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ON A UNIQUENESS THEOREM IN THE INVERSE STURM-LIOUVILLE PROBLEM

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Abstract. We introduce new supplementary data to the set of eigenvalues, to determine uniquely the potential and boundary conditions of the Sturm-Liouville problem. As a corollary we obtain extensions of some known uniqueness theorems in the inverse Sturm-Liouville problem.

1. Introduction and statement of the result

Let $L(q, \alpha, \beta)$ denote the Sturm-Liouville problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0, \pi), \ \ \mu \in \mathbf{C}, \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0, \pi], \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0, \pi), \tag{1.3}$$

where q is a real-valued, summable on $[0, \pi]$ function (we write $q \in L^1_{\mathbf{R}}[0, \pi]$). By $L(q, \alpha, \beta)$ we also denote the self-adjoint operator, generated by the problem (1.1)–(1.3). It is known, that the spectrum of $L(q, \alpha, \beta)$ is discrete and consists of simple eigenvalues (see [1], [2]), which we denote by $\mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, emphasizing the dependence of μ_n on q, α and β.

Let $y = \varphi(x, \mu, \alpha, q)$ and $y = \psi(x, \mu, \beta, q)$ be the solutions of (1.1) with initial values

$$\varphi(0, \mu, \alpha, q) = \sin\alpha, \qquad \varphi'(0, \mu, \alpha, q) = -\cos\alpha$$
$$\psi(\pi, \mu, \beta, q) = \sin\beta, \qquad \psi'(\pi, \mu, \beta, q) = -\cos\beta.$$

The eigenvalues μ_n of $L(q, \alpha, \beta)$ are the solutions of the equation

$$\begin{aligned}\chi(\mu) &\stackrel{\text{def}}{=} \varphi(\pi, \mu, \alpha)\cos\beta + \varphi'(\pi, \mu, \alpha)\sin\beta \\ &= -\left[\psi(0, \mu, \beta)\cos\alpha + \psi'(0, \mu, \beta)\sin\alpha\right] = 0.\end{aligned} \tag{1.4}$$

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It is easy to see, that $\varphi_n(x) \stackrel{\text{def}}{=} \varphi(x, \mu_n(q,\alpha,\beta), \alpha, q)$ and $\psi_n(x) \stackrel{\text{def}}{=} \psi(x, \mu_n(q,\alpha,\beta), \beta, q)$, $n = 0, 1, 2, \dots$, are the eigenfunctions, corresponding to the eigenvalue $\mu_n(q,\alpha,\beta)$. The squares of the L^2-norm of these eigenfunctions:

$$a_n = a_n(q,\alpha,\beta) = \int_0^\pi \varphi_n^2(x)\, dx, \tag{1.5}$$

are usually called the norming constants.

Since all eigenvalues are simple, there exist constants $c_n = c_n(q,\alpha,\beta)$, $n = 0, 1, 2, \dots$, such that

$$\varphi_n(x) = c_n \cdot \psi_n(x). \tag{1.6}$$

The main result of this paper is the following "uniqueness" theorem (in inverse problem):

THEOREM 1. *If for all $n = 0, 1, 2, \dots$*

$$\mu_n(q_1,\alpha_1,\beta_1) = \mu_n(q_2,\alpha_2,\beta_2), \tag{A}$$

$$c_n(q_1,\alpha_1,\beta_1) = c_n(q_2,\alpha_2,\beta_2), \tag{B}$$

then $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $q_1(x) = q_2(x)$ almost everywhere (a.e.) on $[0,\pi]$.

The problem $L(q,\alpha,\beta)$ is called "even" if $\alpha + \beta = \pi$ and $q(\pi - x) = q(x)$ a.e. on $[0,\pi]$.

COROLLARY. *The problem $L(q,\alpha,\beta)$ is even if and only if $c_n(q,\alpha,\beta) = (-1)^n$.*

The inverse Sturm-Liouville problems were stated and solved in different versions (see, for example, [3]–[18]). We will consider below the connections between some of the known uniqueness theorems and our Theorem 1 and its corollary (see §5, Theorems 1′, 2, 2′, 3).

2. Some preliminary results

LEMMA 1. *Let $(\alpha,\beta,q) \in (0,\pi] \times [0,\pi) \times L^1_{\mathbf{R}}[0,\pi]$. Then, for $n \geqslant 1$ (except $\mu_0(\alpha,\beta,q)$)*

$$\mu_n(\alpha,\beta,q) = [n + \delta_n(\alpha,\beta)]^2 + [q] + r_n(\alpha,\beta,q) \tag{2.1}$$

where $[q] = \frac{1}{\pi}\int_0^\pi q(x)\, dx$,

$$\delta_n(\alpha,\beta) = \frac{1}{\pi}\left[\arccos\frac{\cos\alpha}{\sqrt{[n+\delta_n(\alpha,\beta)]^2\sin^2\alpha + \cos^2\alpha}} - \arccos\frac{\cos\beta}{\sqrt{[n+\delta_n(\alpha,\beta)]^2\sin^2\beta + \cos^2\beta}}\right],$$

and $r_n = r_n(\alpha, \beta, q) = o(1)$*, when* $n \to \infty$*, uniformly by* $\alpha, \beta \in [0, \pi]$*, and* q *from bounded subsets of* $L^1_{\mathbf{R}}[0, \pi]$*. The well-known asymptotics*

$$\mu_n(\alpha, \beta, q) = n^2 + \frac{2}{\pi}(\operatorname{ctg}\beta - \operatorname{ctg}\alpha) + [q] + \tilde{r}_n(\alpha, \beta, q), \quad \text{if } \sin\alpha \neq 0, \sin\beta \neq 0, \tag{2.2}$$

$$\mu_n(\pi, \beta, q) = \left(n + \frac{1}{2}\right)^2 + \frac{2}{\pi}\operatorname{ctg}\beta + [q] + \tilde{r}_n(\beta, q), \quad \text{if } \sin\beta \neq 0 \ (\beta \in (0, \pi)), \tag{2.3}$$

$$\mu_n(\alpha, 0, q) = \left(n + \frac{1}{2}\right)^2 - \frac{2}{\pi}\operatorname{ctg}\alpha + [q] + \tilde{r}_n(\alpha, q), \text{if } \sin\alpha \neq 0, \ (\alpha \in (0, \pi)), \tag{2.4}$$

$$\mu_n(\pi, 0, q) = (n+1)^2 + [q] + \tilde{r}_n(q), \tag{2.5}$$

where $\tilde{r}_n = o(1)$ *(but this estimate is not uniform in* $(\alpha, \beta) \in [0, \pi]$*), are the particular cases of (2.1). The sequence* $\{\delta_n(\alpha, \beta)\}_{n=1}^{\infty}$ *has the limit*

$$\delta_\infty(\alpha, \beta) = \begin{cases} 0, & \text{if } \alpha, \beta \in (0, \pi), \\ \frac{1}{2}, & \text{if } \alpha = \pi, \ \beta \in (0, \pi) \ \text{ or } \ \alpha(0, \pi), \ \beta = 0, \\ 1, & \text{if } \alpha = \pi, \ \beta = 0. \end{cases} \tag{2.6}$$

For the proof and the details of Lemma 1 see paper [19].

Let $y_i(x, \mu, q)$, $i = 1, 2$, be the solutions of (1.1) with initial values

$$y_1(0, \mu, q) = y_2'(0, \mu, q) = 1,$$
$$y_1'(0, \mu, q) = y_2(0, \mu, q) = 0.$$

It is clear, that

$$\varphi(x, \mu, \alpha, q) \equiv y_1(x, \mu, q)\sin\alpha - y_2(x, \mu, q)\cos\alpha. \tag{2.7}$$

Lemma 2. *1) Let* $q \in L^1_{\mathbf{C}}[0, \pi]$*. Then*

$$y_1(x, \lambda^2, q) = \cos\lambda x + \frac{\sin\lambda x}{2\lambda}\int_0^x q(s)\,ds + \frac{1}{2\lambda}\int_0^x q(s)\sin\lambda(x - 2s)\,ds + O\left(\frac{e^{|\operatorname{Im}\lambda| x}}{|\lambda|^2}\right), \tag{2.8}$$

$$y_2(x, \lambda^2, q) = \frac{\sin\lambda x}{\lambda} - \frac{\cos\lambda x}{2\lambda^2}\int_0^x q(s)\,ds + \frac{1}{2\lambda^2}\int_0^x q(s)\cos\lambda(x - 2s)\,ds + O\left(\frac{e^{|\operatorname{Im}\lambda| x}}{|\lambda|^3}\right). \tag{2.9}$$

In particular (for real λ*)*

$$y_1(\pi, \lambda^2, q) = \cos\lambda\pi + \frac{\sin\lambda\pi}{2\lambda}\int_0^\pi q(s)\,ds + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to +\infty, \tag{2.10}$$

$$y_2(\pi, \lambda^2, q) = \frac{\sin\lambda\pi}{\lambda} - \frac{\cos\lambda\pi}{2\lambda^2}\int_0^\pi q(s)\,ds + o\left(\frac{1}{\lambda^2}\right), \qquad \lambda \to +\infty. \tag{2.11}$$

Also

$$y_1'(x,\lambda^2,q) = -\lambda \sin \pi x + O\left(e^{|\mathrm{Im}\,\lambda|x}\right), \tag{2.12}$$

$$y_2'(x,\lambda^2,q) = \cos \lambda x + O\left(\frac{e^{|\mathrm{Im}\,\lambda|x}}{|\lambda|}\right). \tag{2.13}$$

2) For $\mu = -t^2 = (it)^2 \to -\infty$ $(t \to +\infty)$

$$\chi(\mu) = \chi(-t^2) = \begin{cases} \frac{te^{\pi t}}{2}\left[\sin\alpha \cdot \sin\beta + O(\frac{1}{t})\right], & \textit{if } \sin\alpha \neq 0,\ \sin\beta \neq 0, \\ \frac{e^{\pi t}}{2}\left[\sin\beta + O(\frac{1}{t})\right], & \textit{if } \sin\beta \neq 0,\ \alpha = \pi, \\ \frac{e^{\pi t}}{2t}\left[1 + O(\frac{1}{t})\right], & \textit{if } \alpha = \pi,\ \beta = 0, \end{cases} \tag{2.14}$$

3) Let $q \in L^1_{\mathbf{R}}[0,\pi]$. *Then*

$$\varphi(\pi,\mu,\alpha,q) = \sum_{n=0}^{\infty} \varphi(\pi,\mu_n,\alpha,q) \cdot \prod_{\substack{m \neq n \\ m=0}}^{\infty} \frac{\mu_m - \mu}{\mu_m - \mu_n}. \tag{2.15}$$

Proof. 1) The asymptotic formulae (2.8)–(2.13) are proved in detail in [19], or they are corollaries of the results of [19] (see also [8]). For $q \in L^2[0,\pi]$ they can be found in [10], [11] and other papers.

2) Relation (2.14) is the corollary of (1.4), (2.7) and (2.8)–(2.13).

3) For $q \in L^2_{\mathbf{R}}[0,\pi]$ (2.15) is proved in [11] (more detailed proof is presented in [17]). For $q \in L^1_{\mathbf{R}}[0,\pi]$ the proof is the same. ■

Now we establish some connections between spectral data. The following formula is well known (see, e.g., [18], (2.8))

$$\int_0^{\pi} \varphi_n^2(x)\,dx = \varphi'(\pi,\mu_n) \cdot \overset{\circ}{\varphi}(\pi,\mu_n) - \overset{\circ}{\varphi}{}'(\pi,\mu_n) \cdot \varphi(\pi,\mu_n)$$

($\overset{\circ}{f}(x,\mu) = \dfrac{\partial}{\partial\mu} f(x,\mu)$) which is equivalent to (see (1.4), (1.5, (1.6))

$$a_n(q,\alpha,\beta) = -c_n(q,\alpha,\beta) \cdot \overset{\circ}{\chi}(\mu_n). \tag{2.16}$$

By definition (1.6) we have, that ($\alpha \in (0,\pi]$)

$$c_n(q,\alpha,\beta) = \frac{\varphi(\pi,\mu_n(q,\alpha,\beta),\alpha,q)}{\sin\beta}, \qquad \sin\beta \neq 0 \ \ (\beta \neq 0) \tag{2.17}$$

and

$$c_n(q,\alpha,0) = -\varphi'\left(\pi,\mu_n(q,\alpha,0),\alpha\right). \tag{2.18}$$

The normalized eigenfunctions h_n we define as

$$h_n(x) = \frac{\varphi_n(x)}{\|\varphi_n\|}. \tag{2.19}$$

Now we present the definitions of spectral data $\ell_n = \ell_n(q,\alpha,\beta)$, which were introduced in [10], [11] and [17] (as supplementary data to eigenvalues), and their connection with our spectral data $c_n = c_n(q,\alpha,\beta)$, that follows from 1.6 and (2.17)–(2.19).

$$\ell_n(q,\alpha,\beta) = \log\left[(-1)^n \cdot \frac{h_n(\pi)}{h_n(0)}\right] = \log\left[(-1)^n c_n(q,\alpha,\beta)\cdot\frac{\sin\beta}{\sin\alpha}\right], \quad \text{if } \sin\alpha\neq 0, \sin\beta\neq 0, \tag{2.20}$$

$$\ell_n(q,\pi,\beta) = \log\left[(-1)^n \cdot \frac{h_n(\pi)}{h'_n(0)}\right] = \log\left[(-1)^n c_n(q,\pi,\beta)\cdot\sin\beta\right], \quad \text{if } \sin\beta\neq 0, \alpha=\pi, \tag{2.21}$$

$$\ell_n(q,\alpha,0) = \log\left[(-1)^{n+1} \cdot \frac{h'_n(\pi)}{h_n(0)}\right] = \log\left[(-1)^n c_n(q,\alpha,0)\cdot\frac{1}{\sin\alpha}\right], \quad \text{if } \sin\alpha\neq 0,\ \beta=0, \tag{2.22}$$

$$\ell_n(q,\pi,0) = \log\left[(-1)^n \cdot \frac{h'_n(\pi)}{h'_n(0)}\right] = \log\left[(-1)^n c_n(q,\pi,0)\right], \text{if } \alpha=\pi,\ \beta=0. \tag{2.23}$$

3. The proof of Theorem 1

We prove Theorem 1 in 4 steps. At first we consider the case $\alpha_1=\pi$, $\beta_1=0$. From condition (A), (2.1) and (2.5) we obtain ($n=0,1,2,\dots$)

$$(n+1)^2 + [q_1] + \tilde r_n(q_1,\pi,0) = (n+\delta_n(\alpha_2,\beta_2))^2 + [q_2] + r_n(q_2,\alpha_2,\beta_2).$$

It follows easily that $\delta_n(\alpha_2,\beta_2)\to 1$, when $n\to\infty$. According to (2.6), it is possible only if $\alpha_2=\pi$, $\beta_2=0$. Then, from condition (B) and (2.23), we obtain $\ell_n(q_1,\pi,0)=\ell_n(q_2,\pi,0)$ for $n=0,1,2,\dots$, and we can repeat the proof of Theorem 5, chapter III, of [10], page 62, to obtain $q_1(x)=q_2(x)$, a.e.

Remark. The uniqueness theorems in [10], [11] and [17] are proved under condition $q_1,q_2\in L^2_{\mathbf R}[0,\pi]$, but they are true also for $q_1,q_2\in L^1_{\mathbf R}[0,\pi]$, because the asymptotic formulae and estimates (see (2.8)–(2.13)) for solutions of (1.1) (which are used particularly to prove that some contour integrals tend to zero) are true also for $q\in L^1[0,\pi]$, as it is proved in details in [20].

Secondly, we consider the case $\alpha_1=\pi$, $\beta\in(0,\pi)$. Then condition (A) gives us

$$\left(n+\frac12\right)^2 + \frac{2}{\pi}\operatorname{ctg}\beta_1 + [q_1] + r_n(q_1,\pi,\beta_1) = [n+\delta_n(\alpha_2,\beta_2)]^2 + [q_2] + r_n(q_2,\alpha_2,\beta_2) \tag{3.1}$$

by (2.1) and (2.3). It easy to prove from (3.1), that $\lim_{n\to\infty}\delta_n(\alpha_2,\beta_2)=\frac12$, and by (2.6) it is possible only if $\alpha_2=\pi$, $\beta_2\in(0,\pi)$ or $\alpha_2\in(0,\pi)$, $\beta_2=0$.

In the case $\alpha_2=\pi$, $\beta_2\in(0,\pi)$ we have

$$\frac{2}{\pi}\operatorname{ctg}\beta_1 + [q_1] = \frac{2}{\pi}\operatorname{ctg}\beta_2 + [q_2]$$

by (3.1) and (2.3). Also

$$\frac{y_2(\pi,\mu_n,q_1)}{\sin\beta_1}=\frac{y_2(\pi,\mu_n,q_2)}{\sin\beta_2}$$

by condition (B) and (2.17). Together with (A) and (2.15) we obtain

$$\frac{y_2(\pi,\mu,q_1)}{\sin\beta_1}=\frac{y_2(\pi,\mu,q_2)}{\sin\beta_2} \tag{3.2}$$

for all $\mu\in\mathbf{C}$. Substituting $\mu=\left(n+\frac{1}{2}\right)^2$ in (3.2), by (2.11) we obtain

$$\frac{y_2\left(\pi,\left(n+\frac{1}{2}\right)^2,q_1\right)}{\sin\beta_1}=\frac{1}{\sin\beta_1}\left[\frac{(-1)^n}{n+\frac{1}{2}}+\frac{o(1)}{\left(n+\frac{1}{2}\right)^2}\right]=\frac{1}{\sin\beta_2}\left[\frac{(-1)^n}{n+\frac{1}{2}}+\frac{o(1)}{\left(n+\frac{1}{2}\right)^2}\right].$$

It follows that $\sin\beta_1-\sin\beta_2=\frac{o(1)}{n+\frac{1}{2}}$, i.e. $\sin\beta_1=\sin\beta_2$. Then, by (2.21), we have $\ell_n(q_1,\pi,\beta_1)=\ell_n(q_2,\pi,\beta_2)$, $n=0,1,2\ldots$, and we can repeat the proof of Theorem 3 in [17] to obtain $\beta_1=\beta_2$ and $q_1(x)=q_2(x)$, a.e.

In the case $\alpha_2\in(0,\pi)$, $\beta_2=0$ from condition (B), according to (2.17) and (2.18) $\frac{y_2(\pi,\mu_n,q_1)}{\sin\beta_1}=-\varphi'(\pi,\mu_n,\alpha_2,q_2)$ and by (2.11), (2.7), (2.12) and (2.13) we obtain

$$\frac{1}{\sin\beta_1}\left\{\frac{\sin\sqrt{\mu_n}\,\pi}{\sqrt{\mu_n}}-\frac{\cos\sqrt{\mu_n}\,\pi}{2\mu_n}\int_0^\pi q_1(s)\,ds+\frac{o(1)}{\mu_n}\right\}=$$
$$=\{-\sqrt{\mu_n}\,\sin\sqrt{\mu_n}\,\pi+O(1)\}\sin\alpha_2+\left\{\cos\sqrt{\mu_n}\,\pi+O\left(\frac{1}{\sqrt{\mu_n}}\right)\right\}\cos\alpha_2.$$

Since $\sin\beta_1\neq 0$ and $\sin\alpha_2\neq 0$, the last equality is impossible (the left-hand side tends to zero, when $n\to\infty$, but the right-hand side does not). Thus in the case $\alpha_1=\pi$, $\beta_1\in(0,\pi)$, Theorem 1 is also proved.

The third case is $\alpha_1\in(0,\pi)$, $\beta_1=0$. In this case from condition (A), (2.1) and (2.4) we obtain

$$\left(n+\frac{1}{2}\right)^2-\frac{2}{\pi}\operatorname{ctg}\alpha_1+[q_1]+r_n(q_1,\alpha_1,0)=[n+\delta_n(\alpha_2,\beta_2)]^2+[q_2]+r_n(q_2,\alpha_2,\beta_2)$$

From this equality it follows easily that $\lim_{n\to\infty}\delta_n(\alpha_2,\beta_2)=\frac{1}{2}$, and therefore, either $\alpha_2=\pi$, $\beta_2\in(0,\pi)$ (as proved above, this case is impossible), or $\alpha_2\in(0,\pi)$, $\beta_2=0$. Similarly to the second case, we prove that $\sin\alpha_1=\sin\alpha_2$ and by (2.16) we obtain that $\ell_n(q_1,\alpha_1,0)=\ell_n(q_2,\alpha_2,0)$. According to Theorem 4 of [17] we get $\alpha_1=\alpha_2$ and $q_1(x)=q_2(x)$, a.e.

The fourth and the last case is $\sin\alpha_1\neq 0$ and $\sin\beta_1\neq 0$, i.e. $\alpha_1,\beta_1\in(0,\pi)$. The cases $\alpha_2=\pi$ or $\beta_2=0$ are impossible, since they reduce to cases I, II or III. Therefore $\alpha_2,\beta_2\in(0,\pi)$. It follows from (A) and (2.2) that $\lim_{n\to\infty}\left(\mu_n(q_1,\alpha_1,\beta_1)-n^2\right)=\frac{2}{\pi}\left(\operatorname{ctg}\alpha_1-\operatorname{ctg}\beta_1\right)+\frac{1}{\pi}\int_0^\pi q_i(t)\,dt=$

$= \lim_{n\to\infty}\left(\mu_n(q_2,\alpha_2,\beta_2)-n^2\right) = \frac{2}{\pi}(\operatorname{ctg}\alpha_2-\operatorname{ctg}\beta_2)+[q_2]$. Also we have by (A), (B) and (2.17) $\dfrac{\varphi(\pi,\mu_n,\alpha_1,q_1)}{\sin\beta_1}=\dfrac{\varphi(\pi,\mu_n,\alpha_2,q_2)}{\sin\beta_2}$. Then, by (2.15) we obtain $\dfrac{\varphi(\pi,\mu,\alpha_1,q_1)}{\sin\beta_1}=\dfrac{\varphi(\pi,\mu,\alpha_2,q_2)}{\sin\beta_2}$ for all $\mu\in\mathbf{C}$. Now, by (2.7), (2.10) and (2.11) for $\mu=n^2$ we have

$$\begin{aligned}\frac{\varphi(\pi,n^2,\alpha_1,q_1)}{\sin\beta_1}&=\frac{\sin\alpha_1}{\sin\beta_1}\left\{(-1)^n+\frac{o(1)}{n}\right\}\\&=\frac{\sin\alpha_2}{\sin\beta_2}\left\{(-1)^n+\frac{o(1)}{n}\right\}=\frac{\varphi(\pi,n^2,\alpha_2,q_2)}{\sin\beta_2}\end{aligned}$$

and it follows easily that $\dfrac{\sin\alpha_1}{\sin\beta_1}=\dfrac{\sin\alpha_2}{\sin\beta_2}$. Thus, by (2.20) we obtain $\ell_n(q_1,\alpha_1,\beta_1)=\ell_n(q_2,\alpha_2,\beta_2)$, $n=0,1,2,\dots$, and by the uniqueness theorem of [11] we have that $\alpha_1=\alpha_2$, $\beta_1=\beta_2$ and $q_1(x)=q_2(x)$, a.e. The proof of Theorem 1 is complete. ■

4. Proof of the Corollary

Let $q^*(x)\overset{\text{def}}{=}q(\pi-x)$. It is easily verified that (see [11])

$$\varphi(\pi-x,\mu,\alpha,q^*)\equiv\psi(x,\mu,\pi-\alpha,q) \tag{4.1}$$

and

$$\mu_n(q,\alpha,\beta)=\mu_n(q^*,\pi-\beta,\pi-\alpha),\qquad n=0,1,2,\dots\ . \tag{4.2}$$

Lemma 3. *For all $n=0,1,2,\dots$, $\alpha\in(0,\pi]$ and $\beta\in[0,\pi)$ the equality*

$$c_n(q,\alpha,\beta)\cdot c_n\left(q^*,\pi-\beta,\pi-\alpha\right)=1 \tag{4.3}$$

is true.

Proof. By (4.1), (4.2) and (1.6)

$$\begin{aligned}\psi(x,\mu_n(q,\alpha,\beta),\beta,q)&\equiv\varphi\left(\pi-x,\mu_n(q,\alpha,\beta),\pi-\beta,q^*\right)\\&\equiv\varphi\left(\pi-x,\mu_n\left(q^*,\pi-\beta,\pi-\alpha\right),\pi-\beta,q^*\right)\\&\equiv c_n\left(q^*,\pi-\beta,\pi-\alpha\right)\psi\left(\pi-x,\mu_n\left(q^*,\pi-\beta,\pi-\alpha\right),\pi-\alpha,q^*\right)\\&\equiv c_n\left(q^*,\pi-\beta,\pi-\alpha\right)\cdot\varphi\left(x,\mu_n\left(q,\alpha,\beta\right),\alpha,q\right)\\&\equiv c_n\left(q^*,\pi-\beta,\pi-\alpha\right)\cdot c_n(q,\alpha,\beta)\cdot\psi\left(x,\mu_n\left(q,\alpha,\beta\right),\beta,q\right).\end{aligned}$$

It follows that (4.3) holds true.

To prove the sufficiency we note that if $c_n(q,\alpha,\beta)=(-1)^n$, then $c_n\left(q^*,\pi-\beta,\pi-\alpha\right)=(-1)^n$ by (4.3) and since $\mu_n(q,\alpha,\beta)=\mu_n(q^*,\pi-\beta,\pi-\alpha)$, then $q(x)=q^*(x)$ and $\alpha=\pi-\beta$ by Theorem 1.

If problem $L(q,\alpha,\beta)$ is even, i.e. $q(\pi-x)=q(x)$ and $\alpha+\beta=\pi$, then $c_n^2(q,\alpha,\beta)=1$ by (4.3). Since the roots μ_n of function $\chi(\mu)$ are simple, then $\mathring{\chi}(\mu_n)$ and $\mathring{\chi}(\mu_{n+1})$ have the different sign and since $a_n>0$, it follows that c_n and c_{n+1}

have the different sign by $a_n = -c_n \cdot \overset{\circ}{\chi}(\mu_n)$ (see (2.16)). If we show that $\overset{\circ}{\chi}(\mu_0) < 0$, we will obtain that $c_0(q,\alpha,\beta) = 1 = (-1)^0$ and therefore $c_n(q,\alpha,\beta) = (-1)^n$.

Really, it follows from (2.14) that when μ changes from $-\infty$ to μ_0, $\chi(\mu)$ changes from $+\infty$ to 0, i.e. $\overset{\circ}{\chi}(\mu_0) < 0$. The proof of corollary is complete. ∎

5. Some extentions

Following reasons, very similar to the proof of Theorem 1, we see that the following holds.

Theorem 2. *Let* $(\alpha_i, \beta_i, q_i) \in (0,\pi] \times [0,\pi) \times L^1_{\mathbf{R}}[0,\pi]$, $i = 1,2$. *If* $\mu_n(q_1,\alpha_1,\beta_1) = \mu_n(q_2,\alpha_2,\beta_2)$ *and* $\ell_n(q_1,\alpha_1,\beta_1) = \ell_n(q_2,\alpha_2,\beta_2)$ *for all* $n = 0,1,2,\dots$, *then* $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ *and* $q_1(x) = q_2(x)$, *a.e.*

If, following [11], we introduce the set

$$M(p,\alpha_0,\beta_0) = \big\{(q,\alpha,\beta) \in L^1_{\mathbf{R}}[0,\pi] \times (0,\pi] \times [0,\pi) : \\ \mu_0(q,\alpha,\beta) = \mu_n(p,\alpha_0,\beta_0),\ n \geqslant 0\big\}\,,$$

then we can formulate next theorem (in terms of [11]), which follows from Theorem 1 and its Corollary.

Theroem 1'. *(i) The mapping*

$$(\alpha,\beta,q) \in (0,\pi] \times [0,\pi) \times L^1_{\mathbf{R}}[0,\pi] \mapsto (\mu_n(q,\alpha,\beta),\ c_n(q,\alpha,\beta)\ \ n \geqslant 0)$$

is one to one. Equivalently, the mapping

$$(q,\alpha,\beta) \in M(p,\alpha_0,\beta_0) \mapsto (c_n(q,\alpha,\beta)\ \ n \geqslant 0)$$

is one to one.

(ii) The mapping

$$(q,\alpha,\beta) \in L^1_{\mathbf{R}}[0,\pi] \times (0,\pi] \times [0,\pi) \mapsto (\mu_n(q,\alpha,\beta);\ n \geqslant 0)\,,$$

is one to one when restricted to the subset of even points (i.e. $\alpha+\beta=\pi$, $q(\pi-x) = q(x)$*) in* $L^1_{\mathbf{R}}[0,\pi] \times (0,\pi] \times [0,\pi)$.

If in Theorem 1' we change $c_n(q,\alpha,\beta)$ to $\ell_n(q,\alpha,\beta)$ we obtain a proposition (call it Theorem 2'), which follows from Theorem 2 and its Corollary (see [11]: $L(q,\alpha,\beta)$ even if and only if $\ell_n(q,\alpha,\beta) = 0$, $n \geqslant 0$), and which not only joins the uniqueness theorems of [10], [11] and [17], but also extend them.

Also the connection (2.16) shows that Theorem 1 is equivalent to

Theorem 3. *Let* $(\alpha_i, \beta_i, q_i) \in (0,\pi] \times [0,\pi) \times L^1_{\mathbf{R}}[0,\pi]$, $i = 1,2$. *If* $\mu_n(q_1,\alpha_1,\beta_1) = \mu_n(q_2,\alpha_2,\beta_2)$ *and* $a_n(q_1,\alpha_1,\beta_1) = a_n(q_2,\alpha_2,\beta_2)$ *for all* $n = 0,1,2,\dots$, *then* $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ *and* $q_1(x) = q_2(x)$, *a.e.*

Of course, it is a variant of the Theorem of Marchenko [8] for finite intervals, which is usually ([9], [16], [21]) formulated for $\alpha_i, \beta_i \in (0,\pi)$, with condition $\dfrac{a_n(q_1,\alpha_1,\beta_1)}{\sin^2\alpha_1} = \dfrac{a_n(q_2,\alpha_2,\beta_2)}{\sin^2\alpha_2}$ instead of $a_n(q_1,\alpha_1,\beta_1) = a_n(q_2,\alpha_2,\beta_2)$.

REFERENCES

[**1**] Levitan, B. M., Sargsyan, I. S., *Sturm-Liouville and Dirac Operators* (in Russian), Nauka, Moscow, 1988.

[**2**] Marchenko, V. A., *The Sturm-Liouville Operators and Their Applications* (in Russian), Naukova Dumka, Kiev, 1977.

[**3**] Ambarzumian, V. A., *Über eine Frage der Eigenwerttheorie*, Z. Physik, **53** (1929), 690–695.

[**4**] Borg, G., *Eine Umkehrung der Sturm-Liouvillschen Eigenwertaufgabe*, Acta Math. **78**,1 (1946), 1–96.

[**5**] Levinson, N., *The inverse Sturm-Lioville problem*, Mat. Tidsskr B. (1949), 25–30.

[**6**] Gel'fand, I. M., Levitan, B. M., *On the determination of a differential equation from its spectral function* (in Russian), Izv. Akad. Nauk SSSR, Ser. Math., **15** (1951), 253–304.

[**7**] Krein, M. G., *Solution of the inverse Sturm-Liouville problem* (in Russian), Dokl. Akad. Nauk SSSR, **76** (1951), 21–24.

[**8**] Marchenko, V. A., *Concerning the theory of differential operators of the second order* (in Russain), Trudy Moskov. Math. Obshch., **1** (1952), 327–420.

[**9**] Marchenko, V. A., *Some problems in the theory of differential operators of the second order* (in Russian), Dokl. Akad. Nauk SSSR, **72** (1950), 457–460.

[**10**] Pöschel, J., Trubowitz E., *Inverse Spectral Theory*, Acad. Press, 1987.

[**11**] Isaacson, E. L., Trubowitz, E., *The inverse Sturm-Liouville problem, I*, Com. Pure Appl. Math., **36** (1983), 767–783.

[**12**] Isaacson, E. L., McKean, H. P., Trubowitz, E., *The inverse Sturm-Liouville problem, II*, Com. Pure Appl. Math., **37** (1984), 1–11.

[**13**] Dahlberg, B. E. I., Trubowitz, E., *The inverse Sturm-Liouville problem, III*, Com. Pure Appl. Math., **37** (1984), 255–267.

[**14**] McLaughlin, J. R., Rundell, W., *A uniqueness theorem for an inverse Sturm-Liouville problem*, J. Math. Phys., **28**, 7 (1984), 1471–1472.

[**15**] McLaughlin, J. R., *On uniqueness theorems for second order inverse eigenvalue problems*, J. Math. Anal. Appl., **118** (1986), 38–41.

[**16**] McLaughlin, J. R., *Analytical methods for recovering coefficients in differential equations from spectral data*, SIAM Review, **28**, 1 (1986), 53–72.

[**17**] Harutyunyan, T. N., Nersesyan, V. V., *A uniqueness theorem in inverse Sturm-Liouville problem*, J. Contemporary Math. Anal., **39**, 6 (2004), 27–36.

[**18**] Koyunbakan, H., Panakhov, E. S., *A uniqueness theorem for inverse nodal problem*, Inverse problems in Science and Engineering, **15**, 6 (2007), 517–524.

[**19**] Harutyunyan, T. N., *The dependence of the eigenvalues of the Sturm-Liouville problem on boundary conditions*, Mat. Vesnik, **60**, 4 (2008), 285–294.

[**20**] Harutyunyan, T. N., Hovsepyan, M. S., *On the solutions of the Sturm-Liouville equation*, Mathem. in Higher School, **I**, 3 (2005), 59–74.

[**21**] Levitan, B. M., *Generalized Translation Operators and Some of Their Applications* (in Russian), Fizmatgiz, Moscow, 1962.

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