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BOUNDS ON ROMAN DOMINATION NUMBERS OF GRAPHS

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Abstract. Roman dominating function of a graph G is a labeling function $f\colon V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(G)$ of G is the minimum of $\Sigma_{v\in V(G)} f(v)$ over such functions. In this paper, we find lower and upper bounds for Roman domination numbers in terms of the diameter and the girth of G.

1. Introduction

For G, a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E), the *open neighborhood* $N(v)$ of the vertex v is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v\in S} N(v)$, and its *closed neighborhood* is $N[S] = N(S) \cup S$. The minimum and maximum vertex degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset S of vertices of G is a *2-packing* if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$.

A *Roman dominating function* (RDF) on a graph $G = (V,E)$ is defined in [13], [15] as a function $f\colon V \longrightarrow \{0,1,2\}$ satisfying the condition that a vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The *weight* of a RDF is defined as $w(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a RDF on G. A $\gamma_R(G)$*-function* is a Roman dominating function of G with weight $\gamma_R(G)$. Observe that a Roman dominating function $f\colon V \to \{0,1,2\}$ can be presented by an ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V \mid f(v) = i\}$.

Cockayne et. al [3] initiated the study of Roman domination, suggested originally in a Scientific American article by Ian Stewart [15]. Since $V_1 \cup V_2$ is a dominating set when f is a RDF, and since placing weight 2 at the vertices of a dominating set yields a RDF, they observed that

$$\gamma(G) \le \gamma_R(G) \le 2\gamma(G). \tag{1}$$

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In a sense, $2\gamma(G)-\gamma_R(G)$ measures "inefficiency" of domination, since the vertices with weight 1 in a RDF serve only to dominate themselves. The authors [3] investigated graph theoretic properties of RDFs and characterized $\gamma_R(G)$ for specific graphs. They found out the graphs G, those with $\gamma_R(G)=\gamma(G)+k$ when $k\leq 2$; and then for larger k by Xing et al. [16]. They also characterized the graphs G with property $\gamma_R(G)=2\gamma(G)$ in terms of 2-packings, referring them to as *Roman* graphs. Henning [9] characterized Roman trees, while Song and Wang [14] identified the trees T with $\gamma_R(T)=\gamma(T)+3$. Computational complexity of $\gamma_R(G)$ is considered in [4]. In [12], linear-time algorithms are given for $\gamma_R(G)$ on interval graphs and on cographs, along with a polynomial-time algorithm for AT-free graphs. Chambers et al. [2] proved that $\gamma_R(G)\leq \frac{4n}{5}$ when G is a connected graph of order $n\geq 3$, and determined when equality holds. They have also obtained sharp upper and lower bounds for $\gamma_R(G)+\gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, where $\overline{G}$ denotes the complement of G. Favaron et al. [7] proved that $\gamma_R(G)+\frac{\gamma(G)}{2}\leq n$ for any connected graph G of order $n\geq 3$. Other related domination models are studied in [1, 5, 6, 10, 11].

The purpose of this paper is to establish sharp lower and upper bounds for Roman domination numbers in terms of the diameter and the girth of G.

Cockayne et al. in [3] proved that:

Theorem A. *For a graph G of order n,*

$$\gamma(G)\leq\gamma_R(G)\leq 2\gamma(G),$$

with equality in lower bound if and only if $G=\overline{K}_n$.

Theorem B. *For paths P_n and cycles C_n,*

$$\gamma_R(P_n)=\gamma_R(C_n)=\left\lceil\frac{2n}{3}\right\rceil.$$

Theorem C. *Let $G=K_{m_1,\dots,m_n}$ be the complete n-partite graph with $m_1\leq m_2\leq\ldots\leq m_n$. If $m_1=2$, then $\gamma_R(G)=3$.*

Theorem D. *Let $f=(V_0^f,V_1^f,V_2^f)$ be a γ_R-function for a simple graph G, such that $|V_1^f|$ is minimum. Then V_1^f is a 2-packing.*

2. Bounds in terms of the diameter

In this section sharp lower and upper bounds for $\gamma_R(G)$ in terms of diam (G) are presented. Recall that the *eccentricity* of vertex v is $ecc(v)=\max\{d(v,w):w\in V\}$ and the *diameter* of G is $\operatorname{diam}(G)=\max\{ecc(v):v\in V\}$. Throughout this section we assume that G is a nontrivial graph of order $n\geq 2$.

Theorem 1. *If a graph G has diameter two, then $\gamma_R(G)\leq 2\delta$. Furthermore, this bound is sharp for infinite family of graphs.*

Proof. Since G has diameter two, $N(u)$ dominates $V(G)$ for all vertex $u \in V(G)$. Now, let $u \in V(G)$ and $\deg(u) = \delta$. Define $f\colon V(G) \longrightarrow \{0,1,2\}$ by $f(x) = 2$ for $x \in N(u)$ and $f(x) = 0$ otherwise. Obviously f is a RDF of G. Thus $\gamma_R(G) \le 2\delta$.

To prove sharpness, let G be obtained from Cartesian product $P_2 \Box K_m$ ($m \ge 3$) by adding a new vertex x and jointing it to exactly one vertex at each copy of K_m. Obviously, $\operatorname{diam}(G) = 2$ and $\gamma_R(G) = 4 = 2\delta$. This completes the proof. ■

Next theorem presents a lower bound for Roman domination numbers in terms of the diameter.

THEOREM 2. *For a connected graph G,*

$$\gamma_R(G) \ge \left\lceil \frac{\operatorname{diam}(G) + 2}{2} \right\rceil.$$

Furthermore, this bound is sharp for P_3 and P_4.

Proof. The statement is obviously true for K_2. Let G be a connected graph of order $n \ge 3$ and $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function. Suppose that $P = v_1 v_2 \dots v_{\operatorname{diam}(G)+1}$ is a diametral path in G. This diametral path includes at most two edges from the induced subgraph $G[N[v]]$ for each $v \in V_1^f \cup V_2^f$. Let $E' = \{v_i v_{i+1} \mid 1 \le i \le \operatorname{diam}(G)\} \cap \bigcup_{v \in V_1^f \cup V_2^f} E(G[N[v]])$. Then the diametral path contains at most $|V_2^f| - 1$ edges not in E', joining the neighborhoods of the vertices of V_2^f. Since G is a connected graph of order at least 3, $V_2^f \ne \emptyset$. Hence,

$$\operatorname{diam}(G) \le 2|V_2^f| + 2|V_1^f| + (|V_2^f| - 1) \le 2\gamma_R(G) - 2,$$

and the result follows. ■

In the following theorem, an upper bound is presented for Roman domination numbers.

THEOREM 3. *For any connected graph G on n vertices,*

$$\gamma_R(G) \le n - \left\lfloor \frac{1 + \operatorname{diam}(G)}{3} \right\rfloor.$$

Furthermore, this bound is sharp.

Proof. Let $P = v_1 v_2 \dots v_{\operatorname{diam}(G)+1}$ be a diametral path in G. Moreover, let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(P)$-function. By Theorem B, the weight of f is $\lceil \frac{2\operatorname{diam}(G)+2}{3} \rceil$. Define $g\colon V(G) \longrightarrow \{0,1,2\}$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = 1$ for $x \in V(G) \setminus V(P)$. Obviously g is a RDF for G. Hence,

$$\gamma_R(G) \le w(f) + (n - \operatorname{diam}(G) - 1) = n - \left\lfloor \frac{1 + \operatorname{diam}(G)}{3} \right\rfloor.$$

To prove sharpness, let G be obtained from a path $P = v_1 v_2 \dots v_{3k}$ ($k \ge 2$) by adding a pendant edge $v_3 u$. Obviously, G achieves the bound and the proof is complete. ■

For a connected graph G with $\delta \geq 3$, the bound in Theorem 3 can be improved as follows.

THEOREM 4. *For any connected graph G of order n with $\delta \geq 3$,*

$$\gamma_R(G) \leq n - \left\lfloor \frac{1 + \operatorname{diam}(G)}{3} \right\rfloor - (\delta - 2) \left\lfloor \frac{\operatorname{diam}(G) + 2}{3} \right\rfloor .$$

Proof. Let $P = v_1 v_2 \dots v_{\operatorname{diam}(G)+1}$ be a diametral path in G and $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(P)$-function for which $|V_1^f|$ is minimized and V_2^f is a 2-packing. Obviously, $|V_2^f| = \lfloor \frac{\operatorname{diam}(G)+2}{3} \rfloor$. Let $V_2^f = \{u_1, \dots, u_k\}$ where $k = \lfloor \frac{\operatorname{diam}(G)+2}{3} \rfloor$. Since P is a diametral path, each vertex of V_2^f has at least $\delta - 2$ neighbors in $V(G) \setminus V(P)$ and $N(u_i) \cap N(u_j) = \emptyset$ if $u_i \neq u_j$. Define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(x) = f(x)$ for $x \in V(P)$, $g(x) = 0$ for $x \in \bigcup_{i=1}^k N(u_i) \cap (V(G) \setminus V(P))$ and $g(x) = 1$ when $x \in V(G) \setminus (V(P) \cup (\bigcup_{i=1}^k N(u_i)))$. Obviously g is a RDF for G and so

$$\gamma_R(G) \leq w(g) = w(f) + n - \operatorname{diam}(G) - 1 - (\delta - 2) \left\lfloor \frac{\operatorname{diam}(G) + 2}{3} \right\rfloor .$$

Now the result follows from $w(f) = \lceil \frac{2\operatorname{diam}(G)+2}{3} \rceil$. ∎

The next theorem speaks of an interesting relationship between the diameter of G and the Roman domination number of $\overline{G}$, the complement of G.

THEOREM 5. *For a connected graph G with* $\operatorname{diam}(G) \geq 3$, $\gamma_R(\overline{G}) \leq 4$.

Proof. Let $P = v_1 v_2 \dots v_m$ be a diametral path in G where $m \geq 4$. Let $S = \{v_1, v_m\}$. Since $\operatorname{diam}(G) \geq 3$, each vertex $v \in V(G) \setminus S$ can be adjacent to at most one vertex of S in G. Consequently, S is a dominating set for $\overline{G}$. By (1), $\gamma_R(\overline{G}) \leq 2\gamma(\overline{G}) \leq 4$ and the proof is complete. ∎

3. Bounds in terms of the girth

In this section we present bounds on Roman domination numbers of a graph G containing cycles, in terms of its girth. Recall that the girth of G (denoted by $g(G)$) is the length of a smallest cycle in G. Throughout this section, we assume that G is a nontrivial graph of order $n \geq 3$ and contains a cycle.

The following result is very crucial for this section.

LEMMA 6. *For a graph G of order n with $g(G) \geq 3$ we have $\gamma_R(G) \geq \lceil \frac{2g(G)}{3} \rceil$.*

Proof. First note that if G is an n-cycle then $\gamma_R(G) = \lceil \frac{2n}{3} \rceil$ by Theorem B. Now, let C be a cycle of length $g(G)$ in G. If $g(G) = 3$ or 4, then we need at least 1 or 2 vertices, respectively, to dominate the vertices of C and the statement follows by Theorem A. Let $g(G) \geq 5$. Then a vertex not in $V(C)$, can be adjacent to at most one vertex of C for otherwise we obtain a cycle of length less than $g(G)$ which is a contradiction. Now the result follows by Theorem A. ∎

Theorem 7. *If $g(G)=4$, then $\gamma_R(G)\geq 3$. Equality holds if and only if G is a bipartite graph with partite sets X and Y with $|X|=2$, where X has one vertex of degree $n-2$ and the other of degree at least two.*

Proof. Let $g(G)=4$. Then $\gamma_R(G)\geq 3$ by Lemma 6. If G is a bipartite graph satisfying the conditions, then obviously $g(G)=4$ and $\gamma_R(G)=3$ by Theorem C. Now let $g(G)=4$ and $\gamma_R(G)=3$ and $f=(V_0^f,V_1^f,V_2^f)$ be a $\gamma_R(G)$-function. Obviously, $|V_1^f|=|V_2^f|=1$. Suppose that $V_1^f=\{u\}$ and $V_2^f=\{v\}$. Since $\gamma_R(G)=3$, $\{u,v\}$ is an independent set and v is adjacent to all vertices in $V(G)\setminus\{u,v\}$. Let $X=\{u,v\}$ and $Y=V(G)\setminus X$. Since $g(G)=4$, Y is an independent set. Henceforth, u and v are contained in each 4-cycle of G. It follows that u has degree at least two. This completes the proof. ■

Theorem 8. *Let G be a simple connected graph of order n, $\delta(G)\geq 2$ and $g(G)\geq 5$. Then $\gamma_R(G)\leq n-\lfloor\frac{g(G)}{3}\rfloor$. Furthermore, the bound is sharp for cycles C_n with $n\geq 5$.*

Proof. Let G be such a graph. Assume C is a cycle of G with $g(G)$ edges. If $G=C$, then the statement is valid by Theorem B. Now let G' be obtained from G by removing the vertices of $V(C)$. Since $g(G)\geq 5$, each vertex of G' can be adjacent to at most one vertex of C which implies $\delta(G')\geq 1$. Thus, $\gamma_R(G')\leq n-g(G)$. Let f and g be a $\gamma_R(G')$-function and $\gamma_R(C)$-function, respectively. Define $h: V(G)\to\{0,1,2\}$ by $h(v)=f(v)$ for $v\in V(G')$ and $h(v)=g(v)$ for $v\in V(C)$. Obviously, h is a RDF of G and the result follows. ■

Theorem 9. *For a simple connected graph G of order n, if $g(G)\geq 5$, then $\gamma_R(G)\geq 2\delta$. The bound is sharp for C_5 and C_6.*

Proof. Let $f=(V_0^f,V_1^f,V_2^f)$ be a $\gamma_R(G)$-function such that $|V_1^f|$ is minimum and let C be a cycle with $g(G)$ edges. If $n=5$, then G is a 5-cycle and $\gamma_R(G)=4=2\delta$. For $n\geq 6$, if $\delta\leq 2$, then $\gamma_R(G)\geq\lceil\frac{2g(G)}{3}\rceil\geq 2\delta$ by Lemma 6. Now, let $\delta\geq 3$. First suppose that $V_1^f=\emptyset$. Assume $v\in V_0^f$ and $N(v)=\{v_1,\dots,v_k\}$ for some $k\geq\delta$. Without loss of generality, one may suppose $v_1,\dots,v_r\in V_2^f$ and $v_{r+1},\dots,v_k\in V_0^f$ and for $j=r+1,\dots,k$, $v_jv_j'\in E(G)$ where $v_j'\in V_2^f$ and $k>r$. Since $g(G)\geq 5$, the vertices of $v_1,\dots,v_r,v_{r+1}',\dots,v_k'$ are distinct. Consequently, $|V_2^f|\geq 2k$ which implies $\gamma_R(G)\geq 2k\geq 2\delta$. For the case $V_1\neq\emptyset$, by definition of f, $|V_1^f|$ is an independent set. Suppose that $u\in V_1^f$ and $N(u)=\{u_1,\dots,u_k\}$ for some $k\geq\delta$. Obviously, $N(u)\subseteq V_0^f$. For each $j=1,\dots,k$, one may consider $u_jv_j\in E(G)$ where $v_j\in V_2^f$. Since $g(G)\geq 5$, the vertices $v_1,\dots,v_k$ are distinct. Hence, $\gamma_R(G)=2|V_2^f|+|V_1^f|\geq 2\delta+1$ and the proof is complete. ■

Theorem 10. *For a simple connected graph G with $\delta\geq 2$ and $g(G)\geq 6$, $\gamma_R(G)\geq 4(\delta-1)$. This bound is sharp for C_6.*

Proof. Let $f=(V_0^f,V_1^f,V_2^f)$ be a $\gamma_R(G)$-function such that $|V_1^f|$ is minimum. Therefore, V_1^f is an independent set and $N(w_1)\cap N(w_2)=\emptyset$ if $w_1\neq w_2$ for

$w_1, w_2 \in V_1^f$. For $V_1^f \neq \emptyset$ and $u \in V_1^f$, $N(u) = \{u_1, \dots, u_{\deg(u)}\} \subseteq V_0^f$. Suppose that $N(u_1) = \{w_1, \dots, w_r\}$ where $u = w_1$. Since $g(G) \geq 6$, $N(u) \cap N(u_1) = \emptyset$ and $N(u_i) \cap N(w_j) = \emptyset$ for each i, j. In this way, each vertex of V_2^f can be adjacent to at most one vertex in $(N(u) \cup N(u_1)) \cap V_0^f$. This implies that $|V_2^f| \geq 2(\delta - 1)$ which follows the statement.

For $V_1^f = \emptyset$, $|V_0^f| \geq 2$ holds clearly. If $G[V_0^f]$ has an edge uv, analogous reasoning proves the statement. Let V_0^f be an independent set in G with $|V_0^f| \geq 2$ and $u, v \in V_0^f$. Since $g(G) \geq 6$ and V_0^f is an independent set, $|N(u) \cap N(v)| \leq 1$ and $N(u) \cup N(v) \subseteq V_2^f$. This implies that $|V_2^f| \geq 2\delta - 1$ and the result follows. ∎

Theorem 11. *For a simple connected graph G with $\delta \geq 2$ and $g(G) \geq 7$, $\gamma_R(G) \geq 2\Delta$. This bound is sharp for $g(G) = 7$.*

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function such that $|V_1^f|$ is minimum and let C be a cycle of G with $g(G)$ edges. Suppose $v \in V(G)$ is a vertex with degree Δ. By Theorem D, V_1^f is an independent set of G and $N(w_1) \cap N(w_2) = \emptyset$ if $w_1 \neq w_2$ for $w_1, w_2 \in V_1^f$. Consider $N(v) = \{v_1, v_2, \dots, v_\Delta\}$. For $v \notin V_2^f$, similar to the proof of Theorem 9, the statement follows. For $v \in V_2^f$, let $A = N[v] \cap V_2^f$ and $B = N(v) \cap V_0^f$. For $u \in B$, three cases might occur.

Case 1. u has a neighbor in $V_2^f - \{v\}$. In this case, consider $x_u \in (V_2^f - \{v\}) \cap N(u)$.

Case 2. u has no neighbor in $V_2^f - \{v\}$ and u has some neighbor in V_0^f. For $y_u \in N(u) \cap V_0^f$, Since $g(G) \geq 7$, $y_u \notin B$. In this case, let $x_u \in V_2^f \cap N(y_u)$.

Case 3. u has no neighbor in $V_0^f \cup (V_2^f - \{v\})$ and u has some neighbor in V_1^f. For $z_u \in V_1^f \cap N(u)$, Since G is connected and $\delta \geq 2$, z_u has a neighbor in $V_0^f - \{u\}$, say y_u. On the other hand y_u has a neighbor in V_2^f, say x_u.

Since $g(G) \geq 7$, it is straightforward to verify that $A \cap \{x_u \mid u \in B\} = \emptyset$ and $x_u \neq x_{u'}$ when $u \neq u'$ and $u, u' \in B$. Thus, $|V_2^f| \geq \Delta$ that implies the statement.

The bound is sharp for the graph $G = (V, E)$, where $V = \{v, u, w, v_i, u_i, w_i \mid 1 \leq i \leq m\}$ and $E = \{vu, uw, w_1w_2, vv_i, v_iu_i, u_iw_i \mid 1 \leq i \leq m\}$ for $m \geq 2$ when $g(G) = 7$. ∎

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