

SOME TOPOLOGICAL PROPERTIES WEAKER THAN LINDELÖFNESS

Yan-Kui Song

Abstract. A space X is C -Lindelöf (weakly C -Lindelöf) if for every closed subset F of X and every open cover \mathcal{U} of F by open subsets of X , there exists a countable subfamily \mathcal{V} of \mathcal{U} such that $F \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}\}$ (respectively, $F \subseteq \overline{\bigcup\mathcal{V}}$). In this paper, we investigate the relationships among C -Lindelöf spaces, weakly C -Lindelöf spaces and Lindelöf spaces, and also study various properties of weakly C -Lindelöf spaces and C -Lindelöf spaces.

1. Introduction

By a space, we mean a topological space. In 1969, Viglino [2] introduced the concept of C -compact spaces that is weaker than compactness. Recall that a space X is C -compact if for every closed subset F of X and every open cover \mathcal{U} of F by open subsets of X , there exists a finite subfamily \mathcal{V} of \mathcal{U} such that $F \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}\}$. It is well-known that a space X is *Lindelöf* if for every open cover of X has a countable subcover. As motivations of the classes of C -compact spaces and Lindelöf spaces, we give the following classes of spaces:

DEFINITION 1.1. A space X is *C -Lindelöf* if for every closed subset F of X and every open cover \mathcal{U} of F by open subsets of X , there exists a countable subfamily \mathcal{V} of \mathcal{U} such that $F \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}\}$.

DEFINITION 1.2. A space X is *weakly C -Lindelöf* if for every closed subset F of X and every open cover \mathcal{U} of F by open subsets of X , there exists a countable subfamily \mathcal{V} of \mathcal{U} such that $F \subseteq \overline{\bigcup\mathcal{V}}$.

From the above definitions, it is clear that if X is Lindelöf, then X is C -Lindelöf and if X is C -Lindelöf, then X is weakly C -Lindelöf. But, the converses do not hold in the class of Hausdorff spaces or the class of Tychonoff spaces (see below Examples 2.3 and 2.4).

AMS Subject Classification: 54D15, 54D20.

Keywords and phrases: Lindelöf, C -Lindelöf, weakly C -Lindelöf.

The author acknowledges the support from the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No 07KJB110055) and NSFC Projects 10571081

The purpose of this paper is to investigate the relationship between C -Lindelöf spaces, weakly C -Lindelöf spaces and Lindelöf spaces, and also study various properties of weakly C -Lindelöf spaces and C -Lindelöf spaces.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal, \aleph_1 the first uncountable cardinal, \mathfrak{c} the cardinality of the continuum. Other terms and symbols that we do not define will be used as in [1].

2. Some examples on C -Lindelöf spaces and weakly C -Lindelöf spaces

In this section, we clarify the relationships of these spaces given in the first section by giving some examples. First, the following theorem can be easily proved:

THEOREM 2.1. *If X is a regular C -Lindelöf space, then every closed subset of X is Lindelöf.*

COROLLARY 2.2. *If X is a regular C -Lindelöf space, then X is Lindelöf.*

In the following, we give an example showing that Corollary 2.2 does not hold for the class of Hausdorff spaces.

EXAMPLE 2.3. *There exists a Hausdorff C -Lindelöf space X which is not Lindelöf.*

Proof. Let

$$A = \{a_\alpha : \alpha < \aleph_1\}, \quad B = \{b_\beta : \beta < \aleph_1\} \text{ and } Y = \{\langle a_\alpha, b_\beta \rangle : \alpha < \aleph_1, \beta < \aleph_1\}$$

Since $|B| = \aleph_1$, we can write B as $B = \cup_{\alpha < \aleph_1} B_\alpha$ such that $|B_\alpha| = \aleph_1$ for each $\alpha < \aleph_1$ and $B_\alpha \cap B_{\alpha'} = \emptyset$ for $\alpha' \neq \alpha$. For each $\alpha < \aleph_1$, let $A_\alpha = \{\langle a_\alpha, b_\beta \rangle : \beta < \aleph_1\}$. Let

$$X = Y \cup A \cup \{a\} \text{ where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighborhood of a point $a_\alpha \in A$ for each $\alpha < \aleph_1$ takes the form

$$U_{a_\alpha}(\gamma) = \{a_\alpha\} \cup \{\langle a_\alpha, b_\beta \rangle : \beta > \gamma\} \cup \{\langle a_\delta, b_\beta \rangle : b_\beta \in B_\alpha, \delta > \gamma\} \text{ for } \gamma < \aleph_1$$

and a basic neighborhood of a takes the form

$$U_a(\alpha) = \{a\} \cup \bigcup_{\beta > \alpha} \{\langle a_\gamma, b_\delta \rangle : b_\delta \in B_\beta, \gamma > \alpha\} \text{ for } \alpha < \aleph_1.$$

Clearly, X is a Hausdorff space by the construction of the topology on X . Moreover, X is not regular, since the point a cannot be separated from the closed subset A by disjoint open subsets of X . Since A is a discrete closed subset of X with $|A| = \aleph_1$, then X is not Lindelöf.

Let us show that X is C -Lindelöf. Let F be a closed subset of X and \mathcal{U} an open cover of F by open subsets of X . Without loss of generality, we can assume that \mathcal{U} consists of basic open sets of X .

Case (1): $a \in F$.

Since $a \in F$, there is a $U_a \in \mathcal{U}$ such that $a \in U_a$. By assumption, there exists a $\alpha_0 < \aleph_1$ such that

$$U_a = U_a(\alpha_0) = \{a\} \cup \cup_{\beta > \alpha_0} \{\langle a_\gamma, b_\delta \rangle : b_\delta \in B_\beta, \gamma > \alpha_0\}.$$

By definition of the topology of X , we have

$$F \cap (\{a_\beta : \beta > \alpha_0\} \cup U_a(\alpha_0)) \subseteq \overline{U_a(\alpha_0)}.$$

Let $A_0 = \{\alpha : a_\alpha \in F \cap \{a_\beta : \beta < \alpha_0 + 1\}\}$ and $A_1 = \{\alpha : a_\alpha \notin F \cap \{a_\beta : \beta < \alpha_0 + 1\}\}$. Then A_0 and A_1 are countable.

For $\alpha \in A_0$, $a_\alpha \in F$ and there is a $U_{a_\alpha} \in \mathcal{U}$ such that $a_\alpha \in U_{a_\alpha}$. By assumption, there is a $\alpha_\gamma < \aleph_1$ such that

$$U_{a_\alpha} = U_{a_\alpha}(\alpha_\gamma) = \{a_\alpha\} \cup \{\langle a_\alpha, b_\beta \rangle : \beta > \alpha_\gamma\} \cup \{\langle a_\delta, b_\beta \rangle : b_\beta \in B_\alpha \text{ and } \delta > \alpha_\gamma\}.$$

For $\alpha \in A_0$, since $F \cap \{\langle a_\alpha, b_\beta \rangle : \beta < \alpha_\gamma + 1\}$ is at most countable, there exists a countable subfamily \mathcal{V}_α of \mathcal{U} such that

$$F \cap \{\langle a_\alpha, b_\beta \rangle : \beta < \alpha_\gamma + 1\} \subseteq \cup \{\overline{V} : V \in \mathcal{V}_\alpha\}.$$

Let $\mathcal{U}_\alpha = \{U_{a_\alpha}\} \cup \mathcal{V}_\alpha$. Then \mathcal{U}_α is a countable subfamily of \mathcal{U} and

$$F \cap (U_{a_\alpha}(\alpha_\gamma) \cup \{\langle a_\alpha, b_\beta \rangle : \beta < \alpha_\gamma + 1\}) \subseteq \cup \{\overline{U} : U \in \mathcal{U}_\alpha\}.$$

If we put $\mathcal{U}' = \bigcup_{\alpha \in A_0} \mathcal{U}_\alpha$, \mathcal{U}' is a countably subfamily of \mathcal{U} and

$$\bigcup_{\alpha \in A_0} (F \cap (U_{a_\alpha}(\alpha_\gamma) \cup \{\langle a_\alpha, b_\beta \rangle : \beta < \alpha_\gamma + 1\})) \subseteq \bigcup \{\overline{U} : U \in \mathcal{U}'\}.$$

On the other hand, for $\alpha \in A_1$, $a_\alpha \notin F$, since F is closed, there exists an open neighborhood $U_{a_\alpha}(\alpha_\gamma)$ of a_α for some $\alpha_\gamma < \aleph_1$ such that

$$U_{a_\alpha}(\alpha_\gamma) \cap F = \emptyset.$$

Therefore, $F \cap \{\langle a_\alpha, b_\beta \rangle : \beta < \aleph_1\}$ is at most countable, and there exists a countable subfamily \mathcal{V}_α of \mathcal{U} such that

$$F \cap A_\alpha \subseteq \cup \{\overline{U} : U \in \mathcal{V}_\alpha\}.$$

If we put $\mathcal{U}'' = \bigcup_{\alpha \in A_1} \mathcal{V}_\alpha$, then \mathcal{U}'' is a countably subfamily of \mathcal{U} and

$$\bigcup_{\alpha \in A_1} (F \cap A_\alpha) \subseteq \bigcup \{\overline{U} : U \in \mathcal{U}''\}$$

Let $\alpha' = \sup\{\alpha_\gamma : \alpha \in A_0 \cup A_1\}$. Then $\alpha' < \aleph_1$, since $A_0 \cup A_1$ is countable. If we put $\mathcal{U}_0 = \mathcal{U}' \cup \mathcal{U}''$, then

$$F \cap \left(\bigcup_{\alpha < \alpha_0 + 1} (\{a_\alpha\} \cup A_\alpha \cup \{\langle a_\delta, b_\beta \rangle : b_\beta \in B_\alpha, \delta > \alpha'\}) \right) \subseteq \bigcup \{\bar{U} : U \in \mathcal{U}_0\}.$$

For each $\alpha_0 < \alpha < \alpha' + 1$, it is not difficult to find a countable subfamily \mathcal{U}_α of \mathcal{U} such that

$$F \cap (\{a_\alpha\} \cup A_\alpha) \subseteq \bigcup \{\bar{U} : U \in \mathcal{U}_\alpha\}.$$

Let $\mathcal{U}_1 = \bigcup_{\alpha_0 < \alpha < \alpha' + 1} \mathcal{U}_\alpha$. Then \mathcal{U}_1 is countable subfamily of \mathcal{U} and

$$F \cap \left(\bigcup_{\alpha_0 < \alpha < \alpha' + 1} (\{a_\alpha\} \cup A_\alpha) \right) \subseteq \bigcup \{\bar{U} : U \in \mathcal{U}_1\}.$$

If we put $\mathcal{V} = \{U_{a_\alpha}\} \cup \mathcal{U}_0 \cup \mathcal{U}_1$, then \mathcal{V} is a countable subfamily of \mathcal{U} and $F \subseteq \bigcup \{\bar{U} : U \in \mathcal{V}\}$, which completes the proof.

Case (2): $a \notin F$.

Since $a \notin F$, there is a basic open neighborhood U_a of a such that $U_a \cap F = \emptyset$. Without loss of generality, we can assume that

$$U_a = U_a(\alpha_0) = \{a\} \cup \bigcup_{\beta > \alpha_0} \{\langle a_\gamma, b_\delta \rangle : b_\delta \in B_\beta, \gamma > \alpha_0\} \text{ for some } \alpha_0 < \aleph_1.$$

As in the previous case, we can find a $\alpha' < \aleph_1$ and a countable subfamily \mathcal{U}_0 of \mathcal{U} such that

$$F \cap \left(\bigcup_{\alpha < \alpha_0 + 1} (\{a_\alpha\} \cup A_\alpha \cup \{\langle a_\delta, b_\beta \rangle : b_\beta \in B_\alpha, \delta > \alpha'\}) \right) \subseteq \bigcup \{\bar{U} : U \in \mathcal{U}_0\}.$$

If $F \cap \{a_\alpha : \alpha > \alpha_0\} = \emptyset$, similarly as in the proof above, we can find a countable subfamily \mathcal{U}_1 of \mathcal{U} such that

$$F \cap \left(\bigcup_{\alpha_0 < \alpha < \alpha' + 1} (\{a_\alpha\} \cup A_\alpha) \right) \subseteq \bigcup \{\bar{U} : U \in \mathcal{U}_1\}.$$

If we put $\mathcal{V} = \mathcal{U}_0 \cup \mathcal{U}_1$, then \mathcal{V} is a countable subfamily of \mathcal{U} and $F \subseteq \bigcup \{\bar{U} : U \in \mathcal{V}\}$.

On the other hand; if $F \cap \{a_\alpha : \alpha > \alpha_0\} \neq \emptyset$, we can pick $a_{\beta_0} \in F \cap \{a_\alpha : \alpha > \alpha_0\}$, and there is $U \in \mathcal{U}$ such that $a_{\beta_0} \in U$, and we can assume

$$U = U_{a_{\beta_0}}(\gamma) = \{a_{\beta_0}\} \cup \{\langle a_{\beta_0}, b_\beta \rangle : \beta \geq \gamma\} \cup \{\langle a_\delta, b_\beta \rangle : b_\beta \in B_{\beta_0}, \delta > \gamma\} \text{ for } \gamma < \aleph_1.$$

Then

$$F \cap \{a_\alpha : \alpha > \gamma\} \subseteq \bar{U}.$$

For $\alpha_0 < \alpha < \max\{\alpha', \gamma + 1\} + 1 = \gamma'$, we can find a countable subfamily \mathcal{U}_α of \mathcal{U} such that

$$F \cap (\{a_\alpha\} \cup A_\alpha) \subseteq \bigcup \{\bar{U} : U \in \mathcal{U}_\alpha\}.$$

If we put $\mathcal{U}_1 = \bigcup_{\alpha_0 < \alpha < \gamma'} \mathcal{U}_\alpha$, then

$$F \cap \left(\bigcup_{\alpha_0 < \alpha < \gamma'} (\{a_\alpha\} \cup A_\alpha) \right) \subseteq \bigcup \{\overline{U} : U \in \mathcal{U}_1\}$$

If we put $\mathcal{V} = \{U\} \cup \mathcal{U}_0 \cup \mathcal{U}_1$, then \mathcal{V} is a countable subfamily of \mathcal{U} and $C \subseteq \bigcup \{\overline{U} : U \in \mathcal{V}\}$, which completes the proof. ■

EXAMPLE 2.4. *There exists a Tychonoff weakly C -Lindelöf space X that is not C -Lindelöf.*

Proof. Let $X = \omega \cup \mathcal{R}$ be the well-known Mrówka space, where \mathcal{R} is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$ (see [3]).

We show that X is not C -Lindelöf. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate \mathcal{R} as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let $F = \{r_\alpha : \alpha < \mathfrak{c}\}$. Then F is a closed subset of X .

Let

$$U_\alpha = \{r_\alpha\} \cup r_\alpha \text{ for each } \alpha < \mathfrak{c}.$$

Then U_α is a closed and open subset of X . Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$$

of F . For any countable subfamily \mathcal{V} of \mathcal{U} , let $\alpha_0 = \sup\{\alpha : U_\alpha \in \mathcal{V}\}$. Then $\alpha_0 < \mathfrak{c}$, since \mathcal{V} is countable. If we pick $\alpha' > \alpha_0$, then $r_{\alpha'} \notin \bigcup \{\overline{U} : U \in \mathcal{V}\}$, since $U_{\alpha'} \notin \mathcal{V}$ and $U_{\alpha'}$ is the only element of \mathcal{U} containing $r_{\alpha'}$ and $U_{\alpha'} \cap U_\alpha$ is finite for each $\alpha < \alpha_0$, which shows that X is not C -Lindelöf.

Next, we show that X is weakly C -Lindelöf. Let F be any closed subset of X and \mathcal{U} any open cover of F by open subsets of X . Without loss of generality, we assume that \mathcal{U} consists of basic open sets of X . Let $A = F \cap \{r_\alpha : \alpha < \mathfrak{c}\}$. For each $r_\alpha \in A$ there is a $V_\alpha \in \mathcal{U}$ such that $r_\alpha \in V_\alpha$. By assumption, there is a finite subset F_α of ω such that

$$V_\alpha = \{r_\alpha\} \cup (r_\alpha \setminus F_\alpha).$$

Let $C = \bigcup \{r_\alpha \setminus F_\alpha : r_\alpha \in A\}$. Then C is a countable subset of ω . For each $n \in C$ we pick $V_n \in \mathcal{U}$ such that $n \in V_n$. Let $\mathcal{V}_1 = \{V_n : n \in C\}$. Then \mathcal{V}_1 is a countable subfamily of \mathcal{U} . By the construction of the Mrówka space, it is not difficult to show that

$$A \subseteq \overline{\bigcup \mathcal{V}_1}.$$

Let $B = F \cap \omega$. Then B is a countable subset of ω , since ω is countable. Hence, there exists a countable subfamily \mathcal{V}_2 of \mathcal{U} such that

$$B \subseteq \overline{\bigcup \mathcal{V}_2}.$$

If we put $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, then \mathcal{V} is a countable subfamily of \mathcal{U} such that $F \subseteq \overline{\bigcup \mathcal{V}}$, which shows that X is weakly C -Lindelöf. ■

3. Various properties of weakly C -Lindelöf spaces and C -Lindelöf spaces

From Example 2.4, it is not difficult to see that the closed subset \mathcal{R} of X is not weakly C -Lindelöf, which shows that a closed subset of a weakly C -Lindelöf space need not be weakly C -Lindelöf. In the following, we give a stronger example that shows that a regular closed subspace of a Tychonoff weakly C -Lindelöf space need not be weakly C -Lindelöf.

EXAMPLE 3.1. *There exists a Tychonoff weakly C -Lindelöf space X having a regular closed subspace which is not weakly C -Lindelöf.*

Proof. Let $S_1 = \omega \cup \mathcal{R}$ be the same Isbell-Mrówka space as in the proof of Example 2.4. Then S_1 is weakly C -Lindelöf.

Let D be a discrete space of cardinality \mathfrak{c} , and let

$$S_2 = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$$

be the subspace of the product of βD and $\omega + 1$.

We show that S_2 is not weakly C -Lindelöf. Since $|D| = \mathfrak{c}$, we can enumerate D as $\{d_\alpha : \alpha < \mathfrak{c}\}$. Let $F = \{\langle d_\alpha, \omega \rangle : \alpha < \mathfrak{c}\}$. Then F is a closed subset of X .

Let

$$U_\alpha = \{d_\alpha\} \times [0, \omega] \quad \text{for each } \alpha < \mathfrak{c}.$$

Then U_α is a closed and open subset of S_2 . Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$$

of F . For any countable subfamily \mathcal{V} of \mathcal{U} let $\alpha_0 = \sup\{\alpha : U_\alpha \in \mathcal{V}\}$. Then $\alpha_0 < \mathfrak{c}$, since \mathcal{V} is countable. If we pick $\alpha' > \alpha_0$, then $\langle d_{\alpha'}, \omega \rangle \notin \overline{\bigcup \mathcal{V}}$, since $U_{\alpha'}$ is the only element of \mathcal{U} containing $\langle d_{\alpha'}, \omega \rangle$ and $U_{\alpha'} \cap U_\alpha = \emptyset$ for each $\alpha < \alpha_0$, which shows that S_2 is not weakly C -Lindelöf.

We assume that $S_1 \cap S_2 = \emptyset$. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate \mathcal{R} as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let $\varphi : D \times \{\omega\} \rightarrow \mathcal{R}$ be a bijection by

$$\varphi(\langle d_\alpha, \omega \rangle) = r_\alpha \quad \text{for each } \alpha < \mathfrak{c}.$$

Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \omega \rangle$ with r_α for each $\alpha < \mathfrak{c}$. Let $\pi : S_1 \oplus S_2 \rightarrow X$ be the quotient map. Let $Y = \pi(S_2)$. Then Y is a regular closed subspace of X , however, it is not weakly C -Lindelöf, since it is homeomorphic to S_2 .

Now, we show that X is weakly C -Lindelöf. For that purpose, let F be a closed subset of X and \mathcal{U} an open cover of F by open subsets of X . Let

$$F' = F \cap \pi(S_1) \quad \text{and} \quad F_n = F \cap \pi(\beta D \times \{n\}) \quad \text{for each } n \in \omega.$$

Since S_1 is weakly C -Lindelöf, $\pi(S_1)$ is weakly C -Lindelöf, hence there exists a countable subfamily \mathcal{U}' of \mathcal{U} such that $F' \subseteq \overline{\bigcup \mathcal{U}'}$. For each $n \in \omega$, since F_n is compact, there exists a finite subfamily \mathcal{U}_n such that $F_n \subseteq \bigcup \overline{\mathcal{U}_n}$. If we put

$\mathcal{V} = \mathcal{U}' \cup \{\mathcal{U}_n : n \in \omega\}$, then \mathcal{V} is a countable subfamily of \mathcal{U} and $F \subseteq \overline{\bigcup \mathcal{V}}$, which shows that X is weakly C -Lindelöf. ■

The following theorem can be easily proved.

THEOREM 3.2. *If A is a closed and open subset of a weakly C -Lindelöf spaces X , then A is weakly C -Lindelöf.*

For a C -Lindelöf space, by Corollary 2.2, every closed subset of a regular C -Lindelöf space is Lindelöf (hence, C -Lindelöf). From Example 2.3, it is not difficult to see that the closed subset $\{a_\alpha : \alpha < \aleph_1\}$ of a Hausdorff space X is not C -Lindelöf which shows that a closed subset of a Hausdorff C -Lindelöf space need not be C -Lindelöf. In the following, we give a stronger example that shows that a regular closed subspace of a Hausdorff C -Lindelöf space need not be C -Lindelöf.

EXAMPLE 3.3. *There exists a Hausdorff C -Lindelöf space X having a regular closed subspace which is not weakly C -Lindelöf.*

Proof. Let $S_1 = X$ be the same space X as in the proof of Example 2.3. Then S_1 is a Hausdorff C -Lindelöf space.

Let D be a discrete space of cardinality \aleph_1 , and let

$$S_2 = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$$

be the subspace of the product of βD and $\omega + 1$. Similar to the proof that S_2 is not weakly C -Lindelöf in Example 3.1, we can prove that S_2 is not C -Lindelöf.

We assume that $S_1 \cap S_2 = \emptyset$. Since $|D| = \aleph_1$, we can enumerate D as $\{d_\alpha < \aleph_1\}$. Let $\varphi : D \times \{\omega\} \rightarrow A$ be a bijection by

$$\varphi(\langle d_\alpha, \omega \rangle) = a_\alpha \text{ for each } \alpha < \aleph_1.$$

Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \omega \rangle$ with a_α for each $\alpha < \aleph_1$. Let $\pi : S_1 \oplus S_2 \rightarrow X$ be the quotient map. Let $Y = \pi(S_2)$. Then Y is a regular closed subspace of X , however it is not C -Lindelöf, since it is homeomorphic to S_2 . Similar to the proof that X is weakly C -Lindelöf in Example 2.3, it is not difficult to show that X is C -Lindelöf, which completes the proof. ■

Now the following theorem can be easily proved.

THEOREM 3.4. *If A is a closed and open subset of a C -Lindelöf spaces X , then A is C -Lindelöf.*

Next, we consider the images of C -Lindelöf spaces and weakly C -Lindelöf spaces under continuous mapping. Since a continuous image of a Lindelöf space is Lindelöf, we give two parallel results for C -Lindelöf spaces and weakly C -Lindelöf spaces.

THEOREM 3.5. *Let $f : X \rightarrow Y$ be a continuous mapping from a C -Lindelöf space X onto a space Y . Then Y is C -Lindelöf.*

Proof. Let F be a closed subset of Y and $\{U_\alpha : \alpha \in \Lambda\}$ an open cover of F by open subsets of Y . Then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is an open cover of $f^{-1}(F)$ by open subsets of X . Since X is C -Lindelöf, there exists a countable subset $\{\alpha_i : i \in \omega\}$ of Λ such that $f^{-1}(F) \subseteq \bigcup_{i \in \omega} \overline{f^{-1}(U_{\alpha_i})}$ and thus

$$F = f(f^{-1}(F)) \subseteq \bigcup_{i \in \omega} f(\overline{f^{-1}(U_{\alpha_i})}) \subseteq \bigcup_{i \in \omega} \overline{f(f^{-1}(U_{\alpha_i}))} \subseteq \bigcup_{i \in \omega} \overline{U_{\alpha_i}}.$$

Hence, Y is C -Lindelöf, which completes the proof. ■

Similar to the proof of Theorem 3.5, we can prove the following theorem.

THEOREM 3.6. *Let $f: X \rightarrow Y$ be a continuous mapping from a weakly C -Lindelöf space X onto a space Y . Then Y is weakly C -Lindelöf.*

Now, we turn to consider preimages. To show that the preimage of a weakly C -Lindelöf space under a closed 2-to-1 continuous map need not be weakly C -Lindelöf, we use the Alexandroff duplicate $A(X)$ of a space X . The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0 \rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup (U \times \{1\}) \setminus \{\langle x, 1 \rangle\}$, where U is a neighborhood of x in X .

EXAMPLE 3.7. *There exists a 2-to-1 closed continuous map f from a space X to a weakly C -Lindelöf space Y such that X is not weakly C -Lindelöf.*

Proof. Let Y be the same space X as in the proof Example 2.4 and consider the space $X = A(Y)$. Let $f: X \rightarrow Y$ be the projection. Then f is a 2-to-1 closed continuous map. The space Y is weakly C -Lindelöf by Example 2.4, but X is not weakly C -Lindelöf, since $\mathcal{R} \times \{1\}$ is a discrete open and closed subset of X with $|\mathcal{R} \times \{1\}| = \mathfrak{c}$. ■

By considering the Alexandroff duplicate of the space Y in Example 2.3, in the same manner we can prove that the preimage of a C -Lindelöf space under a closed 2-to-1 continuous map need not be C -Lindelöf.

REMARK. The author does not know if the product of two C -Lindelöf spaces is C -Lindelöf and the product of two weakly C -Lindelöf spaces is weakly C -Lindelöf even if the product of a C -Lindelöf space and a compact space, and the product of a weakly C -Lindelöf space and a compact space.

ACKNOWLEDGMENTS. The author is most grateful to the referees for their kind help and valuable suggestions.

REFERENCES

- [1] R. Engelking, *General Topology*, Revised and completed edition, Heldermann Verlag, Berlin, 1989.
- [2] G. Viglino, *C-compact spaces*, Duke J. Math **36** (1969), 761–764.
- [3] S. Mrówka, *Set-theoretic constructions in topology*, Fund. Math. **44**, 2 (1977), 83–92.

(received 17.07.2007, in revised form 13.03.2008)

Department of Mathematics, Nanjing Normal University, Nanjing, 210097 P.R. China
E-mail: songyankui@njnu.edu.cn