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ON ORIENTED GRAPH SCORES

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Abstract. In this paper, we obtain some results concerning the scores in oriented graphs. Further, we give a new and direct proof of the Theorem on oriented graph scores due to Avery [1].

1. Introduction

A tournament is an irreflexive, complete, asymmetric digraph, and the score s_v of a vertex v in a tournament is the number of arcs directed away from that vertex. The score sequence (or score structure) $S(T)$ of a tournament T is formed by listing the scores in non-decreasing order. Landau [3] in 1953 characterised the score sequences of a tournament.

THEOREM 1 [3]. *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} s_i \geq \binom{|I|}{2}, \tag{1}$$

with equality when $|I| = n$, where $|I|$ is the cardinality of the set $|I|$.

Since $s_1 \leq \cdots \leq s_n$, the inequality (1), called Landau inequalities, is equivalent to $\sum_i^k s_i \geq \binom{k}{2}$, for $k = 1, 2, \cdots, n-1$, and equality for $k = n$.

An oriented graph is a digraph with no symmetric pairs of directed arcs and without self loops. If D is an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and if $d^+(v)$ and $d^-(v)$ are respectively, the outdegree and indegree of a vertex v, then $a_v = n - 1 + d^+(v) - d^-(v)$ is called the score of v. Clearly, $0 \leq a_v \leq 2n - 2$. The score sequence $A(D)$ of D is formed by listing the scores in non-decreasing order. For any two vertices u and v in an oriented graph D, we have one of the following possibilities.

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(i) An arc directed from u to v, denoted by $u(1-0)v$, (ii) An arc directed from v to u, denoted by $u(0-1)v$, (iii) There is no arc from u to v and there is no arc from v to u, and is denoted by $u(0-0)v$.

If $d^*(v)$ is the number of those vertices u in D which have $v(0-0)u$, then $d^+(v)+d^-(v)+d^*(v)=n-1$. Therefore, $a_v = 2d^+(v)+d^*(v)$. This implies that each vertex u with $v(1-0)u$ contributes two to the score of v. Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$, and each $v(0-0)u$ contributes two(one each at u and v) to scores, therefore the sum total of all the scores is $2\binom{n}{2}$. With this scoring system, player v receives a total of a_v points.

A triple in an oriented graph is an induced oriented subgraph with three vertices. For any three vertices u, v and w, the triples of the form $u(1-0)v(1-0)w(1-0)u$, or $u(1-0)v(1-0)w(0-0)u$ are said to be intransitive, while as the triples of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$ are said to be transitive.

Theorem 2 [1]. *A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of an oriented graph if and only if for each $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i\in I} a_i \geq 2\binom{|I|}{2}, \tag{2}$$

with equality when $|I| = n$.

Since $a_1 \leq a_2 \leq \cdots \leq a_n$, the inequality (2) is equivalent to

$$\sum_{i}^{k} a_i \geq 2\binom{k}{2}, \quad \text{for} \quad k = 1, 2, \cdots, n-1 \tag{3}$$

with equality for $k = n$.

2. Main results

We give a constructive proof of the sufficiency of Theorem 2, based on the proof of Griggs and Reid [2] for Theorem 1 on tournaments. This proof is more direct in comparison to the previous existing ones. First we have the following results.

Theorem 3. *Let $A = [a_i]_1^n$ be a sequence of non-negative integers with $a_1 \leq a_2 \leq \cdots < a_k = a_{k+1} = \cdots = a_{k+m-1} < a_{k+m} \leq a_{k+m+1} \leq \cdots \leq a_n$ and let $A' = [a'_i]_1^n$ where* $a'_i = \begin{cases} a_i - 1, & \text{for } i = k \\ a_i + 1, & \text{for } i = k+m-1 \\ a_i, & \text{otherwise.} \end{cases}$

Then A is a score sequence of some oriented graph if and only if A' is a score sequence of an oriented graph.

Proof. Clearly, $k \geq 1$ and $m \geq 2$, so that either $k+m-1 = n$, or $a_k = a_{k+1} = \cdots = a_{k+m-1} < a_{k+m}$. For $1 \leq i \leq n$ $A' = [a'_i]_1^n$ where $a'_i = \begin{cases} a_i - 1, & \text{for } i = k \\ a_i + 1, & \text{for } i = k+m-1 \\ a_i, & \text{otherwise.} \end{cases}$ Obviously, $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Let A' be the score sequence of some oriented graph D' of order n in which vertex v'_i has score a'_i, $1 \leq i \leq n$. Then $a'_{k+m-1} = a'_k + 2$. If either $v'_{k+m-1}(1-0)v'_k$, or $v'_{k+m-1}(0-0)v'_k$ then making respectively, the transformation $v'_{k+m-1}(0-0)v'_k$, or $v'_k(1-0)v'_{k+m-1}$, gives an oriented graph of order n with score sequence A.

If $v'_k(1-0)v'_{k+m-1}$, since $a'_k \leq a'_{k+m-1}$ there exists at least one vertex v'_j in $V' - \{v'_k, v'_{k+m-1}\}$ such that triple formed by v'_k, v'_{k+m-1} and v'_j is transitive and of the form $v'_k(1-0)v'_{k+m-1}(1-0)v'_j(1-0)v'_k$ or $v'_k(1-0)v'_{k+m-1}(1-0)v'_j(0-0)v'_k$ or $v'_k(1-0)v'_{k+m-1}(0-0)v'_j(1-0)v'_k$. These can be transformed respectively to $v'_k(1-0)v'_{k+m-1}(0-0)v'_j(0-0)v'_k$ or $v'_k(1-0)v'_{k+m-1}(1-0)v'_j(0-1)v'_k$ or $v'_k(1-0)v'_{k+m-1}(0-1)v'_j(0-0)v'_k$, and we obtain an oriented graph of order n with score sequence A.

If for every vertex $V' - \{v'_k, v'_{k+m-1}\}$ the triple formed by v'_k, v'_{k+m-1} and v'_j is transitive, we again get a contradiction.

Now, let A be the score sequence of some oriented graph D of order n in which vertex v_i has score a_i, $1 \leq i \leq n$. We have $a_{k+m-1} = a_k$. If either $v_k(1-0)v_{k+m-1}$, or $v_k(0-0)v_{k+m-1}$, then making respectively, the transformation $v_k(0-0)v_{k+m+1}$, or $v_k(0-1)v_{k+m+1}$, gives an oriented graph of order n with score sequence A'. If $v_{k+m+1}(1-0)v_k$, we claim that there exists at least one vertex $v_j \in V - \{v_{k+m+1}, v_k\}$ such that the triple formed by the vertices v_{k+m+1}, v_k and v_j is intransitive, and of the form $v_{k+m+1}(1-0)v_k(1-0)v_j(1-0)v_{k+m+1}$, or $v_{k+m+1}(1-0)v_k(1-0)v_j(0-0)v_{k+m+1}$, or $v_{k+m+1}(1-0)v_k(0-0)v_j(1-0)v_{k+m+1}$. These can be transformed respectively to $v_{k+m+1}(1-0)v_k(0-0)v_j(0-0)v_{k+m+1}$, or $v_{k+m+1}(1-0)v_k(0-0)v_j(0-1)v_{k+m+1}$, or $v_{k+m+1}(1-0)v_k(0-1)v_j(0-0)v_{k+m+1}$ and we obtain an oriented graph of order n with score sequence A'.

In case for every vertex $v_j \in V - \{v_k, v_{k+m-1}\}$ the triple formed by v_{k+m-1}, v_k and v_j is transitive, we again get a contradiction.

Thus, A' is a score sequence if and only if A is a score sequence. ∎

THEOREM 4. *Let $A = [a_i]_1^n$ be a sequence of non-negative integers in non-decreasing order with at least two odd terms a_k and a_m (say) with $a_k < a_m$ and let* $A' = [a'_i]_1^n$ *with* $a'_i = \begin{cases} a_i - 1, & \textit{for } i = k \\ a_i + 1, & \textit{for } i = k+m-1 \\ a_i, & \textit{otherwise.} \end{cases}$

Then A is a score sequence if and only if A' is a score sequence.

Proof. Let a_k be the lowest odd term, and a be the greatest odd term and let

$A' = [a'_1, a'_2, \cdots, a'_n]$, where $a'_i = \begin{cases} a_i - 1, & \text{for } i = k \\ a_i + 1, & \text{for } i = k + m - 1 \\ a_i, & \text{otherwise.} \end{cases}$

Clearly, $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Let A' be the score sequence of some oriented graph D' of order n in which vertex v'_i has score a'_i, $1 \leq i \leq n$. Then, $a'_m \geq a'_k + 2$ with equality appearing when the two odd terms are same. Therefore, it follows by the argument used in Theorem 3, that A is the score sequence of some oriented graph D of order n in which vertex v_i has score a_i, $1 \leq i \leq n$. We have $a_m \geq a_k$. The equality appears when the two odd terms are same, and in this case A' is a score sequence of some oriented graph of order n, again by Theorem 3. If $a_m > a_k$, then $a_m \geq a_k + 2$, since $a_m = a_k + 1$ implies that one of a_k or a_m is even, which contradicts the choice of a_k and a_m. Thus, by using again the argument as in Theorem 3, it follows that A' is a score sequence of some oriented graph of order n. ∎

Lemma 5. *(a) Let A and A' be given as in Theorem 3. Then A satisfies (3) if and only if A' satisfies (3).*

(b) Let A and A' be given as in Theorem 4. Then A satisfies (3) if and only if A' satisfies (3).

Proof. (a) If A satisfies (3), then $\sum_{i=1}^{j} a'_i = \sum_{i=1}^{j} a_i$, or $\sum_{i=1}^{k-1} a_i + (a_k - 1) + \sum_{i=k+1}^{j} a_i$, or $\sum_{i=1}^{k} a_i + (a_k - 1) + \sum_{i=k+1}^{k+m-2} + (a_{k+m-1} + 1) + \sum_{i=k+m}^{j} a_i$ according to $j \leq k - 1$, or $k \leq j \leq k + m - 2$, or $j \geq k + m - 1$ respectively.

If $j \leq k - 1$ and $j \geq k + m - 1$, then $\sum_{i=1}^{j} a'_i \geq j(j - 1)$. If $k \leq j \leq k + m - 2$, claim $\sum_{i=1}^{j} a_i > j(j - 1)$, for $k \leq j \leq k + m - 2$.

Assume to the contrary, that for some j, $k \leq j < k+m-2$, $\sum_{i=1}^{j} a_i \leq j(j-1)$. For (3), we have $\sum_{i=1}^{j} a_i \geq j(j-1)$. Combining the two, we obtain $\sum_{i=1}^{j} a_i = j(j-1)$. Therefore, again by (3), we have $a_{j+1} + j(j-1) = a_{j+1} + \sum_{i=1}^{j} a_i = \sum_{i=1}^{j+1} a_i \geq j(j+1) = j(j-1+2) = j(j-1)+2j$. That is, $a_{j+1} \geq 2j$. Also, $a_j = a_{j+1}$ implies that $a_j \geq 2j$. Thus, $\sum_{i=1}^{j} a_i = \sum_{i=1}^{j-1} a_i + a_j \geq (j-1)(j-2) + 2j = j(j-1) - (j-1) + 2j$. Therefore $\sum_{i=1}^{j} a_i \geq j(j - 1) + 2 > j(j - 1)$, contradicting the assumption. Hence,

$$\sum_{i=1}^{j} a_i > j(j - 1), \text{for } k \leq j \leq k + m - 2. \qquad (4)$$

Thus, when $k \leq j \leq k+m-2$, using (4), we obtain $\sum_{i=1}^{j} a'_i = \sum_{i=1}^{j} a_i - 1 > j(j-1)$.

Therefore in all cases A' satisfies (3). Now, if A' satisfies (3), it can be easily seen that A also satifies (3). Proof of (b) follows similarly. ∎

Proof of Theorem 2. Necessity. It can be seen in [4].

Sufficiency. Let the sequence $A = [a_i]_1^n$ of non-negative integers n non-decreasing order satisfy (3). Clearly, the sequence $A = [0, 2, 4, \cdots, 2n - 2]$ satisfies (3), since it is the score sequence of the transitive tournament of order n. Now, if any sequence $A \neq A_n$ satisfies (3), then $a_1 \geq 0$ and $a_n \leq 2n-2$. We claim that A contains either (a) a repeated term, or (b) at least two odd terms, or both (a) and (b). To verify the claim, suppose that there is no repeated term. If at least one term is odd, then a parity argument shows that there are at least two odd terms. So assume that all terms are even. Therefore, $a_1 \geq 0$, $a_2 > a_1$, and a_2 even imply that $a_2 \geq 2$. And $a_2 \geq 2$ $a_3 > a_2$, and a_3 even imply that $a_3 \geq 4$. Inductively, $a_i \geq 2(i-1)$, for all $1 \leq i \leq n$. Thus, $n(n-1) = \sum_{i=1}^{n} a_i \geq 2\sum_{i=1}^{j}(i-1) = n(n-1)$. This implies that equality holds throughout. Thus, $a_i = 2(i-1)$, for all $1 \leq i \leq n$, and $A = A_n$, a contradiction. Consequently, if there is no repeated term, then at least two terms are odd.

We produce a new sequence A' from A which also satisfies (3), A' is closer to A_n than A, and A' is a score sequence if and only if A is a score sequence. When A contains a repeated term, reduce the first occurrence of that of that repeated term in A by one and increase the last occurrence of that repeated term by one to form A'. If A contains at least two odd terms, reduce the first odd term by one and increase the last odd term by one to form A'. The process is repeated until the sequence A_n is obtained. Let the total order on the non-negative integer sequences be defined by $X = [x_1, x_2, \cdots, x_n] \preceq Y = [y_1, y_2, \cdots, y_n]$ if either $X = Y$, or $x_i < y_i$ for some i, $1 \leq i \leq n$, and $x_{i+1} = y_{i+1}, \cdots, x_n = y_n$. Clearly, $\preceq$ is reflexive, antisymmetric and satisfies comparability. We write $X \prec Y$, if $X \prec Y$ but $X \neq Y$. For any sequence $A \neq A_n$, satisfies (3), $A \prec A_n$, where $A_n = [0, 2, 4, \cdots, 2n-2]$, the score sequence of a transitive tournament of order n. Thus, we have shown that for any sequence A' satisfies (3), we can form another sequence A' satisfying (3)(By Lemma 5) such that $A \prec A'$, and A is a score sequence if and only if A' is a score sequence (By Theorem 3 and 4). Therefore, by the repeated application of this transformation, starting from the original sequence satisfying (3), we reach A_n. Hence, A is a score sequence. ■

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