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NOTES ON ALMOST OPEN MAPPINGS

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Abstract. In this paper, we give some characterizations of almost open mappings defined on first countable spaces, which corrects some errors for almost open mappings.

1. Introduction

Recently, some weak forms of open mappings have attracted considerable attention, and many interesting results have been obtained [5, 6, 7, 13]. In [5], Y. Ge gave the following results without proofs (see [5, Proposition 2.14], [5, Theorem 2.15] and [5, Corollary 2.16]).

PROPOSITION 1.1. *Let $f\colon X \longrightarrow Y$ be an almost sn-open mapping. If one of the following two conditions is satisfied, then f is almost open.*

(1) *Y is a sequential space.*

(2) *X is a sequential space and f is a quotient mapping.*

PROPOSITION 1.2. *Let $f\colon X \longrightarrow Y$ be a mapping. If X is first countable (especially, if X is metric), then the following are equivalent.*

(1) *f is an almost open mapping.*

(2) *f is an almost weak-open mapping.*

(3) *f is an almost sn-open, quotient mapping.*

(4) *f is a 1-sequence-covering, quotient mapping.*

PROPOSITION 1.3. *The following are equivalent for a space X.*

(1) *X is an almost open, P-image of a metric space.*

(2) *X is an almost weak-open, P-image of a metric space.*

(3) *X is an almost sn-open, quotient, P-image of a metric space.*

(4) *X is a 1-sequence-covering, quotient, P-image of a metric space.*

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Unfortunately, the above propositions are not true. In this paper, we correct these propositions.

Throughout this paper, all spaces are assumed to be regular T_1 and all mappings are continuous and onto. $\mathbf{N}$ denotes the set of all natural numbers. $\{x_n\}$ denotes a sequence, where the n-th term is x_n. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbf{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of X. Then $\bigcup \mathcal{P}$ and $\bigcap \mathcal{P}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the intersection $\bigcap\{P : P \in \mathcal{P}\}$ respectively. For terms which are not defined here, we refer to [3].

2. The main results

Definition 2.1. Let X be a space.

(1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to the point x, then $\{x_n\}$ is eventually in P.

(2) Let $P \subset X$. P is called a sequentially open subset in X if P is a sequential neighborhood of x in X for each $x \in P$.

(3) X is called a sequential space if each sequentially open subset in X is open.

(4) X is called a Fréchet space if for each $P \subset X$ and for each $x \in \overline{P}$, there exists a sequence $\{x_n\}$ in P converging to the point x.

Remark 2.2. (1) P is a sequential neighborhood of x if and only if each sequence $\{x_n\}$ converging to x is frequently in P.

(2) The intersection of finitely many sequential neighborhoods of x is a sequential neighborhood of x.

(3) It is well known that first countable $\Longrightarrow$ Fréchet $\Longrightarrow$ sequential.

Definition 2.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that for each $x \in X$, the following conditions (a) and (b) are satisfied.

(a) $\mathcal{P}_x$ is a network at x in X, i.e., $x \in \bigcap \mathcal{P}_x$ and for each neighborhood U of x in X, $P \subset U$ for some $P \in \mathcal{P}_x$.

(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \bigcap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is called a weak base [2] of X if whenever $G \subset X$, G is open in X if and only if for each $x \in G$ there exists $P \in \mathcal{P}_x$ with $P \subset G$, where $\mathcal{P}_x$ is called a wn-network (i.e., weak neighborhood network) at x in X.

(2) $\mathcal{P}$ is called an sn-network [4] of X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$, where $\mathcal{P}_x$ is called an sn-network at x in X.

Remark 2.4. [11] For a space, weak base $\Longrightarrow$ sn-network. An sn-network for a sequential space is a weak base.

Definition 2.5. Let $f\colon X \longrightarrow Y$ be a mapping.

(1) f is called an almost-open mapping [9] if for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y for each neighborhood U of x.

(2) f is called an almost weak-open mapping [5] if there exists a weak base $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y satisfying the condition: for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x, $P \subset f(U)$ for some $P \in \mathcal{P}_y$.

(3) f is called an almost sn-open mapping [5] if there exists an sn-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y satisfying the condition: for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x, $P \subset f(U)$ for some $P \in \mathcal{P}_y$.

(4) f is called pseudo-open [2] if for each $y \in Y$ and each neighborhood U of $f^{-1}(y)$ in X, $f(U)$ is a neighborhood of y in Y.

(5) f is called a quotient mapping [3] if U is open in Y if and only if $f^{-1}(U)$ is open in X.

(6) f is called a 1-sequence-covering mapping [11] if for each $y \in Y$ there exists $x \in f^{-1}(y)$, such that whenever $\{y_n\}$ is a sequence converging to y in Y, there exists a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(7) f is called a sequence-covering mapping [11] if whenever $\{y_n\}$ is a convergent sequence in Y, there exists a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$.

(8) f is called a compact mapping [11] if $f^{-1}(y)$ is a compact subset of X for each $y \in Y$.

Remark 2.6. The following implications hold (see [5, Remark 2.7], [13, Proposition 3.2], [9, Remark 1.2.2]).

$$\begin{array}{ccccccc}
\text{almost-open} & \Longrightarrow & \text{almost weak-open} & \Longrightarrow & \text{almost } sn\text{-open} & \Longleftarrow & \text{1-sequence-covering} \\
\Downarrow & & \Downarrow & & & & \\
\text{pseudo-open} & \Longrightarrow & \text{quotient} & & & &
\end{array}$$

Remark 2.7. (1) Quotient mappings preserve sequential spaces [9, Lemma 1.4.3(2)].

(2) Pseudo-open mappings preserve Fréchet spaces [9, Lemma 1.4.3(3)].

(3) Every sequence-covering, compact mapping from a metric space is 1-sequence-covering [11, Theorem 4.4].

Lemma 2.8. [5, Proposition 2.13] *Let $f\colon X \longrightarrow Y$ be a mapping. Then the following hold.*

(1) *If f is 1-sequence-covering, then f is almost sn-open.*

(2) *If f is almost sn-open and X is first countable, then f is 1-sequence-covering.*

By Remark 2.4 and Remark 2.7, we have the following lemma.

Lemma 2.9. *Let $f\colon X \longrightarrow Y$ be an almost sn-open mapping. If one of the following two conditions is satisfied, then f is almost weak-open.*

(1) *Y is a sequential space.*

(2) *X is a sequential space and f is a quotient mapping.*

Corollary 2.10. *Let $f\colon X \longrightarrow Y$ be a mapping. If X is a sequential space, then the following are equivalent.*

(1) *f is an almost weak-open mapping.*

(2) *f is an almost sn-open, quotient mapping.*

Remark 2.11. "X is a sequential space" in Corollary 2.10 cannot be omitted. In fact, S. Lin gave a 1-sequence-covering, quotient mapping $f\colon X \longrightarrow Y$ such that f is not almost weak-open [10, Example 1]. By Lemma 2.8, f is almost *sn*-open, quotient.

The following theorem is obtained from Lemma 2.8 and Corollary 2.10.

Theorem 2.12. *Let $f\colon X \longrightarrow Y$ be a mapping. If X is first countable (especially, if X is metric), then the following are equivalent.*

(1) *f is an almost weak-open mapping.*

(2) *f is an almost sn-open, quotient mapping.*

(3) *f is a 1-sequence-covering, quotient mapping.*

Corollary 2.13. *The following are equivalent for a space X, where P denotes some mapping property.*

(1) *X is an almost weak-open, P-image of a metric space.*

(2) *X is an almost sn-open, quotient, P-image of a metric space.*

(3) *X is a 1-sequence-covering, quotient, P-image of a metric space.*

Lemma 2.14. *Let X be a Fréchet space and $x \in X$. If P is a sequential neighborhood of x in X, then $x \in P^\circ$, where P° is the interior of P.*

Proof. Let P be a sequential neighborhood of x in X. If $x \notin P^\circ$, then $x \in X - P^\circ = \overline{X - P}$. There is a sequence $\{x_n\}$ in $X - P$ converging to x because X is Fréchet. This contradicts that P is a sequential neighborhood of x in X. So $x \in P^\circ$. ∎

Proposition 2.15. *Let $f\colon X \longrightarrow Y$ be an almost sn-open mapping. If one of the following two conditions is satisfied, then f is almost open.*

(1) *Y is Fréchet.*

(2) *f is pseudo-open and X is Fréchet.*

Proof. Pseudo-open mappings preserve Fréchet spaces from Remark 2.7(2), so condition (2) implies condition (1). Thus we only need to prove that f is almost-open if condition (1) is satisfied.

Let Y be Fréchet. Since f is almost *sn*-open there is an *sn*-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y such that for each $y \in Y$, there is $x \in f^{-1}(y)$ satisfying the condition in Definition 2.5(6). Whenever U is a neighborhood of x in X, then $P \subset f(U)$ for some $P \in \mathcal{P}_y$. Note that P is a sequential neighborhood of y in Y. So $f(U)$ is a sequential neighborhood of y in Y. By Lemma 2.14, $y \in (f(U))^\circ$, so $f(U)$ is a neighborhood of y in Y. This proves that f is almost-open. ∎

Corollary 2.16. *Let $f\colon X \longrightarrow Y$ be a mapping. If X is a Fréchet space, then the following are equivalent.*

(1) *f is an almost open mapping.*

(2) *f is an almost sn-open, pseudo-open mapping.*

The following theorem is obtained from Theorem 2.12 and Corollary 2.16.

Theorem 2.17. *Let $f\colon X \longrightarrow Y$ be a mapping. If X is first countable, then the following are equivalent.*

(1) *f is an almost-open mapping.*

(2) *f is an almost weak-open, pseudo-open mapping.*

(3) *f is an almost sn-open, pseudo-open mapping.*

(4) *f is a 1-sequence-covering, pseudo-open mapping.*

Proposition 2.18. *The following are equivalent for a space X, where P denotes some mapping property.*

(1) *X is an almost open, P-image of a metric space.*

(2) *X is an almost weak-open, pseudo-open, P-image of a metric space.*

(3) *X is an almost sn-open, pseudo-open, P-image of a metric space.*

(4) *X is a 1-sequence-covering, pseudo-open, P-image of a metric space.*

3. The Arens space S_2

In this section, we investigate the Arens space S_2.

Definition 3.1. Let $L_0 = \{a_n : n \in \mathbf{N}\}$ be a sequence converging to ∞, where $\infty \notin L_0$. For each $n \in \mathbf{N}$, let L_n be a sequence converging to b_n, where $b_n \notin L_n$. Put $T_0 = L_0 \bigcup\{\infty\}$ and $T_n = L_n \bigcup\{b_n\}$ for each $n \in \mathbf{N}$. Let M be the topological sum of $\{T_n : n \geq 0\}$. Let S_2 be the quotient space obtained from the topological sum M by identifying a_n with b_n for each $n \in \mathbf{N}$. Then S_2 is called the Arens space [1, 8, 12]. Let $q\colon M \longrightarrow S_2$ be the natural mapping.

It is clear that M is a metric space. The following proposition comes from [12, Proposition 1.6(2)].

Proposition 3.2. *S_2 is a sequential space. But S_2 is not Fréchet.*

Proposition 3.3. *q is 1-sequence-covering, quotient.*

Proof. q is quotient because every natural mapping from a space on its quotient space is quotient. It is clear that q is a compact mapping. By Remark 2.7(3), we only need to prove that q is sequence-covering. Let $\{y_n\}$ be a sequence converging to y in S_2. Without loss of generality, we can assume that $\{y_n\}$ is a nontrivial sequence. Thus $y = q(\infty)$ or $y = q(a_m) = q(b_m)$ for some $m \in \mathbf{N}$.

Case 1: $y = q(\infty) \in S_2$. Since $\{y_n\}$ is a nontrivial sequence, there exists $k \in \mathbf{N}$ such that $y_n \in q(L_0)$ for all $n > k$. For each $n \leq k$, choose $x_n \in q^{-1}(y_n)$; for each $n > k$, choose $x_n \in q^{-1}(y_n) \bigcap L_0$. It is easy to check that $\{x_n\}$ is a sequence converging to ∞ in M.

Case 2: $y = q(a_m) = q(b_m)$ for some $m \in \mathbf{N}$. Since $\{y_n\}$ is a nontrivial sequence, there exists $k \in \mathbf{N}$ such that $y_n \in q(L_m)$ for all $n > k$. For each $n \leq k$, choose $x_n \in q^{-1}(y_n)$; for each $n > k$, choose $x_n \in q^{-1}(y_n) \bigcap L_m$. It is easy to check that $\{x_n\}$ is a sequence converging to b_m in M. ■

COROLLARY 3.4. (1) *q is almost weak-open, and so q is almost sn-open.*

(3) *q is not pseudo-open, and so q is not almost open.*

Proof. (1) By Proposition 3.3, q is 1-sequence-covering, quotient. Note that M is first countable. So q is almost weak-open from Theorem 2.12.

(3) By Proposition 3.2, S_2 is not a Fréchet space. Note that M is Fréchet. So q is not pseudo-open from Remark 2.7(2). ■

REMARK 3.5. By Proposition 3.2, Proposition 3.3 and Corollary 3.4, we have the following facts, which show that Proposition 1.1, Proposition 1.2 and Proposition 1.3 are not true.

(1) "f is almost weak-open" in Lemma 2.9 cannot be replaced by "f is almost open".

(2) "Pseudo-open" in Theorem 2.17(2)(resp. Corollary 2.18(2)) cannot be omitted.

(3) "Pseudo-open" in Theorem 2.17(3)(4) (resp. Corollary 2.18(3)(4)) cannot be relaxed to "quotient".

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