

## ON BITOPOLOGICAL PARACOMPACTNESS

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**Abstract.** A new definition of pairwise paracompactness is given. Among other results, an analogue of Michael's characterization of regular paracompact spaces is proved. This notion of pairwise paracompactness is more general than the notion of pairwise compactness.

The study of bitopological spaces was initiated by J.C. Kelly [5]. Since then many works on bitopological spaces have been done by several authors: Lane [7], Fletcher et al [3], Datta [1], Raghavan and Reilly [9], Ganster and Reilly [4], Kovár [6], Srivastava and Bhatia [10] and others. Fletcher et al [3] defined pairwise paracompactness. But in presence of pairwise Hausdorffness [5], the two topologies in a bitopological space pairwise paracompact in the sense of [3] coincide, and the bitopological space becomes a single topological space. This was proved by Fletcher et al, and they did not proceed further with this definition. Later Datta [1] and Raghavan and Reilly [9] defined the notion of pairwise paracompactness in different ways. Datta's definition of pairwise paracompactness is a generalization of pairwise compactness [3]. He attempted to get a bitopological version of the following theorem of Michael [8] on paracompactness. But a complete analogue of the theorem could not be obtained.

**THEOREM 1.** (Dugundji [2], p. 163) *Let  $X$  be a regular topological space. Then the following statements are equivalent.*

- (a)  $X$  is paracompact.
- (b) Every open cover of  $X$  has an open refinement  $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ , where each  $\mathcal{V}_n$  is a locally finite collection of open sets.
- (c) Every open cover of  $X$  has a locally finite refinement.
- (d) Every open cover of  $X$  has a locally finite closed refinement.

Raghavan and Reilly [9] proved an analogue of Michael's theorem on their notion of  $\delta$ -pairwise paracompactness. But  $\delta$ -pairwise paracompactness is not a

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generalization of pairwise compactness. In this paper we introduce a definition of pairwise paracompactness as a generalization of pairwise compactness and prove an analogue of Michael's theorem. This notion of pairwise paracompactness is slightly different from that of Datta [1].

Let  $(X, \mathcal{P}_1, \mathcal{P}_2)$  be a bitopological space. Fletcher et al [3] defined a pairwise open cover of  $X$ : a cover  $\mathcal{U}$  of  $X$  is a pairwise open cover if  $\mathcal{U} \subset \mathcal{P}_1 \cup \mathcal{P}_2$  and for  $i = 1, 2$ ,  $\mathcal{U} \cap \mathcal{P}_i$  contains a non-empty set. If a set  $E$  is open in the space  $(X, \mathcal{P}_i)$ , we write ' $E$  is  $(\mathcal{P}_i)$ open'. Similarly we define  $(\mathcal{P}_i)$ closed sets,  $(\mathcal{P}_i)F_\sigma$  sets,  $(\mathcal{P}_i)$ locally finite collection of sets,  $(\mathcal{P}_i)$ paracompactness etc. A pairwise open cover  $\mathcal{V}$  of  $X$  is said to be a parallel refinement (Datta [1]) of a pairwise open cover  $\mathcal{U}$  if every  $(\mathcal{P}_i)$ open set of  $\mathcal{V}$  is contained in some  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ . If  $x \in X$  and  $E \subset X$ , by ' $E$  is  $(\mathcal{P}_{ux})$ open', we mean ' $E$  is  $(\mathcal{P}_1)$ open (resp.  $(\mathcal{P}_2)$ open)' if  $x$  belongs to a  $(\mathcal{P}_1)$ open (resp.  $(\mathcal{P}_2)$ open) set of  $\mathcal{U}$ .

We introduce the following definitions:

**DEFINITION 2.** A refinement  $\mathcal{V}$  of a pairwise open cover  $\mathcal{U}$  of  $X$  is said to be locally finite if for each  $x \in X$ , there exists a  $(\mathcal{P}_{ux})$ open nbd of  $x$  intersecting a finite number of members of  $\mathcal{V}$ .

**DEFINITION 3.** The bitopological space  $X$  is pairwise paracompact if every pairwise open cover  $\mathcal{U}$  of  $X$  has a locally finite parallel refinement.

In the above definition, if some sets  $U \in \mathcal{U}$  are both  $(\mathcal{P}_1)$ open and  $(\mathcal{P}_2)$ open, then for each such set  $U$ , we select one of  $\mathcal{P}_1$  and  $\mathcal{P}_2$  with respect to which  $U$  is open and for this choice, we have a locally finite refinement of  $\mathcal{U}$ . Changing the choice, we get a class of locally finite refinements of  $\mathcal{U}$ .

From the definition, it follows that if  $X$  is pairwise compact [3], then it is pairwise paracompact.

**DEFINITION 4.** The bitopological space  $X$  is said to be strongly pairwise regular if  $X$  is pairwise regular, and if both the topological spaces  $(X, \mathcal{P}_1)$  and  $(X, \mathcal{P}_2)$  are regular.

The above notion is substantiated by giving examples.

We denote the set of real numbers and set of natural numbers by  $R$  and  $N$  respectively. The open interval  $\{x \in R \mid a < x < b\}$  is denoted by  $(a, b)$ .

Now we prove three results on pairwise paracompactness. The second one (Theorem 6) is an analogue of Theorem 1.

**THEOREM 5.** *If  $X$  is pairwise Hausdorff and pairwise paracompact, then it is pairwise normal.* [5]

*Proof.* Let  $x \in X$  and  $F$  be a  $(\mathcal{P}_i)$ closed set with  $x \notin F$ . For  $\xi \in F$ , there exist  $U_\xi \in \mathcal{P}_i$  and  $V_\xi \in \mathcal{P}_j$ ,  $i \neq j$  such that  $x \in U_\xi$  and  $\xi \in V_\xi$  and  $U_\xi \cap V_\xi = \emptyset$ . Then the collection  $\{V_\xi \mid \xi \in F\} \cup \{X - F\}$  forms a pairwise open cover of  $X$ . Therefore it has a locally finite parallel refinement  $\mathcal{W}$ . Let  $H = \cup\{W \in \mathcal{W} \mid W \cap F \neq \emptyset\}$ . Then  $H \in \mathcal{P}_j$  and  $F \subset H$ . Since  $x \in X - F$  and  $X - F$  is a  $(\mathcal{P}_i)$ open set, there exists a  $(\mathcal{P}_i)$ open nbd  $D$  of  $x$  intersecting a finite number of sets  $W_1, W_2, \dots, W_n$

of  $\mathcal{W}$  with  $W_k \cap F \neq \emptyset, k = 1, 2, \dots, n$ . Let  $W_k \subset V_{\xi_k}, \xi_k \in F, k = 1, 2, \dots, n$ . Then  $G = D \cap (\bigcap_{k=1}^n U_{\xi_k}) \in \mathcal{P}_i$ . Also  $x \in G$  and  $G \cap H = \emptyset$ . Thus  $X$  is pairwise regular. Therefore, given two disjoint sets  $A$  and  $B$  which are  $(\mathcal{P}_j)$ closed and  $(\mathcal{P}_i)$ closed ( $i \neq j$ ) respectively, for  $x \in B$ , we get two sets  $U_x \in \mathcal{P}_i$  and  $V_x \in \mathcal{P}_j$  such that  $A \subset U_x, x \in V_x$  and  $U_x \cap V_x = \emptyset$ . Then the pairwise open cover  $\mathcal{C} = \{V_x \mid x \in B\} \cup \{X - B\}$  of  $X$  has a locally finite parallel refinement  $\mathcal{G}$ . If  $V = \bigcup \{G \in \mathcal{G} \mid G \cap B \neq \emptyset\}$ , then  $V \in \mathcal{P}_j$  and  $B \subset V$ . Now consider a point  $y \in A$ . Then  $y$  belongs to the  $(\mathcal{P}_i)$ open set  $X - B$  of the cover  $\mathcal{C}$  and so there exists a  $(\mathcal{P}_i)$ open nbd  $D_y$  of  $y$  intersecting a finite number of elements  $G_1(y), G_2(y), \dots, G_m(y)$  of  $\mathcal{G}$  such that for  $k = 1, 2, \dots, m, B \cap G_k(y) \neq \emptyset$ . If  $G_k(y) \subset V_{x_k}, x_k \in B$ , then  $U_y \cap V = \emptyset$  and  $y \in U_y$  where  $U_y = D_y \cap (\bigcap_{k=1}^m U_{x_k}) \in \mathcal{P}_i$ . Now if  $U = \bigcup_{y \in A} U_y$ , then  $U \in \mathcal{P}_i, A \subset U$  and  $U \cap V = \emptyset$ . Therefore  $X$  is pairwise normal. ■

**THEOREM 6.** *If the bitopological space  $X$  is strongly pairwise regular, then the following statements are equivalent.*

- (a)  $X$  is pairwise paracompact.
- (b) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a parallel refinement  $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ , where for each  $n$  and for each  $x$ , there exists a  $(\mathcal{P}_{u_x})$ open nbd of  $x$  intersecting a finite number of members of  $\mathcal{V}_n$ .
- (c) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a locally finite refinement.
- (d) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a locally finite refinement  $\mathcal{B}$  such that if  $B \subset U \in \mathcal{U}, B \in \mathcal{B}$ , then  $((\mathcal{P}_1)clB) \cup ((\mathcal{P}_2)clB) \subset U$ .

*Proof.* (a)  $\Rightarrow$  (b): Straightforward.

(b)  $\Rightarrow$  (c): Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . Suppose  $\mathcal{H}$  is a parallel refinement of  $\mathcal{U}$  such that  $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ , where for each  $n$  and for each  $x$ , there exists a  $(\mathcal{P}_{u_x})$ open nbd of  $x$  intersecting a finite number of members of  $\mathcal{H}_n$ . Let  $\mathcal{H}_n = \{H_{n\alpha} \mid \alpha \in A\}$ . If  $W_n = \bigcup_{\alpha} H_{n\alpha}$ , then the collection  $\{W_n \mid n \in N\}$  is a cover of  $X$ . We write  $E_n = W_n - \bigcup_{k < n} W_k$ . If  $n(x)$  is the first  $n$  for which  $x \in W_{n(x)}$ , then  $x \in E_{n(x)}$ . So  $\{E_n \mid n \in N\}$  is a refinement of  $\{W_n \mid n \in N\}$ . It is also locally finite. Indeed, there exists an  $\alpha_0$  such that  $x \in H_{n(x)\alpha_0}$ . Then  $H_{n(x)\alpha_0}$  is a  $(\mathcal{P}_{u_x})$ open nbd of  $x$ , which intersects only a finite number of members of  $\{E_n \mid n \in N\}$ . Now the collection  $\{E_n \cap H_{n\alpha} \mid n \in N, \alpha \in A\}$  is a refinement of  $\mathcal{U}$ . It is also locally finite. In fact, each  $x \in X$  has a  $(\mathcal{P}_{u_x})$ open nbd intersecting a finite number of members of  $\{E_n \mid n \in N\}$ , and for each such  $n$ , the point  $x$  has a  $(\mathcal{P}_{u_x})$ open nbd intersecting at most a finite number of members of  $\{H_{n\alpha} \mid \alpha \in A\}$ .

(c)  $\Rightarrow$  (d): For  $x \in X$ , consider  $U_x \in \mathcal{U}$  with  $x \in U_x$ . Suppose  $U_x$  is  $(\mathcal{P}_i)$ open. By the strong pairwise regularity of  $X$ , there exists a  $(\mathcal{P}_i)$ open nbd  $G_x$  of  $x$  such that

$$((\mathcal{P}_1)clG_x) \cup ((\mathcal{P}_2)clG_x) \subset U_x.$$

Then  $\{G_x \mid x \in X\}$  is a pairwise open cover of  $X$ . Therefore by (c), there is a locally finite refinement  $\mathcal{B}$  of  $\{G_x \mid x \in X\}$ , and hence of  $\mathcal{U}$ . If  $B \in \mathcal{B}$ , then for some  $G_x$ , we have  $B \subset G_x \subset U_x$ , and so

$$((\mathcal{P}_1)clB) \cup ((\mathcal{P}_2)clB) \subset ((\mathcal{P}_1)clG_x) \cup ((\mathcal{P}_2)clG_x) \subset U_x.$$

(d)  $\Rightarrow$  (a): Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . It is sufficient to consider the case when there are no sets  $\in \mathcal{U}$ , open with respect to both  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Let  $\mathcal{A}$  be a locally finite refinement of  $\mathcal{U}$ . For  $x \in X$ , let  $x \in U \in \mathcal{U}$  and  $U$  be  $(\mathcal{P}_i)$ open. Suppose  $W_x$  is a  $(\mathcal{P}_i)$ open nbd of  $x$  intersecting a finite number of members of  $\mathcal{A}$ . Then  $\mathcal{W} = \{W_x \mid x \in X\}$  is a pairwise open cover of  $X$ . Let  $\mathcal{E} = \{E_\lambda \mid \lambda \in \Lambda\}$  be a locally finite refinement of  $\mathcal{W}$  such that if  $E_\lambda \subset W_x$ , then

$$((\mathcal{P}_1)clE_\lambda) \cup ((\mathcal{P}_2)clE_\lambda) \subset W_x.$$

For  $A \in \mathcal{A}$ , we choose  $U_A \in \mathcal{U}$  such that  $A \subset U_A$ . If  $U_A$  is  $(\mathcal{P}_i)$ open, we write

$$F_A = \bigcup \{(\mathcal{P}_i)clE \mid E \in \mathcal{E}, ((\mathcal{P}_i)clE) \cap A = \emptyset\}.$$

We define  $G_A = X - F_A$  and  $H_A = U_A \cap G_A$ . Obviously the collection  $\{H_A \mid A \in \mathcal{A}\}$  covers  $X$ . Since  $U_A$  is  $(\mathcal{P}_i)$ open and since for each point of  $U_A$ , we get a  $(\mathcal{P}_i)$ open nbd of  $x$  intersecting a finite number of sets  $\in \mathcal{E}$ , it follows that  $H_A$  is  $(\mathcal{P}_i)$ open. Also  $H_A \subset U_A$ . Thus  $\{H_A \mid A \in \mathcal{A}\}$  is a parallel refinement of  $\mathcal{U}$ .

Now we show that  $\{H_A \mid A \in \mathcal{A}\}$  is locally finite. It is sufficient to show that the collection  $\{G_A \mid A \in \mathcal{A}\}$  is locally finite. Let  $x \in X$  and  $D_x$  be a  $(\mathcal{P}_{u_x})$ open nbd of  $x$  intersecting a finite number of  $E_{\lambda_1}, E_{\lambda_2}, \dots, E_{\lambda_n}$  of  $\mathcal{E}$ . Now

$$\begin{aligned} D_x \cap G_A \neq \emptyset, &\Rightarrow E_{\lambda_k} \cap G_A \neq \emptyset \text{ for some } k = 1, 2, \dots, n, \\ &\Rightarrow ((\mathcal{P}_i)clE_{\lambda_k}) \cap A \neq \emptyset \text{ if } U_A \text{ is } (\mathcal{P}_i)\text{open.} \end{aligned}$$

Since for each  $k$ ,  $(\mathcal{P}_i)clE_{\lambda_k}$  is contained in some  $W_{x_k} \in \mathcal{W}$ , and  $W_{x_k}$  intersects a finite number of sets  $\in \mathcal{A}$ , it follows that  $(\mathcal{P}_i)clE_{\lambda_k}$  intersects a finite number of sets  $\in \mathcal{A}$ , and hence  $D_x$  intersects a finite number of sets of  $\{G_A \mid A \in \mathcal{A}\}$ . Thus  $\{G_A \mid A \in \mathcal{A}\}$  is locally finite. ■

**THEOREM 7.** *Let  $X$  be pairwise paracompact. If the topological space  $(X, \mathcal{P}_j)$  is regular, and if for  $i \neq j$ ,  $F$  is a  $(\mathcal{P}_i)F_\sigma$  proper subset of  $X$ , then  $F$  is  $(\mathcal{P}_j)$ paracompact.*

*Proof.* Let  $F = \bigcup_{n=1}^\infty F_n$ , where for each  $n$ ,  $F_n$  is a  $(\mathcal{P}_i)$ closed set. Suppose  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  is a  $(\mathcal{P}_j)$ open cover of  $F$ . Then  $U_\alpha = F \cap V_\alpha$ , where  $V_\alpha$  is  $(\mathcal{P}_j)$ open in  $X$ . For each fixed  $n$ ,  $\{V_\alpha\} \cup \{X - F_n\}$  is a pairwise open cover of  $X$ . Hence it has a locally finite parallel refinement  $\{W_\alpha^n\}$ . Let  $\mathcal{G}_n = \{W_\alpha^n \cap F \mid W_\alpha^n \cap F_n \neq \emptyset\}$ . Then for each  $n$ ,  $\mathcal{G}_n$  is  $(\mathcal{P}_j)$ locally finite. Also  $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$  is a  $(\mathcal{P}_j)$ open cover of  $F$ , and is a refinement of  $\mathcal{U}$ . Therefore by Theorem 1,  $F$  is  $(\mathcal{P}_j)$ paracompact. ■

Now we present two examples of bitopological spaces: one is strongly pairwise regular, and the other is pairwise regular but not strongly pairwise regular.

**EXAMPLE 8.** Let  $a, b \in R$  with  $a < b$ , and  $\tau$  be the collection of subsets  $G$  of  $R$  for which  $(a, b) \subset R - G$  or  $G \cap (a, b)$  is the union of some open subintervals of  $(a, b)$ . Then  $(R, \tau)$  is a regular topological space. If  $\sigma$  is the usual topology on  $R$ , then the bitopological space  $(R, \sigma, \tau)$  is pairwise regular. Since the space  $(R, \sigma)$  is regular, the space  $(R, \sigma, \tau)$  is strongly pairwise regular.

EXAMPLE 9. Suppose  $\mathcal{T} = \{G \subset R \mid R - G \text{ is a bounded set}\} \cup \{\emptyset\}$ . Then  $\mathcal{T}$  forms a topology on  $R$ . If  $\sigma$  is the usual topology on  $R$ , then the bitopological space  $(R, \sigma, \mathcal{T})$  is pairwise regular. But the space  $(R, \mathcal{T})$  is not regular and so  $(R, \sigma, \mathcal{T})$  is not strongly pairwise regular.

Lastly we note that the bitopological space considered in Example 4 [3, p. 330] is pairwise compact and, hence, pairwise paracompact. It is also pairwise Hausdorff but the topologies do not coincide.

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