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A CLASS OF MULTIVALENT HARMONIC FUNCTIONS INVOLVING A GENERALIZED RUSCHEWEYH TYPE OPERATOR

Waggas Galib Atshan, S. R. Kulkarni and R. K. Raina

Abstract. A class of p-valent harmonic functions associated with a certain generalized Ruscheweyh type operator is introduced. Among the various properties investigated for this class of functions are the results giving the coefficient bounds, distortion properties and extreme points.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbf{C}$ if both u and v are real harmonic in $\mathbf{C}$. In any simply-connected domain $D \subset \mathbf{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D. See Clunie and Sheil-Small [3].

Denote by $\mathcal{H}(p)$ the class of functions $f = h+\overline{g}$ that are harmonic multivalent and sense-preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. For $f = h + \overline{g} \in \mathcal{H}(p)$, we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad g(z) = \sum_{n=p}^{\infty} b_n z^n, \quad |b_p| < 1. \tag{1.1}$$

Let $W(p)$ denote the subclass of $\mathcal{H}(p)$ consisting of functions $f = h + \overline{g}$, where h and g are given by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n, \quad g(z) = -\sum_{n=p}^{\infty} |b_n| z^n, \quad |b_p| < 1. \tag{1.2}$$

We introduce here a new class $\mathcal{H}_{\lambda}^{k}(p,\alpha,\beta)$ of harmonic functions of the form (1.1) that satisfy the inequality

$$\operatorname{Re}\left\{(1-\beta)\frac{D_{\lambda}^{k+p-1} f(z)}{z^p} + \beta\frac{(D_{\lambda}^{k+p-1} f(z))'}{pz^{p-1}}\right\} \geq \frac{\alpha}{p}, \tag{1.3}$$

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where $0 \leq \alpha < p$, $p \in \mathbf{N} = \{1, 2, \dots\}$, $\lambda \geq 0, \beta \geq 0, k \in \mathbf{N}_0$ and

$$D_\lambda^{k+p-1} f(z) = D_\lambda^{k+p-1} h(z) + \overline{D_\lambda^{k+p-1} g(z)}. \tag{1.4}$$

The operator D_λ^{k+p-1} denotes the generalized Ruscheweyh derivative operator introduced in [2]. For h and g given by (1.1), we obtain

$$D_\lambda^{k+p-1} h(z) = z^p + \sum_{n=p+1}^{\infty} (1 + \lambda(n-p)) C(k, n, p) a_n z^n, \tag{1.5}$$

$$D_\lambda^{k+p-1} g(z) = \sum_{n=p}^{\infty} (1 + \lambda(n-p)) C(k, n, p) b_n z^n, \tag{1.6}$$

where $\lambda \geq 0$, $p \in \mathbf{N}$, $k > -p$ and

$$C(k, n, p) = \binom{n+k-1}{k+p-1}. \tag{1.7}$$

We deem it worthwhile to point here the relevance of the function class $\mathcal{H}_\lambda^k(p, \alpha, \beta)$ with those classes of functions which have been studied recently. Indeed, we observe that:

(i) $\mathcal{H}_0^0(1, \alpha, 1) \equiv N_H(\alpha)$ (Ahuja and Jahangiri [1]);

(ii) $\mathcal{H}_\lambda^k(p, \alpha, 1) \equiv \mathcal{H}_\lambda^k(p, \alpha)$ (Al Shaqsi and Darus [2]);

(iii) $\mathcal{H}_\lambda^k(1, 0, 1) \equiv \mathcal{H}_\lambda^k$ (Darus and Al Shaqsi [4]);

(iv) $\mathcal{H}_0^0(1, 0, 1) \equiv S_{\mathcal{H}}^*$ (Silverman [6]);

(v) $\mathcal{H}_\lambda^0(1, 0, 1) \equiv H(\lambda)$ (Yalçin and Öztürk][7]).

Also, we note that the analytic part of the class $\mathcal{H}_0^k(p, \alpha, 1)$ was introduced and studied by Goel and Sohi [5].

We further denote by $W_\lambda^k(p, \alpha, \beta)$ the subclass of $\mathcal{H}_\lambda^k(p, \alpha, \beta)$ that satisfies the relation

$$W_\lambda^k(p, \alpha, \beta) = W(p) \cap \mathcal{H}_\lambda^k(p, \alpha, \beta). \tag{1.8}$$

In this paper we study a class of p-valent harmonic functions involving a certain generalized Ruscheweyh type operator. We obtain the coefficient bounds, distortion properties and extreme points for this class of functions.

2. Coefficient bounds

THEOREM 1. *Let $f = h + \overline{g}$ (h and g being given by (1.1)). If*

$$\sum_{n=p+1}^{\infty} ((n-p)\beta + p)(1 + \lambda(n-p)) C(k, n, p) |a_n| + \\ + \sum_{n=p}^{\infty} ((n-p)\beta + p)(1 + \lambda(n-p)) C(k, n, p) |b_n| \leq p - \alpha, \tag{2.1}$$

where $\lambda \geq 0$, $\beta \geq 0$, $0 \leq \alpha < p$, $p \in \mathbf{N}$ and $k \in \mathbf{N}_0$, then f is harmonic p-valent sense-preserving in $\mathcal{U}$ and $f \in \mathcal{H}_\lambda^k(p, \alpha, \beta)$.

Proof. Let $w(z) = (1-\beta)\frac{D_\lambda^{k+p-1}f(z)}{z^p} + \beta\frac{(D_\lambda^{k+p-1}f(z))'}{pz^{p-1}}$. To prove that $\operatorname{Re}\{w\} \geq \frac{\alpha}{p}$, it is sufficient to show equivalently that $|p-\alpha+pw(z)| \geq |p+\alpha-pw(z)|$. Substituting for $w(z)$ and making use of (1.4) to (1.6), and resorting to simple calculations, we find that

$$|p-\alpha+pw(z)| \geq 2p-\alpha - \sum_{n=p+1}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|a_n||z^{n-p}| - \sum_{n=p}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|b_n||z^{n-p}| \quad (2.2)$$

and

$$|p+\alpha-pw(z)| \leq \alpha + \sum_{n=p+1}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|a_n||z^{n-p}| + \sum_{n=p}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|b_n||z^{n-p}|, \quad (2.3)$$

where $C(k,n,p)$ is given by (1.7). Evidently, (2.2) and (2.3) in conjunction with (2.1) yields

$$|p-\alpha+pw(z)| - |p+\alpha-pw(z)| \geq 0.$$

The harmonic functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty}\frac{x_n}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)}z^n + \sum_{n=p}^{\infty}\frac{\overline{y}_n}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)}(\overline{z})^n \quad (2.4)$$

$(\sum_{n=p+1}^{\infty}|x_n| + \sum_{n=p}^{\infty}|y_n| = p-\alpha)$ show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.4) are in $\mathcal{H}_\lambda^k(p,\alpha,\beta)$ because in view of (2.1), we infer that

$$\sum_{n=p+1}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|a_n| + \sum_{n=p}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|b_n| = \sum_{n=p+1}^{\infty}|x_n| + \sum_{n=p}^{\infty}|y_n| = p-\alpha.$$

The restriction imposed in Theorem 1 on the moduli of the coefficients of $f = h+\overline{g}$ implies that for arbitrary rotation of the coefficients of f, the resulting functions would still be harmonic multivalent and $f \in \mathcal{H}_\lambda^k(p,\alpha,\beta)$. ∎

The following theorem shows that the condition (2.1) is also necessary for function f to belong to $W_\lambda^k(p,\alpha,\beta)$.

Theorem 2. *Let $f = h+\overline{g}$ with h and g are given by (1.2). Then $f \in W_\lambda^k(p,\alpha,\beta)$ if and only if*

$$\sum_{n=p+1}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|a_n| + \sum_{n=p}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|b_n| \leq p-\alpha, \quad (2.5)$$

where $\lambda \geq 0$, $\beta \geq 0$, $0 \leq \alpha < p$, $p \in \mathbf{N}$ *and* $k \in \mathbf{N}_0$.

Proof. By noting that $W_\lambda^k(p,\alpha,\beta) \subset \mathcal{H}_\lambda^k(p,\alpha,\beta)$, the sufficiency part of Theorem 2 follows at once from Theorem 1. To prove the necessary part, let us assume that $f \in W_\lambda^k(p,\alpha,\beta)$. Using (1.3), we get

$$\operatorname{Re}\left\{(1-\beta)\left(\frac{D_\lambda^{k+p-1}h(z)+\overline{D_\lambda^{k+p-1}g(z)}}{z^p}\right)+\right.$$
$$\left.+\beta\left(\frac{(D_\lambda^{k+p-1}h(z))'+\overline{(D_\lambda^{k+p-1}g(z))'}}{pz^{p-1}}\right)\right\}$$
$$=\operatorname{Re}\left\{1-\sum_{n=p+1}^{\infty}((\frac{n}{p}-1)\beta+1)(1+\lambda(n-p))C(k,n,p)|a_n|z^{n-p}-\right.$$
$$\left.-\sum_{n=p}^{\infty}((\frac{n}{p}-1)\beta+1)(1+\lambda(n-p))C(k,n,p)|b_k|(\overline{z})^{n-p}\right\}\geq\frac{\alpha}{p}.$$

If we choose z to be real and let $z \to 1^-$, we obtain

$$1-\sum_{n=p+1}^{\infty}((\frac{n}{p}-1)\beta+1)(1+\lambda(n-p))C(k,n,p)|a_n|-$$
$$-\sum_{n=p}^{\infty}((\frac{n}{p}-1)\beta+1)(1+\lambda(n-p))C(k,n,p)|b_n|\geq\frac{\alpha}{p}.$$

Hence

$$\sum_{n=p+1}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|a_n|+$$
$$+\sum_{n=p}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)|b_n|\leq p-\alpha,$$

which completes the proof of Theorem 2. ■

3. Distortion bounds and extreme points

In this section we obtain the distortion bounds for functions belonging to the class $W_\lambda^k(p,\alpha,\beta)$ and also provide extreme points for this class $W_\lambda^k(p,\alpha,\beta)$.

Theorem 3. *If* $f \in W_\lambda^k(p,\alpha,\beta)$, *for* $\lambda \geq 0$, $\beta \geq 0$, $0 \leq \alpha < p$, $p \in \mathbf{N}$, $k \in \mathbf{N}_0$ *and* $|z| = r > 1$, *then*

$$|f(z)| \leq (1+|b_p|)r^p + \frac{(p-\alpha)-|b_p|}{(\beta+p)(\lambda+1)(p+k)}r^{p+1}, \tag{3.1}$$

and

$$|f(z)| \geq (1-|b_p|)r^p - \frac{(p-\alpha)-|b_p|}{(\beta+p)(\lambda+1)(p+k)}r^{p+1}. \tag{3.2}$$

Proof. We only prove the first inequality (3.1). The proof for the second inequality (3.2) is similar, and is hence omitted.

Suppose $f \in W_\lambda^k(p,\alpha,\beta)$. Using (1.1) and (2.1) of Theorem 1, we find that

$$
\begin{aligned}
|f(z)| &\le (1+|b_p|)r^p + \sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^n \le (1+|b_p|)r^p + \sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^{p+1} \\
&= (1+|b_p|)r^p + \frac{1}{(\beta+p)(1+\lambda)(p+k)}\times \\
&\quad \sum_{n=p+1}^{\infty}(\beta+p)(1+\lambda)(p+k)(|a_n|+|b_n|)r^{p+1} \\
&\le (1+|b_p|)r^p + \frac{1}{(\beta+p)(1+\lambda)(p+k)}\times \\
&\quad \sum_{n=p+1}^{\infty}((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)(|a_n|+b_n|)r^{p+1} \\
&\le (1+|b_p|)r^p + \frac{1}{(\beta+p)(1+\lambda)(p+k)}[(p-\alpha)-|b_p|]r^{p+1}.
\end{aligned}
$$

The bounds given in Theorem 3 (for the functions $f = h + \overline{g}$ of the form (1.2)) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

$$f(z) = z^p + |b_p|(\overline{z})^p + \frac{(p-\alpha)-|b_p|}{(\beta+p)(1+\lambda)(p+k)}(\overline{z})^{p+1} \tag{3.3}$$

and

$$f(z) = z^p - |b_p|(\overline{z})^p - \frac{(p-\alpha)-|b_p|}{(\beta+p)(1+\lambda)(p+k)}(\overline{z})^{p+1} \tag{3.4}$$

for $|b_p| < 1$ show that the bounds given in Theorem 3 are sharp. ∎

The covering result given below in Corollary 1 follows from the inequality (3.2) of Theorem 3.

Corollary 1. *If $f \in W_\lambda^k(p,\alpha,\beta)$, then*

$$\left\{ w : |w| < (1-|b_p|) - \frac{(p-\alpha)-|b_p|}{(\beta+p)(\lambda+1)(k+p)} \right\} \subset f(\mathcal{U}). \tag{3.5}$$

The next theorem gives the extreme points of the closed convex hulls of $W_\lambda^k(p,\alpha,\beta)$, denoted by $clcoW_\lambda^k(p,\alpha,\beta)$

Theorem 4. *$f \in clcoW_\lambda^k(p,\alpha,\beta)$ if and only if*

$$f(z) = \sum_{n=p}^{\infty}(\sigma_n h_n + \mathcal{E}_n g_n), \tag{3.6}$$

where $z \in \mathcal{U},\ h_p(z) = z^p,$

$$h_n(z) = z^p - \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} z^n, \quad (n = p+1, p+2\dots) \tag{3.7}$$

$$g_n(z) = z^p - \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} (\overline{z})^n, \quad (n = p, p+1, \dots) \tag{3.8}$$

and

$$\sum_{n=p}^{\infty} (\sigma_n + \mathcal{E}_n) = 1 (\sigma_n \geq 0,\ \mathcal{E}_n \geq 0).$$

In particular, the extreme points of $W_\lambda^k(p,\alpha,\beta)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. Suppose $f(z)$ is of the form (3.6). Using (3.7) and (3.8), we get

$$\begin{aligned} f(z) &= \sum_{n=p}^{\infty} (\sigma_n h_n + \mathcal{E}_n g_n) \\ &= \sum_{n=p}^{\infty} (\sigma_n + \mathcal{E}_n) z^p - \sum_{n=p+1}^{\infty} \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} \sigma_n z^n \\ &\quad - \sum_{n=p}^{\infty} \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} \mathcal{E}_n (\overline{z})^n \\ &= z^p - \sum_{n=p+1}^{\infty} \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} \sigma_n z^n \\ &\quad - \sum_{n=p}^{\infty} \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} \mathcal{E}_n (\overline{z})^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=p+1}^{\infty} [((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)] \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} \sigma_n \\ &+ \sum_{n=p}^{\infty} [((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)] \frac{p-\alpha}{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)} \mathcal{E}_n \\ &\qquad = (p-\alpha) \left(\sum_{n=p}^{\infty} (\sigma_n + \mathcal{E}_n) - \sigma_p \right) = (p-\alpha)(1-\sigma_p) \leq p-\alpha \end{aligned}$$

which implies that $f \in clcoW_\lambda^k(p,\alpha,\beta)$.

Conversely, assume that $f \in W_\lambda^k(p,\alpha,\beta)$. Putting

$$\sigma_n = \frac{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)}{p-\alpha} |a_n| \quad (n = p+1, p+2, \dots),$$

$$\mathcal{E}_n = \frac{((n-p)\beta+p)(1+\lambda(n-p))C(k,n,p)}{p-\alpha} |b_n| \quad (n = p, p+1, p+2, \dots),$$

we get

$$f(z) = \sum_{n=p}^{\infty} (\sigma_n h_n + \mathcal{E}_n g_n),$$

and this completes the proof of Theorem 4. ∎

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Waggas Galib Atshan, Department of Mathematics, College of Computer Science and Mathematics, Univesrity of AL_Qadisiya, Diwaniya, Iraq

E-mail: `waggashnd@yahoo.com`

S. R. Kulkarni, Department of Mathematics, Fergusson College, Pune – 411004, India

E-mail: `kulkarni_ferg@yahoo.com`

R. K. Raina, 10/11, Ganpati Vihar, Opposite Sector 5, Udaipur 313002, Rajasthan, India

E-mail: `rainark_7@hotmail.com`