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WEIGHTED COMPOSITION OPERATORS BETWEEN TWO L^p-SPACES

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Abstract. In this paper, we study the boundedness of weighted composition operators between two L^p-spaces. We present equivalent conditions for the compactness of weighted composition operators between two L^p-spaces. We also give equivalent conditions for weighted composition operators with closed range.

1. Introduction

Let (X, Σ, μ) and (Y, Γ, ν) be two σ-finite measure spaces. A measurable transformation $T\colon Y \to X$ is said to be non-singular if $\nu(T^{-1}(A)) = 0$, whenever $A \in \Sigma$ with $\mu(A) = 0$. In this case, we write $\nu \circ T^{-1} \ll \mu$. Let $u\colon Y \to \mathbf{C}$ be an essentially bounded measurable function. We assume that the support u is the domain of T. Then the linear transformation $W = W_{u,T}\colon L(X, \Sigma, \mu) \to L(Y, \Gamma, \nu)$ is defined as

$$Wf = W_{u,T} f = u \cdot f \circ T, \quad \text{for each} \quad f \in L(X, \Sigma, \mu),$$

where $L(X, \Sigma, \mu)$ and $L(Y, \Gamma, \nu)$ are the linear spaces of all μ-measurable and ν-measurable functions on X and Y, respectively. In the case when W maps $L^p(X) = L^p(X, \Sigma, \mu)$ into $L^q(Y) = L^q(Y, \Gamma, \nu)$, for $p, q \in [1, \infty]$, we call $W = W_{u,T}$ a weighted composition operator induced by the pair (u, T).

Note that the pair (u, T) induces a weighted composition operator while T may fail to induce a composition operator from $L^p(X)$ into $L^q(Y)$. For example if $u(y) = 0$, for each $y \in Y$, then $W_{u,T}$ induces a weighted composition operator whether T induces the corresponding composition operator or not.

For $q \neq \infty$, we define a measure $\mu_{u,T}$ on Σ as

$$\mu_{u,T} = \int_{T^{-1}(A)} |u(y)|^q \, d\nu, \quad \text{for each} \quad A \in \Sigma.$$

Clearly $\mu_{u,T} \ll \nu \circ T^{-1} \ll \mu$. Let $f_{u,T}$ denote the Radon-Nikodym derivative of $\mu_{u,T}$ with respect to μ and let $h = (f_{u,T})^{\frac{1}{q}}\colon X \to \mathbf{C}$. We know that if the

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Radon-Nikodym derivative of $\nu \circ T^{-1}$ with respect to μ, denoted as f_T, is essentially bounded, then $W_{u,T}$ is a bounded operator from $L^p(X)$ into $L^q(Y)$, for $p, q \in [1, \infty]$, but the converse need not be true.

We recall the next result from [16] for the sake of clarity of usage of symbols throughout the paper.

PROPOSITION 1.1. *A σ-finite measure space X can be uniquely decomposed as*

$$X = \left(\bigcup_{n \in N} A_n \right) \cup B,$$

where $\{A_n\}_{n \in N}$ is a collection of disjoint atoms and B is a non-atomic part of X.

Note that we identify two sets whose symmetric difference has μ-measure zero. We have $\mu(E \setminus F) = 0$.

Note that the surjective isometries between the spaces of continuous functions are weighted composition operators. The study of weighted composition operators is very interesting since such operators provide a common ground for the study of composition operators and multiplication operators on various function spaces, in particular, on measurable function spaces. The study of composition operators and multiplication operators on various functions spaces is still in progress while the study of weighted composition operators on measurable function spaces is carried out in this paper.

Some notable works in this direction can be seen in [2], [3], [6], [7], [9], [10], [15] and [16]. For a beautiful exposition of the study of composition operators on L^2-spaces, see [1] and the references therein.

Using Proposition 1.1 and Proposition 3.1 from [16], the next result easily follows.

PROPOSITION 1.2. *For $p, q \in [1, \infty]$, every weighted composition operator $W = W_{u,T} \colon L^p(X) \to L^q(Y)$ is a bounded linear operator.*

DEFINITION 1. A linear operator $U \colon A \to C$ from a Banach space A into a Banach space C is said to be

(i) compact, if for each bounded set S in A, $T(S)$ is contained in some compact set in C.

(ii) completely continuous, if for each weakly convergent sequence $(f_n)_{n \geq 1}$ in A, the sequence $(U(f_n))_{n \geq 1}$ is strongly convergent in C.

Note that every compact operator $U \colon A \to C$ is completely continuous, but the converse may not be true in general. The converse holds provided the Banach space A is reflexive. See [4, p. 173] for details on this issue.

This paper is motivated by the interesting work of H. Takagi, K. Narita and K. Yokouchi in [10], [15] and [16]. As in [16], we divide our results into three cases as Case I: $p = q$, Case II: $p < q$, and Case III: $p > q$.

2. Boundedness

In the next theorem, we characterize weighted composition operators between two L^p-spaces.

Theorem 2.1. *For $p, q \in [1, \infty]$, the pair (u, T) induces a bounded weighted composition operator $W\colon L^p(X) \to L^q(Y)$ if and only if $M_h\colon L^p(X) \to L^q(X)$ is a bounded multiplication operator, where $h = (f_{u,T})^{\frac{1}{q}}\colon X \to \mathbf{C}$ denotes the Radon-Nikodym derivative of $\mu_{u,T}$ with respect to μ.*

Proof. For each $f \in L^p(X)$, we have

$$\begin{aligned}
\|Wf\|^q_{L^q(Y)} &= \int_Y |u(y)|^q |f(T(y))|^q \, d\nu(y) \\
&= \int_X |u(T^{-1}(y))|^q |f(x)|^q \, d\nu(T^{-1}(y)) \\
&= \int_X |f(x)|^q \, d\mu_{u,T}(x) = \int_X |f(x)|^q f_{u,T}(x) \, d\mu(x) \\
&= \int_X |h(x) f(x)|^q \, d\mu(x) = \int_X |(M_h f)(x)|^q \, d\mu(x) \\
&= \|M_h f\|^q_{L^q(X)}.
\end{aligned}$$

This proves the result. ∎

Using Theorem 1.1 in [16], the next result follows from the above result.

Theorem 2.2. *Let $W = W_{u,T}\colon L^p(X) \to L^p(Y)$ be a weighted composition operator. Then the following statements are equivalent.*

(i) $W\colon L^p(X) \to L^p(X)$ is a compact operator.

(ii) $M_h\colon L^p(X) \to L^p(X)$ is a compact operator.

(iii) For each $\epsilon > 0$, Z_h^ϵ is a finite dimensional subspace of $L^p(X)$, where

$$Z_h^\epsilon = \{f\chi_A : f \in L^p(X)\},$$

and $A = A_{h,\epsilon} = \{x \in X : |h(x)| > \epsilon\}$.

Using [16, Thm. 1.2] and Theorem 2.1, the next result is an easy exercise for the reader.

Theorem 2.3. *For $1 \le q < p < \infty$, the weighted composition operator $W = W_{u,T}\colon L^p(X) \to L^q(Y)$ is continuous if and only if $h \in L^r(X)$, where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.*

Combining Theorem 2.1 and Theorem 2.3, we have

Theorem 2.4. *For $1 \le q < p < \infty$, let $W = W_{u,T}\colon L^p(X) \to L^q(Y)$ be a continuous weighted composition operator. Then the following statements are equivalent.*

(*i*) *W is a compact operator.*
(*ii*) *W is a completely continuous operator.*
(*iii*) $M_h\colon L^p(X) \to L^q(X)$ *is a compact operator.*
(*iv*) $M_h\colon L^p(X) \to L^q(X)$ *is a completely continuous operator.*
(*v*) $h(x) = 0$ *for* μ*-almost all* $x \in B$*, and* $\sum_{n\in N} |h(A_n)|^r < \infty$.

THEOREM 2.5. *For* $1 \leq p < q < \infty$*, the pair* (u, T) *induces a continuous weighted composition operator* $W = W_{u,T}\colon L^p(X) \to L^q(Y)$ *if and only if we have*
(*i*) $h(x) = 0$ *for* μ*-almost all* $x \in B$*, the non-atomic part of* X.
(*ii*) $\sup_{n\in N} \frac{|h(A_n)|^s}{\mu(A_n)} < \infty$*, where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.
In this case, we have

$$\|W\| = \|M_h\| = \sup_{n\in N} \frac{|h(A_n)|}{\mu(A_n)^{\frac{1}{s}}}.$$

COROLLARY 2.6. *Let* $1 \leq p < q < \infty$ *and* X *be non-atomic. Then there is no non-zero weighted composition operator from* $L^p(X)$ *into* $L^q(Y)$.

Combining Theorem 2.1 and Theorem 2.5, we have

THEOREM 2.7. *For* $1 \leq p < q < \infty$*, let* $W = W_{u,T}\colon L^p(X) \to L^q(Y)$ *be a continuous weighted composition operator. Then the following statements are equivalent.*
(*i*) *W is a compact operator.*
(*ii*) *W is a completely continuous operator.*
(*iii*) $M_h\colon L^p(X) \to L^q(X)$ *is a compact operator.*
(*iv*) $M_h\colon L^p(X) \to L^q(X)$ *is a completely continuous operator.*
(*v*) $h(x) = 0$ *for* μ*-almost all* $x \in B$*, and* $\frac{h(A_n)}{\mu(A_n)^{\frac{1}{p}-\frac{1}{q}}} \longrightarrow 0$.

3. Weighted composition operators with closed range

In this section, we discuss the weighted composition operators with closed range. Using Theorem 2.1 the next result follows along the similar lines as in [16, Lemma 4.1].

THEOREM 3.1. *For* $p \in [1, \infty]$*, let* $W = W_{u,T}\colon L^p(X) \to L^q(Y)$ *be a weighted composition operator. Then* W *has closed range if and only if the multiplication operator* $M_h\colon L^p(X) \to L^q(X)$ *has closed range.*

Using Theorem 2.1, we can easily prove the following lemma. For details, see [16].

LEMMA 3.2. *Under the hypothesis of the above theorem, operator* W *has closed range if and only if there is a constant* $c > 0$ *such that*

$$\|W_{u,T} f\|_{L^q(S)} \geq c\|f\|_{L^p(S)},$$

for each $h \in L^p(S)$*, where* $S = S_h = \mathrm{supp}(h)$.

Let $\Sigma_{u,T}$ be the collection of all $E \in \Sigma$ with

(i) $\mu(E) < \infty$.

(ii) If $F \in \Sigma$ with $F \subseteq E$ and $\mu_{u,T}(F) = 0$, then $\mu(F) = 0$.

LEMMA 3.3. *Suppose $E \in \Sigma$ and $\mu(E) < \infty$. Then $E \in \Sigma_{u,T}$ if and only if $E \in \text{supp}(h) = \text{supp}(f_{u,T})$.*

Proof. Let $E \in \text{supp}(h)$, let $F \in \Sigma$ be that $F \subseteq E$ and $\mu_{u,T}(F) = 0$. We write $F = \{x \in F : h(x) > 0\} = \{x \in F : f_{u,T}(x) > 0\}$.

If $\mu(F) > 0$, there is some $\delta > 0$ such that $\mu(G) > 0$, where $G = \{x \in F : f_{u,T}(x) > \delta\}$ and we have

$$0 = \mu_{u,T}(F) = \int_F f_{u,T}\, d\mu \geq \int_G f_{u,T}\, d\mu \geq \delta\mu(G) > 0$$

and this contradiction proves that $\mu(F) = 0$ so that $F \in \Sigma_{u,T}$.

Conversely, for $E \in \Sigma_{u,T}$, let $F = E \setminus \text{supp}(h)$. Then $F \subseteq E$ and since $f_{u,T}(x) = 0$ for μ-almost all $x \in F$, we have

$$\mu_{u,T}(F) = \int_F f_{u,T}\, d\mu = 0,$$

which implies that $\mu(F) = 0$ so that $E \subseteq \text{supp}(h)$. This proves the lemma. ■

THEOREM 3.4. *For $p = q \in [1, \infty)$, let $W\colon L^p(X) \to L^q(Y)$ be a weighted composition operator. Then the following statements are equivalent.*

(i) *The operator W has closed range.*

(ii) *There is some $\delta > 0$such that $|h(x)| \geq \delta$ for μ-almost all $x \in \text{supp}(h) = S_h$.*

(iii) *There is some $c > o$ such that $\mu_{u,T}(E) \geq c\mu(E)$ for all $E \in \Sigma_{u,T}$.*

Proof. Using Theorem 2.1 and Thm. 2.1 in [16], we have $(i) \Leftrightarrow (ii)$. Next we prove that (ii) and (iii) are equivalent.

$(ii) \Rightarrow (iii)$. Let $E \in \Sigma_{u,T}$ be arbitrary. Then by the above lemma $E \in \text{supp}(h)$ so that $f_{u,T} \geq \delta$ for μ-almost all $x \in E$ and we have

$$\mu_{u,T}(E) = \int_E f_{u,T}\, d\mu \geq \delta\mu(E).$$

$(iii) \Rightarrow (ii)$. Let $A = \{x \in \text{supp}(h) : f_{u,T}(x) < \delta\}$. We prove that $\mu(A) = 0$. Suppose $\mu(A) > 0$. Since μ-is σ-finite, there exists $E \in \Sigma_{u,T}$ and $0 < \delta_\circ < \delta$ with $E \subseteq \{x \in \text{supp}(h) : f_{u,T}(x) < \delta_\circ\})$ and $0 < \mu(E) < \infty$, using Lemma 3.3, we see that $E \in \Sigma$. Then for $c = \delta$, we have

$$c\mu(E) \leq \mu_{u,T}(E) = \int_E f_{u,T}\, d\mu \leq \delta_\circ\mu(E) < \delta\mu(E) = c\mu(E).$$

This proves that $\mu(A) = 0$. Thus $f_{u,T}(x) \geq \delta$ for μ-almost all $x \in \text{supp}(h)$. ■

THEOREM 3.5. *For $1 \leq q < p < \infty$, let W be a weighted composition from $L^p(X)$ into $L^q(Y)$. Then the following statements are equivalent.*

(i) *The operator W has closed range.*

(ii) *The operator W has finite rank.*

(iii) *The operator $M_h\colon L^p(X) \to L^q(X)$ has closed range.*

(iv) *The operator $M_h\colon L^p(X) \to L^q(X)$ has finite rank.*

(v) *$h(x) = 0$ for μ-almost all $x \in B$ and the set $\{n \in N : h(A_n) \neq 0\}$ is finite.*

(vi) *$\mu_{u,T}(B) = 0$ for μ-almost all $x \in B$ and the set $\{n \in N : \mu_{u,T}(A_n) \neq 0\}$ is finite.*

Proof. Using Theorem 2.1 and Thm. 2.2 in [16], we see that

$$(i) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v).$$

Note that we have

$$\mu_{u,T}(A_n) = \int_{A_n} f_{u,T}\, d\mu = f_{u,T}(A_n)\mu(A_n). \tag{3.1}$$

Using equation (3.1), we see that $(v) \Leftrightarrow (vi)$. Clearly, we have $(ii) \Rightarrow (i)$. We prove that $(i) \Rightarrow (ii)$.

Suppose that (i) holds. Then, clearly, (vi) holds since (i) and (vi) are equivalent. Clearly, the range $W(L^p(X))$ is contained in the subspace of $L^q(Y)$ generated by the collection $\{\chi_{T^{-1}(A_n)}\}_{n \in N_{u,T}}$, where $N_{u,T} = \{x \in N : \mu_{u,T}(A_n) \neq 0\}$. Also, we have

$$\mu_{u,T}(A_n) = \int_{T^{-1}(A_n)} |u(x)|^q\, d\nu(x) = u(T^{-1}(A_n))^q \nu(T^{-1}(A_n)).$$

Since $N_{u,T}$ is finite, we see that $W(L^p(X))$ is finite-dimensional. This proves that W has finite rank. ∎

COROLLARY 3.6. *For $1 \leq q < p < \infty$. If X is non-atomic and $\nu(Y) > 0$, then there is no non-zero weighted composition operator from $L^p(X)$ into $L^q(Y)$ with closed range.*

Using Theorem 2.1 and Thm. 2.3 from [16], we have the next result.

THEOREM 3.7. *Let $W_{u,T}$ be a weighted composition operator from $L^p(X)$ into $L^q(Y)$, for $1 \leq p < q < \infty$. Then the following statements are equivalent.*

(i) *The operator W has closed range.*

(ii) *The operator $M_h\colon L^p(X) \to L^q(X)$ has closed range.*

(iii) *The operator $M_h\colon L^p(X) \to L^q(X)$ has finite rank.*

(iv) *The set $\{n \in N : h(A_n) \neq 0\}$ is finite.*

In the following results we include the case when $p = \infty$ or $q = \infty$.

THEOREM 3.8. *Suppose* $u \in L^\infty(Y,\Gamma,\nu)$ *and* $T\colon Y \to X$ *is a non-singular measurable transformation. Then* $W\colon L^\infty(X) \to L^\infty(Y)$ *is a continuous weighted composition operator and it has closed range if and only if* $|u(x)| \geq \delta$ *for* μ*-almost all* $x \in \operatorname{supp}(u)$.

THEOREM 3.9. *For* $p = \infty$, $q \in [1,\infty)$ *and* $\mu_{u,T}(X) < \infty$, *the pair* (u,T) *induces the weighted composition operator* $W\colon L^\infty(X) \to L^q(Y)$ *and its norm is given by*

$$\|W_{u,T}\| = (\mu_{u,T}(X))^{\frac{1}{q}}.$$

Moreover, the following statements are equivalent.

(i) *The operator* $W_{u,T}$ *has closed range.*

(ii) *The operator* $W_{u,T}$ *has finite rank.*

(iii) $\mu_{u,T}(B) = 0$ *and the set* $\{n \in N : \mu_{u,T}(A_n) \neq 0\}$ *is finite.*

THEOREM 3.10. *For* $p \in [1,\infty)$ *and* $q = \infty$, *the pair* (u,T) *induces the weighted composition operator* $W\colon L^p(X) \to L^\infty(Y)$ *if and only if* $\nu(T^{-1}(B)) = 0$ *and* $\inf_{n\in N} \mu(A_n) > 0$, *where*

$$N = \{n \in N : \nu(T^{-1}(A_n)) \neq 0\}.$$

In this case, we have

$$\|W_{u,T}\| = (\sup_{n\in N} \frac{1}{\mu(A_n)})^{\frac{1}{p}}.$$

Also, the following statements are equivalent.

(i) *The operator* $W_{u,T}$ *has closed range.*

(ii) *The operator* $W_{u,T}$ *has finite rank.*

(iii) *The set* $\{n \in N : \nu(T^{-1}(A_n)) \neq 0\}$ *is finite.*

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