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CHARACTERIZATIONS OF δ-STRATIFIABLE SPACES

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Abstract. In this paper, we give some characterizations of δ-stratifiable spaces by means of g-functions and semi-continuous functions. It is established that:

(1) A topological space X in which every point is a regular G_δ-set is δ-stratifiable if and only if there exists a g-function $g : N \times X \to \tau$ satisfies that if $F \in RG(X)$ and $y \notin F$, then there is an $m \in N$ such that $y \notin \overline{g(m, F)}$;

(2) If there is an order preserving map $\varphi : USC(X) \to LSC(X)$ such that for any $h \in USC(X), 0 \le \varphi(h) \le h$ and $0 < \varphi(h)(x) < h(x)$ whenever $h(x) > 0$, then X is δ-stratifiable space.

1. Introduction

It is one of the questions in general topology how to characterize the generalized metric spaces [2, 5]. Recently, the problem of monotone insertions of generalized metric spaces has been studied [4]. Lane, Nyikos and Pan [7] proved that a topological space X is stratifiable if and only if there is an order-preserving map $\psi : UL(X) \to C(X)$ such that for any $(g, h) \in UL(X)$, $g \le \psi(g, f) \le h$ and $g(x) < \psi((g, h))(x) < h(x)$ whenever $g(x) < h(x)$.

As a generalization of stratifiable spaces, Good and Haynes [3] defined δ-stratifiable spaces. Just as for stratifiablity, they discussed the products of compact metrizable spaces and δ-stratiable spaces. It is natural to pose the following question.

Question 1.1. How to characterize δ-stratifiable spaces by the g-functions, or by the monotone insertion functions?

In this paper, we give some characterizations of δ-stratifiable spaces by means of g-functions and semi-continuous functions.

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All spaces in this paper are assumed to be T_1. For a topological space X, τ denotes the topology on X, and $\tau^c = \{X - O : O \in \tau\}$. We refer the reader to [1, 8] for undefined terms.

A real-valued function f defined on a space X is *lower (upper) semi-continuous* if for each $x \in X$ and each real number r with $f(x) > r$ ($f(x) < r$), there exists a neighborhood U of x in X such that $f(x') > r$ ($f(x') < r$) for every $x' \in U$. We write $LSC(X)$ ($USC(X)$) for the set of all real-valued lower (upper) semi-continuous functions on X into $I = [0, 1]$.

Let X be a space, if $A \subset X$, we write χ_A for the *characteristic function* on A. Then $\chi_A \in USC(X)$ if A is a closed subset of X, and $\chi_A \in LSC(X)$ if A is an open subset in X.

A g-function on a topological space (X, τ) is a mapping $g : N \times X \to \tau$ such that $x \in g(n, x)$ for each $n \in N$ [6]. We define $g(n, F) = \bigcup\{g(n, x) : x \in F\}$ for each $F \subset X$ and each $n \in N$.

We recall some basic concepts about the δ-stratifiable spaces.

Definition 1.2. [10] A subset G of a topological space X is called a regular G_δ-set if G is an intersection of a sequence of closed sets whose interiors contain G, i.e. if $G = \bigcap_{n\in N} F_n = \bigcap_{n\in N} F_n^\circ$, where F_n is a closed set of X. Equivalently, there exists a sequence $\{U_n\}$ of open sets such that $G = \bigcap_{n\in N} U_n = \bigcap_{n\in N} \overline{U}_n$. The complement of a regular G_δ-set is called a regular F_σ-set. Clearly, a set M is a regular F_σ-set if and only if there exists a sequence $\{F_n\}$ of closed sets such that $M = \bigcup_{n\in N} F_n = \bigcup_{n\in N} F_n^\circ$.

For a topological space X, $RG(X)$ denotes the set of all regular G_δ-sets of X, and $RF(X) = \{X - G : G \in RG(X)\}$ are the sets of all regular F_σ-sets of X.

Definition 1.3. [3] A topological space X is δ-stratifiable if and only if there is an operator U assigning to each $n \in N$ and $D \in RG(X)$, an open set $U(n, D)$ containing D such that

(1) If $E, D \in RG(X)$ and $E \subset D$, then $U(n, E) \subset U(n, D)$ for each $n \in N$;

(2) $D = \bigcap_{n\in N} \overline{U(n, D)}$.

We may assume that the operator U is also monotonic with respect to n, so that $U(n + 1, D) \subset U(n, D)$ for each $n \in N$ and each $D \in RG(X)$.

The following lemma, included for convenience, is clearly just another way of stating the definition.

Lemma 1.4. *A topological space X is δ-stratifiable if and only if there is an operator $V : N \times RF(X) \to \tau^c$, such that*

(1) $F \supset V(n, F)$ for each $F \in RF(X)$ and all $n \in N$;

(2) If $E, F \in RF(X)$ and $E \subset F$, then $V(n, E) \subset V(n, F)$ for each $n \in N$;

(3) $F = \bigcup_{n\in N} V^\circ(n, F)$.

We may assume that $V(n, F) \subset V(n + 1, F)$ for each $n \in N$ and each $F \in RF(X)$.

2. Main results and their proofs

First, we give a characterization of δ-stratifiable spaces by the g-functions.

LEMMA 2.1. *If every point of X is a regular G_δ-set, X is δ-stratifiable if only and if there exists a g-function $g : N \times X \to \tau$ satisfying that if $F \in RG(X)$ and $y \notin F$, then there is an $m \in N$ such that $y \notin \overline{g(m,F)}$.*

Proof. Let X be δ-stratifiable and U an operator on X which satisfies conditions (1) and (2) in Definition 1.3. For each $x \in X$, let $g(n,x) = U(n,\{x\})$, then $g : N \times X \to \tau$ is a g-function. Let $F \in RG(X)$ and $y \notin F$; we have $y \notin F = \bigcap_{n\in N} \overline{U(n,F)}$ by condition (2) of Definition 1.3. Thus there is an $m \in N$ such that $y \notin \overline{U(m,F)}$, and therefore $y \notin \overline{g(m,F)}$.

Conversely, suppose there exists a g-function $g : N \times X \to \tau$ that satisfies the conditions given in the theorem. For each $D \in RG(X)$, let

$$U(n,D) = g(n,D) = \bigcup\{g(n,t) : t \in D\}.$$

Then U is an operation on X which satisfies the conditions (1) and (2) in Definition 1.3. In fact, it is clear for (1). For (2), if $D \in RG(X)$ and $D \neq \bigcap_{n\in N} \overline{U(n,D)}$, there exists $y \in \bigcap_{n\in N} \overline{U(n,D)} - D$. Since $y \notin D$, there exists $m \in N$ such that $y \notin \overline{U(m,D)}$ by the condition of the theorem; this is a contradiction with $y \in \overline{U(n,D)}$ for each $n \in N$. ■

Next, we characterize δ-stratifiable spaces by semi-continuous functions.

THEOREM 2.2. *A space X is δ-stratifiable if and only if for any partially ordered set $(\mathbb{H},<)$ and map $F : N \times \mathbb{H} \to RG(X)$ such that*

(1) *$F(n+1,h) \subset F(n,h)$ for all $h \in H$ and all $n \in N$;*

(2) *for any $h_1, h_2 \in H$, if $h_1 \leq h_2$ then $F(n,h_2) \subset F(n,h_1)$,*

there is a map $G : N \times \mathbb{H} \to \tau$, such that (1) and (2) hold for G, $F(n,h) \subset G(n,h)$ for all $h \in \mathbb{H}$, $n \in N$ and $\bigcap_{n\in N} F(n,h) = \bigcap_{n\in N} \overline{G(n,h)}$ for all $h \in \mathbb{H}$.

Proof. Let X be a δ-stratifiable space and V an operator as in Lemma 1.4. We show that the map $G : N \times \mathbb{H} \to \tau$ defined by

$$G(n,h) = X - V(n, X - F(n,h)),$$

satisfies the conditions of the theorem. By the properties of V and F, one can easily verify that the conditions (1) and (2) hold for G. Since $F(n,h) \in RG(X)$ for each $h \in \mathbb{H}$ and all $n \in N$, then $X - F(n,h) \in RF(X)$. By the condition (1) in Lemma 1.4, $X - F(n,h) \supset V(n, X - F(n,h))$ and so $F(n,h) \subset G(n,h)$ for each $h \in \mathbb{H}$ and all $n \in N$.

So we need only to show that $\bigcap_{n\in N} F(n,h) = \bigcap_{n\in N} \overline{G(n,h)}$ for all $h \in \mathbb{H}$. If $x \notin \bigcap_{n\in N} F(n,h)$, then $x \notin F(m_0,h)$ for some $m_0 \in N$. Consequently, $x \in V^\circ(n_0, X - F(m_0,h))$ for some $n_0 \in N$ since $X - F(m_0,h) = \bigcup_{n\in N} V^\circ(n, X -$

$F(m_0,h))$. Let $m = max\{n_0, m_0\}$, then $x \in V^\circ(n_0, X - F(m_0,h)) \subset V(m, X - F(m_0,h)) \subset V(m, X - F(m,h))$, and $x \in V^\circ(m, X - F(m,h))$. But $V(m, X - F(m,h)) \cap G(m,h) = \emptyset$, hence $x \notin \overline{G(m,h)}$, so $x \notin \bigcap_{n\in N} \overline{G(n,h)}$, which proves the necessity.

Conversely, for each $D \in RF(X)$, consider the map $F : N \times RF(X) \to RG(X)$ defined by $F(n,D) = X - D$. One can easily verify that F satisfies the conditions (1) and (2) above. So there is a map $G : N \times RF(X) \to \tau$ such that the conditions (1) and (2) hold for G. Moreover, $F(n,D) \subset G(n,D)$ for all $n \in N$ and all $D \in RF(X)$ and $\bigcap_{n\in N} F(n,D) = \bigcap_{n\in N} \overline{G(n,D)}$. Let $V(n,D) = X - G(n,D)$, then the map $V : N \times RF(X) \to \tau^c$ satisfies the conditions in Lemma 1.4. In fact, it is clear that the condition (2) holds; $V(n,D) \subset D$ because $V(n,D)$ is a subset of $X - G(n,D)$, which is a closed subset of $X - F(n,D)$=D, the condition (1) holds. We now show that the condition (3) holds. We only need to show that $D \subset \bigcup_{n\in N} V^\circ(n,D)$. If $x \notin \bigcup_{n\in N} V^\circ(n,D)$, then $x \notin V^\circ(n,D) = X - \overline{X - V(n,D)} = X - \overline{G(n,D)}$ for all $n \in N$. This implies that $x \in \bigcap_{n\in N} \overline{G(n,D)} = \bigcap_{n\in N} F(n,D) = X - D$, hence $x \notin D$. So X is δ-stratfiable. ■

Let $(X,<)$ and $(Y,<')$ be a partially ordered sets. A map $\psi : X \to Y$ is said to be *order-preserving* [1] if $\psi(x) <' \psi(y)$ for every pair $x, y \in X$ with $x < y$.

Theorem 2.3. *Let X be a topological space. If there is an order preserving map $\varphi : USC(X) \to LSC(X)$ such that for any $h \in USC(X), 0 \le \varphi(h) \le h$ and $0 < \varphi(h)(x) < h(x)$ whenever $h(x) > 0$, then X is δ-stratifiable.*

Proof. Suppose that there is a map $\varphi : USC(X) \to LSC(X)$ that satisfies the conditions of the theorem. For any $F \in RF(X)$, $F = \bigcup_{n\in N} W_n = \bigcup_{n\in N} W_n^\circ$, W_n is a closed subset of X by Definition 1.2. Let $h_{W_n} = \chi_{W_n}$; then $h_{W_n} \in USC(X)$. Let

$$h_F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} h_{W_n}(x);$$

then $h_F \in USC(X)$ by Theorem 2.4 in [11] and so $\varphi(h_F) \in LSC(X)$. For each $n \in N$, let

$$V(n,F) = \{x \in X : \varphi(h_F)(x) > 1/2^n\} \text{ and } \overline{V}(n,F) = \overline{\{x \in X : \varphi(h_F)(x) > 1/2^n\}}.$$

Then the equality above defines a map $\overline{V} : N \times RF(X) \to \tau^c$. We shall show that the map V satisfies the conditions (1) through (3) in Lemma 1.4.

For each $n \in N$, if $x \in V(n,F)$, then $1/2^n < \varphi(h_F)(x) \le h_F(x)$. So

$$h_F(x) = \sum_{k=1}^{n} \frac{1}{2^k} h_{W_k}(x) + \sum_{k=n+1}^{\infty} \frac{1}{2^k} h_{W_k}(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} h_{W_n}(x) > 1/2^n > 0,$$

but

$$\sum_{k=n+1}^{\infty} \frac{1}{2^k} h_{W_k}(x) \le \sum_{k=n+1}^{\infty} \frac{1}{2^k} = 1/2^n.$$

Thus $\sum_{k=1}^{n}\frac{1}{2^k}h_{W_k}(x)>0$. Hence there is $k\in\{1,2,\cdots,n\}$ such that $x\in W_k\subset\bigcup_{1\le k\le n}W_k$, and $V(n,F)\subset\bigcup_{1\le k\le n}W_k$. This implies that

$$\overline{V}(n,F)\subset\overline{\bigcup_{1\le k\le n}W_k}\subset\bigcup_{1\le k\le n}\overline{W}_k=\bigcup_{1\le k\le n}W_k\subset F$$

for each $n\in N$ and so

$$\bigcup_{n\in N}V(n,F)\subset\bigcup_{n\in N}\overline{V}(n,F)\subset F.$$

We show that reverse inclusion. If $x\notin\bigcup_{n\in N}V(n,F)$, then

$$x\in\bigcap_{n\in N}\{t\in X:\varphi(h_F)(t)\le 1/2^n\}=\{t\in X:\varphi(h_F)(t)=0\},$$

thus $\varphi(h_F)(x)=0$. We have $h_F(x)=0$ by the property of the map φ. Hence $x\notin F$, and this implies that $F\subset\bigcup_{n\in N}V(n,F)$. So $F=\bigcup_{n\in N}V(n,F)=\bigcup_{n\in N}\overline{V}(n,F)$.

If $E,F\in RG(X)$, and $E\subset F$, then $h_E\le h_F$, and $\varphi(h_E)\le\varphi(h_F)$. Hence $V(n,E)\subset V(n,F)$ for all $n\in N$.

By Lemma 1.4, X is δ-strtifiable. ■

In the same manner as in Theorem 2.3, we can prove the following corollary.

Corollary 2.4. *Let X be a topological space. If for each $F\in RF(X)$, there is an $f_F\in LSC(X)$ that satisfies the following conditions:*

(1) $X-F=f_F^{-1}(0)$ *and*

(2) $f_U\le f_V$ *whenever* $U\subset V$,

then X is δ-stratifiable.

For Theorem 2.3, we have a following question.

Question 2.5. Is there an order preserving map $\varphi:USC(X)\to LSC(X)$ such that for any $h\in USC(X),0\le\varphi(h)\le h$ and $0<\varphi(h)(x)<h(x)$ whenever $h(x)>0$, if X is δ-stratifiable?

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