNCTIONS IN THE UNIT DISC

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Abstract. For analytic function $f(z) = z + a_2z^2 + \cdots$ in the open unit disc $\mathbb{D}$, a new fractional operator $\mathrm{D}^\lambda f(z)$ is defined. Applying this fractional operator $\mathrm{D}^\lambda f(z)$ and the principle of subordination, we give new proofs for some classical results concerning the class $\mathcal{S}^*_\lambda(A, B, \alpha)$ of functions $f(z)$.

1. Introduction

Let Ω be the family of functions $w(z)$ regular in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$.

Let $g(z) = z + b_2z^2 + \cdots$ and $h(z) = z + c_2z^2 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $g(z)$ is subordinate to $h(z)$, written as $g \prec h$, if

$$g(z) = h(w(z)),\ w(z) \in \Omega, \quad \text{and for all } z \in \mathbb{D}.$$

In particular if $h(z)$ is univalent in $\mathbb{D}$, then $g \prec h$ if and only if $g(0) = h(0)$, $g(\mathbb{D}) \subset h(\mathbb{D})$ ([1], [3]).

For arbitrary fixed numbers A, B, α, $-1 \le B < A \le 1$, $0 \le \alpha < 1$, let $\mathcal{P}(A, B, \alpha)$ denote the family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ regular in $\mathbb{D}$ and such that $p(z) \in \mathcal{P}(A, B, \alpha)$ if and only if

$$p(z) \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz} \iff p(z) = \frac{1 + [(1-\alpha)A + \alpha B]w(z)}{1 + Bw(z)}$$

for some function $w(z)$ and all $z \in \mathbb{D}$.

Using the fractional calculus, we define the fractional operator $\mathrm{D}^\lambda f(z)$ by

$$\mathrm{D}^\lambda f(z) = \Gamma(2-\lambda) z^\lambda \mathrm{D}_z^\lambda f(z),$$

where $\mathrm{D}_z^\lambda f(z)$ is the fractional derivative of order λ which will be defined below.

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Furthermore, let $\mathcal{S}^*_\lambda(A,B,\alpha)$ denote the family of functions $f(z)=z+a_2z^2+\cdots$ regular in $\mathbb{D}$ and such that $f(z)$ is in $\mathcal{S}^*_\lambda(A,B,\alpha)$ if and only if

$$z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}=p(z)$$

for some $p(z)$ in $\mathcal{P}(A,B,\alpha)$ and for all $z\in\mathbb{D}$.

The fractional integral of order λ is defined for a function $f(z)\in\mathcal{S}^*_\lambda(A,B,\alpha)$, by

$$\mathrm{D}_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\,d\zeta \quad (\lambda>0),$$

where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$ ([4], [5]).

The fractional derivative of order λ is defined for a function $f(z)\in\mathcal{S}^*_\lambda(A,B,\alpha)$, by

$$\mathrm{D}_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\lambda}}\,d\zeta \quad (0\le\lambda<1),$$

where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in the definition of the fractional integral ([4], [5]).

Under the hypotheses of the fractional derivative of order λ, the fractional derivative of order $(n+\lambda)$ is defined for a function $f(z)$, by

$$\mathrm{D}_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}\mathrm{D}_z^{\lambda}f(z) \quad (0\le\lambda<1,\ \ n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}).$$

By means of the definitions above, we see that

$$\begin{aligned}\mathrm{D}_z^{-\lambda}z^k&=\frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)}z^{k+\lambda} \quad (\lambda>0),\\ \mathrm{D}_z^{\lambda}z^k&=\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}z^{k-\lambda} \quad (0\le\lambda<1)\end{aligned} \tag{1.1}$$

and

$$\mathrm{D}_z^{n+\lambda}z^k=\frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)}z^{k-n-\lambda} \quad (0\le\lambda<1, n\in\mathbb{N}_0).$$

Therefore, we conclude that, for any real λ

$$\mathrm{D}_z^{\lambda}z^k=\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}z^{k-\lambda}. \tag{1.2}$$

The following lemma, due to Jack [2], plays an important rôle in our proofs.

Lemma 1.1 *Let $w(z)$ be a non-constant function analytic in $\mathbb{D}(r)=\{z\mid |z|<r\}$ with $w(0)=0$. If*

$$|w(z_1)|=\mathrm{Max}\,\{|w(z)|\mid |z|\le|z_1|\} \quad (z_1\in\mathbb{D}(r)),$$

then there exists a real number k $(k\ge1)$, such that $z_1w'(z_1)=kw(z_1)$.

2. Main Results

LEMMA 2.1. *Let $f(z) = z + a_2z^2 + \cdots$ be analytic in the open unit disc $\mathbb{D}$. Then the λ-fractional operator $\mathrm{D}^\lambda f(z)$ satisfies the following equalities*

(i) $\mathrm{D}^\lambda f(z) = \Gamma(2-\lambda)z^\lambda \mathrm{D}_z^\lambda f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n;$

(ii) for $\lambda = 1$, $\mathrm{D}f(z) = \lim_{\lambda\to 1} \mathrm{D}^\lambda f(z) = zf'(z);$

(iii) for $\lambda < 1$, $\delta < 1$, $\mathrm{D}^\lambda(\mathrm{D}^\delta f(z)) = \mathrm{D}^\delta(\mathrm{D}^\lambda f(z))$

$$= z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(2-\delta)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\delta)} z^n;$$

(iv) $\mathrm{D}(\mathrm{D}^\lambda f(z)) = z + \sum_{n=2}^{\infty} na_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n = z(\mathrm{D}^\lambda f(z))'$

$$= \Gamma(2-\lambda)z^\lambda(\lambda \mathrm{D}_z^\lambda f(z) + z\mathrm{D}_z^{\lambda+1} f(z));$$

(v) $$\frac{\mathrm{D}(\mathrm{D}^\lambda f(z))}{\mathrm{D}^\lambda f(z)} - 1 = \begin{cases} z\dfrac{f'(z)}{f(z)} - 1, & \text{for } \lambda = 0, \\ z\dfrac{f''(z)}{f'(z)}, & \text{for } \lambda = 1. \end{cases}$$

Proof. Making use of the fractional derivative rules (1.1) and (1.2), we obtain

$$\mathrm{D}_z^\lambda f(z) = \frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + a_2 \frac{\Gamma(3)}{\Gamma(3-\lambda)} z^{2-\lambda} + \cdots + a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^{n-\lambda} + \cdots$$

wherefrom

$$\mathrm{D}^\lambda f(z) = \Gamma(2-\lambda)z^\lambda \mathrm{D}_z^\lambda f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n. \tag{2.1}$$

Other equalities follow directly from (2.1). ■

Lemma 2.2. *Let $f(z) = z + a_2z^2 + \cdots$ and $g(z) = z + b_2z^2 + \cdots$ be analytic functions in the open unit disc $\mathbb{D}$. Then the solution of the fractional differential equation*

$$\mathrm{D}_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z)$$

is

$$f(z) = z + \sum_{n=2}^{\infty} b_n \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} z^n.$$

Proof. Using the definition of fractional integral, fractional derivative and fractional calculus of order $(n+\lambda)$, we get

$$\begin{aligned} \mathrm{D}_z^\lambda f(z) &= \frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + a_2 \frac{\Gamma(3)}{\Gamma(3-\lambda)} z^{2-\lambda} + \cdots + a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^{n-\lambda} + \cdots \\ &= \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) \\ &= \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + b_2 \frac{1}{\Gamma(2-\lambda)} z^{2-\lambda} + \cdots + b_n \frac{1}{\Gamma(2-\lambda)} z^{n-\lambda} + \cdots . \end{aligned}$$

Therefore, we have

$$\frac{\Gamma(2)}{\Gamma(2-\lambda)}z^{1-\lambda}+a_2\frac{\Gamma(3)}{\Gamma(3-\lambda)}z^{2-\lambda}+\cdots+a_n\frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)}z^{n-\lambda}+\cdots$$
$$=\frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}+b_2\frac{1}{\Gamma(2-\lambda)}z^{2-\lambda}+\cdots+b_n\frac{1}{\Gamma(2-\lambda)}z^{n-\lambda}+\cdots. \quad (2.2)$$

Comparing the coefficient of $z^{n-\lambda}$ in both sides of (2.2) we obtain

$$a_n=\frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)}b_n. \quad \blacksquare$$

Theorem 2.3. *Let $f(z)=z+a_2z^2+\cdots$ be analytic in the open unit disc $\mathbb{D}$. If $f(z)$ satisfies*

$$\left(\frac{\mathrm{D}(\mathrm{D}^\lambda f(z))}{\mathrm{D}^\lambda f(z)}-1\right)\prec\begin{cases}\dfrac{(1-\alpha)(A-B)z}{1+Bz}=F_1(z), & B\neq 0,\\ (1-\alpha)Az=F_2(z), & B=0,\end{cases} \quad (2.3)$$

then $f(z)\in\mathcal{S}^*_\lambda(A,B,\alpha)$ and this result is sharp as the function

$$\mathrm{D}^\lambda f(z)=\begin{cases}z(1+Bz)^{\frac{(1-\alpha)(A-B)}{B}}, & B\neq 0,\\ ze^{(1-\alpha)Az}, & B=0.\end{cases}$$

Proof. We define the function $w(z)$ by

$$\frac{\mathrm{D}^\lambda f(z)}{z}=\begin{cases}(1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}, & B\neq 0,\\ e^{(1-\alpha)Aw(z)}, & B=0,\end{cases} \quad (2.4)$$

where $(1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}$ and $e^{(1-\alpha)Aw(z)}$ have the value 1 at the origin (we consider the corresponding Riemann branch). Then $w(z)$ is analytic in $\mathbb{D}$ and $w(0)=0$. If we take the logarithmic derivative of the equality (2.4), simple calculations yield

$$\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-1\right)=\begin{cases}\dfrac{(1-\alpha)(A-B)zw'(z)}{1+Bw(z)}, & B\neq 0,\\ (1-\alpha)Azw'(z), & B=0.\end{cases}$$

Now, it is easy to realize that the subordination (2.3) is equivalent to $|w(z)|<1$ for all $z\in\mathbb{D}$. Indeed, assume the contrary; then, there exists $z_1\in\mathbb{D}$ such that $|w(z_1)|$ attains its maximum value on the circle $|z|=r$ at the point z_1, that is $|w(z_1)|=1$. Then, by I.S. Jack's lemma, $z_1w'(z_1)=kw(z_1)$ for some real $k\geq 1$. For such z_1 we have

$$\left(z_1\frac{(\mathrm{D}^\lambda f(z_1))'}{\mathrm{D}^\lambda f(z_1)}-1\right)=\begin{cases}\dfrac{(1-\alpha)(A-B)kw(z_1)}{1+Bw(z_1)}=F_1(w(z_1))\notin F_1(\mathbb{D}), & B\neq 0,\\ (1-\alpha)Akw(z_1)=F_2(w(z_1))\notin F_2(\mathbb{D}), & B=0,\end{cases}$$

because $|w(z)| = 1$ and $k \geq 1$. But this contradicts (2.3), so assumption is wrong, i.e., $|w(z)| < 1$ for every $z \in \mathbb{D}$.

The sharpness of the result follows from the fact that

$$\mathrm{D}^\lambda f(z) = \begin{cases} z(1+Bz)^{\frac{(1-\alpha)(A-B)}{B}}, & B \neq 0, \\ ze^{(1-\alpha)Az}, & B = 0, \end{cases} \Longrightarrow$$

$$\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)} - 1 \right) = \begin{cases} \dfrac{(1-\alpha)(A-B)z}{1+Bz}, & B \neq 0, \\ (1-\alpha)Az, & B = 0. \end{cases} \qquad \blacksquare$$

Corollary 2.4. *If $f(z) \in \mathcal{S}^*_\lambda(A, B, \alpha)$, then*

$$\left| \left(\frac{\Gamma(2-\lambda)\mathrm{D}^\lambda_z f(z)}{z^{1-\lambda}} \right)^{\frac{B}{(1-\alpha)(A-B)}} - 1 \right| < 1, \quad B \neq 0, \tag{2.5}$$

$$\left| \log \left(\frac{\Gamma(2-\lambda)\mathrm{D}^\lambda_z f(z)}{z^{1-\lambda}} \right)^{\frac{1}{(1-\alpha)A}} \right| < 1, \quad B = 0. \tag{2.6}$$

Proof. This corollary is a simple consequence of Theorem 2.3. ■

Remark 2.5. We note that the inequalities (2.5) and (2.6) are the λ-fractional Marx-Strohhacker inequalities. Indeed, for $A = 1$, $B = -1$, $\alpha = 0$, we have $\left| \left(\frac{\Gamma(2-\lambda)\mathrm{D}^\lambda_z f(z)}{z^{1-\lambda}} \right)^{-\frac{1}{2}} - 1 \right| < 1$, which yields

a) $\left| \sqrt{\dfrac{z}{f(z)}} - 1 \right| < 1$ for $\lambda = 0$: this is the Marx-Strohhacker inequality for starlike functions [1];

b) $\left| \dfrac{1}{\sqrt{f'(z)}} - 1 \right| < 1$ for $\lambda = 1$: this is the Marx-Strohhacker inequality for convex functions [1].

Moreover, assigning special values to A, B α and λ, we obtain Marx-Strohhacker inequalities for the all the subclasses $\mathcal{S}^*_\lambda(A, B, \alpha)$ of analytic functions in the unit disc where $0 \leq \lambda < 1$, $0 \leq \alpha < 1$, $-1 \leq B < A \leq 1$.

THEOREM 2.6. *If $f(z)\in\mathcal{S}^*_\lambda(A,B,\alpha)$, then*

$$\begin{aligned}\frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}(1-Br)^{\frac{(1-\alpha)(A-B)}{B}} &\le |\mathrm{D}_z^\lambda f(z)|\\ &\le \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}(1+Br)^{\frac{(1-\alpha)(A-B)}{B}}, \quad B\neq 0,\\ \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}e^{-(1-\alpha)Ar} &\le |\mathrm{D}_z^\lambda f(z)|\\ &\le \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}e^{(1-\alpha)Ar}, \quad B=0. \end{aligned} \tag{2.7}$$

These bounds are sharp, because the extremal function is the solution of the λ-fractional differential equation

$$\mathrm{D}_z^\lambda f(z)=\begin{cases}\frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}(1+Bz)^{\frac{(1-\alpha)(A-B)}{B}}, & B\neq 0,\\ \frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}e^{(1-\alpha)Az}, & B=0.\end{cases}$$

Proof. The set of the values $\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right)$ is the closed disc centered at

$$\begin{cases}C(r)=\frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-Br^2}, & B\neq 0,\\ C(r)=(1,0), & B=0,\end{cases}$$

with radius

$$\begin{cases}\rho(r)=\frac{(1-\alpha)(A-B)r}{1-B^2r^2}, & B\neq 0,\\ \rho(r)=(1-\alpha)|A|r, & B=0.\end{cases}$$

By using the definition of the class $\mathcal{S}^*_\lambda(A,B,\alpha)$ and the definition of the subordination we can write

$$\left|z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-\frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-Br^2}\right|\le\frac{(1-\alpha)(A-B)r}{1-B^2r^2}. \tag{2.8}$$

After simple calculations from (2.8) we get

$$\begin{aligned}\frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2} &\le \mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right)\\ &\le \frac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, \quad B\neq 0,\\ 1-(1-\alpha)|A|r\le \mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right) &\le 1+(1-\alpha)|A|r, \quad B=0.\end{aligned} \tag{2.9}$$

On the other hand we have

$$\mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right)=r\frac{\partial}{\partial r}\log|\mathrm{D}^\lambda f(z)|, \quad |z|=r. \tag{2.10}$$

If we substitute (2.9) into (2.10) we get

$$\begin{cases} \dfrac{1}{r} - \dfrac{(1-\alpha)(A-B)}{1-Br} \leq \dfrac{\partial}{\partial r}\log|\mathrm{D}^{\lambda}f(z)| \leq \dfrac{1}{r} + \dfrac{(1-\alpha)(A-B)}{1+Br}, & B \neq 0, \\ \dfrac{1}{r} - (1-\alpha)|A| \leq \dfrac{\partial}{\partial r}\log|\mathrm{D}^{\lambda}f(z)| \leq \dfrac{1}{r} + (1-\alpha)|A|, & B = 0. \end{cases} \tag{2.11}$$

Integrating both sides (2.11) and substituting $\mathrm{D}^{\lambda}f(z) = \Gamma(2-\lambda)z^{\lambda}\mathrm{D}_z^{\lambda}f(z)$ into the result of integration we obtain (2.7). ■

REMARK 2.7. Similarly, if we give special values to A, B, α and λ we obtain the distortions of the subclasses $\mathcal{S}_{\lambda}^{*}(A, B, \alpha)$.

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