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ON BITOPOLOGICAL FULL NORMALITY

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Abstract. The notion of bitopological full normality is introduced. Along with other results, we prove a bitopological version of A. H. Stone's theorem on paracompactness: A Hausdorff topological space is paracompact if and only if it is fully normal.

1. Introduction

A bitopological space is a set equipped with two topologies. Kelly [5] initiated the systematic study of such spaces. Since then considerable works have been done on bitopological spaces. Generalizing the notion of pairwise compactness (Fletcher, Hoyle III and Patty [4]), Bose, Roy Choudhury and Mukharjee [1] introduced a notion of pairwise paracompactness and obtained an analogue of Michael's theorem (Michael [6]). In this paper, we introduce the notions of pairwise full normality and a-pairwise full normality. For a pairwise Hausdorff topological space X, we prove that X is a-pairwise fully normal if it is pairwise paracompact, and conversely, X is pairwise paracompact if it is pairwise fully normal. To prove the converse part, we use the above Michael's theorem on pairwise paracompactness.

2. Definitions

Let $(X, \mathcal{P}_1, \mathcal{P}_2)$ be a bitopological space.

Definition 2.1. [4] A cover $\mathcal{U}$ of X is a pairwise open cover if $\mathcal{U} \subset \mathcal{P}_1 \cup \mathcal{P}_2$ and for each $i = 1, 2$, $\mathcal{U} \cap \mathcal{P}_i$ contains a nonempty set.

Definition 2.2. [2] A pairwise open cover $\mathcal{V}$ of X is said to be a parallel refinement of a pairwise open cover $\mathcal{U}$ of X if every $(\mathcal{P}_i)$-open set of $\mathcal{V}$ is contained in some $(\mathcal{P}_i)$-open set of $\mathcal{U}$.

We also recall the following known definitions:

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(a) X is said to be pairwise Hausdorff (Kelly [5]) if for each pair of distinct points x and y of X, there exist $U \in \mathcal{P}_1$ and $V \in \mathcal{P}_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

(b) $\mathcal{P}_i$ is said to be regular with respect to $\mathcal{P}_j, i \neq j$ if for each $x \in X$ and each $(\mathcal{P}_i)$-closed set A with $x \notin A$, there exist $U \in \mathcal{P}_i$ and $V \in \mathcal{P}_j$ such that $x \in U, A \subset V$ and $U \cap V = \emptyset$. X is said to be pairwise regular (Kelly [5]) if $\mathcal{P}_i$ is regular with respect to $\mathcal{P}_j$ for both $i = 1$ and $i = 2$.

(c) X is said to be pairwise normal (Kelly [5]) if for any pair of a $(\mathcal{P}_i)$-closed set A and a $(\mathcal{P}_j)$-closed set B with $A \cap B = \emptyset$, $i \neq j$, there exist $U \in \mathcal{P}_j$ and $V \in \mathcal{P}_i$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

(d) A cover $\{E_\alpha \mid \alpha \in A\}$ of X is said to be point finite (Dugundji [3]) if for each $x \in X$, there are at most finitely many indices $\alpha \in A$ such that $x \in E_\alpha$.

The following definitions are introduced in Bose, Roy Choudhury and Mukharjee [1].

Definition 2.3. A subcollection $\mathcal{C}$ of a refinement $\mathcal{V}$ of a pairwise open cover $\mathcal{U}$ of X is $\mathcal{U}$-locally finite if for each $x \in X$, there exists a neighbourhood of x intersecting a finite number of members of $\mathcal{C}$, the neighbourhood being $(\mathcal{P}_i)$-open if x belongs to a $(\mathcal{P}_i)$-open set of $\mathcal{U}$.

Definition 2.4. The bitopological space X is pairwise paracompact if every pairwise open cover $\mathcal{U}$ of X has a $\mathcal{U}$-locally finite parallel refinement.

If in the above definition, some sets $U \in \mathcal{U}$ are both $(\mathcal{P}_1)$-open and $(\mathcal{P}_2)$-open, then for each such set U, we select one of $\mathcal{P}_1$ and $\mathcal{P}_2$ with respect to which U is open. For this choice, we have a $\mathcal{U}$-locally finite refinement of $\mathcal{U}$. Changing the choice, we get a class of $\mathcal{U}$-locally finite refinements of $\mathcal{U}$. If there are two distinct sets $U_1, U_2 \in \mathcal{U}$ such that for $i = 1, 2$, U_i is $(\mathcal{P}_i)$-open and $U_1 \cap U_2 \neq \emptyset$, then for $\mathcal{U}$-local finiteness of a subcollection $\mathcal{C}$ of the refinement $\mathcal{V}$ of $\mathcal{U}$ at the points $x \in U_1 \cap U_2$, we must get two neighbourhoods $N_i, i = 1, 2$ of x such that N_i is $(\mathcal{P}_i)$-open and each intersects a finite number of members of $\mathcal{C}$.

Definition 2.5. The bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is strongly pairwise regular if it is pairwise regular, and if both the topological spaces $(X, \mathcal{P}_1)$ and $(X, \mathcal{P}_2)$ are regular.

If $\mathcal{U}$ is a pairwise open cover of X, then for each $i = 1, 2, \mathcal{U}^i$ denotes the class of $(\mathcal{P}_i)$-open sets belonging to $\mathcal{U}$. For a point $x \in X$, a set $A \subset X$ and a collection $\mathcal{C}$ of subsets of X, we write

$$St(x, \mathcal{C}) = \bigcup \{C \in \mathcal{C} \mid x \in C\},$$
$$St(A, \mathcal{C}) = \bigcup \{C \in \mathcal{C} \mid A \cap C \neq \emptyset\}.$$

Let $\mathcal{P}$ be the topology on X generated by the subbase $\mathcal{A} = \mathcal{P}_1 \cup \mathcal{P}_2$.

We now introduce the following definitions.

Definition 2.6. Let $\mathcal{U}$ be a pairwise open cover of X. A parallel refinement $\mathcal{V}$ of $\mathcal{U}$ is said to be a parallel star (resp. barycentric) refinement of $\mathcal{U}$ whenever it satisfies the following conditions: (1) if there are two distinct sets $U_1, U_2 \in \mathcal{U}$ such that U_i is $(\mathcal{P}_i)$-open and $U_1 \cap U_2 \neq \emptyset$, then for $x \in U_1 \cap U_2$, there are two sets $V_1, V_2 \in \mathcal{V}$ such that $V_i \subset U_i, V_i$ is $(\mathcal{P}_i)$-open and $x \in V_1 \cap V_2$; (2) for any $V \in \mathcal{V}$ (resp. $x \in X$), there exists a $U \in \mathcal{U}$ such that $St(V, \mathcal{V}) \subset U$ (resp. $St(x, \mathcal{V}) \subset U$).

A $(\mathcal{P})$-open refinement $\mathcal{V}$ of $\mathcal{U}$ is said to be a $(\mathcal{P})$-open barycentric refinement of $\mathcal{U}$ if for any $x \in X$, there exists a $U \in \mathcal{U}$ such that $St(x, \mathcal{V}) \subset U$.

Definition 2.7. A set $G \in \mathcal{P}$ is said to be $(\mathcal{P}_j^*)$-open if it is a union of a $(\mathcal{P}_i)$-open set and a nonempty $(\mathcal{P}_j)$-open set. The complement of a $(\mathcal{P}_j^*)$-open set is called a $(\mathcal{P}_j^*)$-closed set.

Definition 2.8. X is said to be α-pairwise normal if for any pair of a $(\mathcal{P}_i)$-closed set A and a $(\mathcal{P}_j^*)$-closed set B with $A \cap B = \emptyset, i \neq j$, there exist a set $U \in \mathcal{P}$ and a set $V \in \mathcal{P}_i$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

It is easy to see that X is α-pairwise normal if and only if for any $(\mathcal{P}_j^*)$-closed set K and any $(\mathcal{P}_i)$-open set U with $K \subset U$, there exists a $(\mathcal{P}_i)$-open set V such that $K \subset V \subset (\mathcal{P})clV \subset U$.

Definition 2.9. A pairwise open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is said to be shrinkable if there exists a pairwise open cover $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ such that for each $\alpha \in A, (\mathcal{P})clV_\alpha \subset U_\alpha$. $\mathcal{V}$ is then called a shrinking of $\mathcal{U}$.

Definition 2.10. X is said to be pairwise (resp. a-pairwise) fully normal if for every pairwise open cover $\mathcal{U}$ of X, there is a pairwise open (resp. $(\mathcal{P})$-open) cover $\mathcal{V}$ of X such that $\mathcal{V}$ is a parallel (resp. $(\mathcal{P})$-open) star (resp. barycentric) refinement of $\mathcal{U}$.

We denote the set of natural numbers by N and the set of real numbers by R.

3. Theorems

Theorem 3.1. *X is pairwise fully normal if and only if for every pairwise open cover $\mathcal{U}$ of X, there is a pairwise open cover $\mathcal{V}$ of X such that $\mathcal{V}$ is a parallel barycentric refinement of $\mathcal{U}$.*

The above theorem can be proved with standard arguments.

Theorem 3.2. *If X is pairwise fully normal, then it is α-pairwise normal and pairwise normal.*

Proof. Let A and B be two disjoint subsets of X which are $(\mathcal{P}_i)$-closed and $(\mathcal{P}_j^*)$-closed respectively with $i \neq j$. Then there exist a $(\mathcal{P}_i)$-open set G_1 and a nonempty $(\mathcal{P}_j)$-open set G_2 such that $X - B = G_1 \cup G_2$. So $\{X - A, G_1, G_2\}$ is a pairwise open cover of X. Therefore there exists a parallel star refinement

$\mathcal{V}$ of $\{X - A, G_1, G_2\}$. Then $G = St(A, \mathcal{V})$ and $H = St(B, \mathcal{V})$ are $(\mathcal{P})$-open and $(\mathcal{P}_i)$-open respectively, $A \subset G$ and $B \subset H$. We claim $G \cap H = \emptyset$. If $G \cap H \neq \emptyset$, then there exist $V', V'' \in \mathcal{V}$ with $A \cap V' \neq \emptyset, B \cap V'' \neq \emptyset$ and $V' \cap V'' \neq \emptyset$, and so $St(V', \mathcal{V})$ intersects both A and B which is impossible. Thus X is α-pairwise normal. Similarly, we can show that it is pairwise normal. ■

EXAMPLE 3.3. For any $a \in R$, the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ where $\mathcal{P}_1 = \{\emptyset, R, (-\infty, a], (a, \infty)\}$ and $\mathcal{P}_2 = \{\emptyset, R, (-\infty, a), [a, \infty)\}$ is α-pairwise normal but not pairwise normal.

EXAMPLE 3.4. Let $p \in R$, $\mathcal{P}_1 = \{\emptyset, R\} \cup \{E \cup (x, \infty) \mid p \notin E \subset R, x \in R$ and $x \geq p+1\}$ and $\mathcal{P}_2 =$ the usual topology of R. Then the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise normal, since for any $(\mathcal{P}_1)$-closed set $A(\neq \emptyset, R)$, we have

$$A = E \cap (-\infty, x],\ p \in E \subset R, x \geq p+1$$

and for any $(\mathcal{P}_2)$-closed set B with $A \cap B = \emptyset$, we have $p \notin B$, one can take for $y > x$,

$$U = (X - B) \cap (-\infty, y) \in \mathcal{P}_2,$$
$$V = B \cup (y, \infty) \in \mathcal{P}_1$$

so that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

But $(R, \mathcal{P}_1, \mathcal{P}_2)$ is not α-pairwise normal, since for the $(\mathcal{P}_1)$-closed set

$$F = ((p-1, p+1) \cup (\text{the set of rationals})) \cap (-\infty, x],\ x \geq p+1,$$

and the $(\mathcal{P}_2^*)$-closed set

$$K = M \cap ((-\infty, p-1] \cup [p+1, \infty))$$

where M is the $(\mathcal{P}_1)$-closed set

$$((p-1, p+1) \cup (\text{the set of irrationals})) \cap (-\infty, x],\ x \geq p+1,$$

there exists no pair of a $(\mathcal{P})$-open set U and a $(\mathcal{P}_1)$-open set V with $F \subset U, K \subset V$ and $U \cap V = \emptyset$.

From the above two examples, it follows that the notions of pairwise normality and α-pairwise normality are independent.

THEOREM 3.5. *If X is pairwise Hausdorff and pairwise paracompact, then X is α-pairwise normal.*

Proof. Let us consider a $(\mathcal{P}_i)$-closed set A and a $(\mathcal{P}_j^*)$-closed set B with $A \cap B = \emptyset$ and $i \neq j$. Let $\xi \in B$. Then $\xi \notin A$. Since X is pairwise Hausdorff and pairwise paracompact, it is pairwise regular (Theorem 5, Bose et al. [1]). Therefore there exist a set $U_\xi \in \mathcal{P}_j$ and a set $V_\xi \in \mathcal{P}_i$ such that $A \subset U_\xi, \xi \in V_\xi$ and $U_\xi \cap V_\xi = \emptyset$. The set $X - B$ is $(\mathcal{P}_j^*)$-open, and so there exist a $(\mathcal{P}_i)$-open set G_1 and a nonempty $(\mathcal{P}_j)$-open set G_2 such that $X - B = G_1 \cup G_2$. Therefore the family $\mathcal{V} = \{V_\xi \mid \xi \in B\} \cup \{G_1, G_2\}$ is a pairwise open cover of X. Since X is pairwise paracompact, there exists a $\mathcal{V}$-locally finite parallel refinement $\mathcal{D}$ of $\mathcal{V}$. Let $V = \bigcup\{D \in \mathcal{D} \mid D \cap B \neq \emptyset\}$. Then $V \in \mathcal{P}_i$ and $B \subset V$. Now let $x \in A \subset X - B$. Since $X - B = G_1 \cup G_2$ and $G_1, G_2 \in \mathcal{V}$, it follows that there exists a neighbourhood W_x of x such that $W_x \in \mathcal{P}_i$ (resp. $W_x \in \mathcal{P}_j$) if $x \in G_1$ (resp. $x \in G_2$) and W_x intersects finite number of sets $D_x^1, D_x^2, \dots, D_x^m$ with $B \cap D_x^k \neq \emptyset$ and $D_x^k \in \mathcal{D}$ for $k = 1, 2, \dots, m$. If $D_x^k \subset V_{\xi_k}, \xi_k \in B$, then $U_x \cap V = \emptyset$ and $x \in U_x$ where $U_x = W_x \cap (\bigcap_{k=1}^m U_{\xi_k}) \in \mathcal{P}$. If $U = \bigcup_{x \in A} U_x$, then $U \in \mathcal{P}, A \subset U$ and $U \cap V = \emptyset$. Therefore X is α-pairwise normal. ∎

THEOREM 3.6. *If X is α-pairwise normal, then every point finite pairwise open cover is shrinkable.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be a point finite pairwise open cover of X. We well-order the index set A, and write $A = \{1, 2, \dots, \alpha, \dots\}$. By transfinite induction, we now construct a pairwise open cover $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ which is a shrinking of $\mathcal{U}$. We write $F_1 = X - \bigcup\{U_\alpha \mid \alpha > 1\}$. Since $\mathcal{U}$ is a pairwise open cover, it follows that if U_1 is $(\mathcal{P}_i)$-open, then F_1 is $(\mathcal{P}_j^*)$-closed and $F_1 \subset U_1$. Therefore there exists a $(\mathcal{P}_i)$-open set V_1 such that $F_1 \subset V_1 \subset (\mathcal{P}) cl V_1 \subset U_1$. Assume that V_β is defined for every $\beta < \alpha$, and consider the set

$$F_\alpha = X - \Big((\bigcup\{V_\beta \mid \beta < \alpha\}) \cup (\bigcup\{U_\gamma \mid \gamma > \alpha\}) \Big).$$

If U_α is $(\mathcal{P}_i)$-open, then F_α is $(\mathcal{P}_j^*)$-closed. Also $F_\alpha \subset U_\alpha$. Therefore there exists a set $V_\alpha \in \mathcal{P}_i$ such that

$$F_\alpha \subset V_\alpha \subset (\mathcal{P}) cl V_\alpha \subset U_\alpha. \tag{1}$$

If $x \in X$, then there exist a finite number of sets $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ such that $x \in U_{\alpha_i}$ for all $i = 1, 2, \dots, n$. If $\alpha = \max(\alpha_1, \alpha_2, \dots, \alpha_n)$, then for $\gamma > \alpha$, $x \notin U_\gamma$. Therefore $x \in F_\alpha \subset V_\alpha$ if $x \notin V_\beta$ for all $\beta < \alpha$. So $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ is a pairwise open cover of X. Hence it follows from (1) that $\mathcal{V}$ is a shrinking of $\mathcal{U}$. ∎

Now we prove an analogue (Theorem 3.8) of A. H. Stone's theorem on paracompactness (Stone [7]).

For this, we require the following result.

THEOREM 3.7. [1] *If X is strongly pairwise regular, then X is pairwise paracompact if and only if every pairwise open cover $\mathcal{U}$ of X has a parallel refinement $\mathcal{V} = \bigcup_{n=1}^\infty \mathcal{V}_n$, where each $\mathcal{V}_n$ is $\mathcal{U}$-locally finite.*

THEOREM 3.8. *Suppose X is pairwise Hausdorff. If X is pairwise paracompact, then it is a-pairwise fully normal. Conversely, if X is pairwise fully normal, then it is pairwise paracompact.*

Proof. At first we suppose that X is pairwise Hausdorff and pairwise paracompact.

Let $\mathcal{U}$ be a pairwise open cover of X. Then there exists a $\mathcal{U}$-locally finite parallel refinement $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ of $\mathcal{U}$. Since $\mathcal{V}$ is $\mathcal{U}$-locally finite, it is point finite. Again by Theorem 3.5, X is α-pairwise normal, and so by Theorem 3.6, there exists a shrinking $\mathcal{W} = \{W_\alpha \mid \alpha \in A\}$ of $\mathcal{V}$. $\mathcal{W}$ is a pairwise open cover of X such that for each α,

$$(\mathcal{P})clW_\alpha \subset V_\alpha. \tag{2}$$

For $x \in X$, we write

$$D_x = \bigcap\{V_\alpha \mid x \in (\mathcal{P})clW_\alpha\}. \tag{3}$$

From (2) and point finiteness of $\mathcal{V}$, it follows that there are finite number of V_α in the intersection (3). Hence $D_x \in \mathcal{P}$. Now let

$$K_x = \bigcup\{(\mathcal{P})clW_\alpha \mid x \notin (\mathcal{P})clW_\alpha\}.$$

Since $\mathcal{V}$ is $\mathcal{U}$-locally finite, $\{(\mathcal{P})clW_\alpha\}$ is $(\mathcal{P})$-locally finite. Therefore by 9.2 (Dugundji [3], p. 82), K_x is a $(\mathcal{P})$-closed set. Therefore $G_x = X - K_x$ is a $(\mathcal{P})$-open set. Hence the collection $\mathcal{B} = \{D_x \cap G_x \mid x \in X\}$ is a $(\mathcal{P})$-open cover of X. For $y \in X$, let $y \in (\mathcal{P})clW_\alpha$. If $y \in D_x \cap G_x$, then $x \in (\mathcal{P})clW_\alpha$, since otherwise $(\mathcal{P})clW_\alpha \subset K_x$ and hence $y \notin G_x$. Again if $x \in (\mathcal{P})clW_\alpha$, then $D_x \subset V_\alpha \Rightarrow D_x \cap G_x \subset V_\alpha$. Therefore $\mathcal{B}$ is a $(\mathcal{P})$-open barycentric refinement of $\mathcal{V}$ and hence of $\mathcal{U}$. Therefore X is a-pairwise fully normal.

Conversely, suppose X is pairwise Hausdorff and pairwise fully normal. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be a pairwise open cover of X. By Theorem 3.1, we can construct a sequence $\{\mathcal{U}_n\}$ of pairwise open covers of X such that $\mathcal{U}_1$ is a parallel barycentric refinement of $\mathcal{U}$, and for each $n \in N$, $\mathcal{U}_{n+1}$ is a parallel barycentric refinement of $\mathcal{U}_n$. For $\alpha \in A$, let

$$V_\alpha^n = \{x \in U_\alpha \mid St(x,\mathcal{U}_n) \subset U_\alpha\},$$
$$V_\alpha = \bigcup_{n=1}^{\infty} V_\alpha^n.$$

If $x \in V_\alpha$, then $x \in V_\alpha^n$ for some n, and so $St(x,\mathcal{U}_n) \subset U_\alpha$. Now let $y \in St(x,\mathcal{U}_{n+1})$, then $x \in St(y,\mathcal{U}_{n+1})$. Since $\mathcal{U}_{n+1}$ is a barycentric refinement of $\mathcal{U}_n$, it follows that, $St(y,\mathcal{U}_{n+1}) \subset St(x,\mathcal{U}_n) \subset U_\alpha$. So $y \in V_\alpha^{n+1} \subset V_\alpha$. Thus $St(x,\mathcal{U}_{n+1}) \subset V_\alpha$. Since $\mathcal{U}_1$ is a barycentric refinement of $\mathcal{U}$, for any $x \in X$, there exists a U_α such that $St(x,\mathcal{U}_1) \subset U_\alpha$ and so $x \in V_\alpha^1 \subset V_\alpha$. Therefore $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ is a refinement of $\mathcal{U}$. We now well-order $\mathcal{V}$ as $V_1, V_2, \ldots, V_\alpha, \ldots$. For a fixed $n \in N$, we define

$$B_1^n = X - St(X - V_1,\mathcal{U}_n),$$
$$B_\alpha^n = X - St\left((X - V_\alpha) \cup (\bigcup_{\beta<\alpha} B_\beta^n),\mathcal{U}_n\right) \quad \text{if } \alpha > 1.$$

It is easy to see that

$$St(B_\alpha^n,\mathcal{U}_n) \subset V_\alpha \text{ for all } \alpha,$$
$$St(B_\alpha^n,\mathcal{U}_n) \cap B_\beta^n = \emptyset \text{ for all } \beta \neq \alpha. \tag{4}$$

Let $x \in X$. Since $\{V_\alpha \mid \alpha \in A\}$ is a cover of X, there is a first index α such that $x \in V_\alpha$. Then $St(x,\mathcal{U}_m) \subset V_\alpha$ for some m. We now show $x \in B_\alpha^m$. If possible, suppose $x \notin B^m_\alpha$. Then

$$x \in St\Big((X - V_\alpha) \cup (\bigcup\nolimits_{\beta<\alpha} B^m_\beta), \mathcal{U}_m\Big)$$
$$\Rightarrow St(x, \mathcal{U}_m) \cap \Big((X - V_\alpha) \cup (\bigcup\nolimits_{\beta<\alpha} B^m_\beta)\Big) \neq \emptyset$$
$$\Rightarrow St(x, \mathcal{U}_m) \cap B^m_\beta \neq \emptyset \text{ for some } \beta < \alpha \text{ (since } St(x, \mathcal{U}_m) \subset V_\alpha)$$
$$\Rightarrow x \in St(B^m_\beta, \mathcal{U}_m) \subset V_\beta.$$

This contradicts the fact that α is the first index for which $x \in V_\alpha$. Therefore $x \in B^m_\alpha$. Hence $\{B^n_\alpha \mid n \in N, \alpha \in A\}$ is a cover of X. We now define

$$G^n_\alpha = St(B^n_\alpha, \mathcal{U}^i_{n+2}), n \in N, \alpha \in A \text{ if } U_\alpha \text{ is } (\mathcal{P}_i)\text{open.}$$

Then G^n_α is $(\mathcal{P}_i)$-open. Since $St(B^n_\alpha, \mathcal{U}_n) \subset V_\alpha$, we have $St(B^n_\alpha, \mathcal{U}_{n+2}) \subset V_\alpha$ and hence $G^n_\alpha \subset V_\alpha$. Now let $x \in X$. Then $x \in B^n_\alpha$ for some pair of n and α and so $x \in U_\alpha$, since $B^n_\alpha \subset St(B^n_\alpha, \mathcal{U}_n) \subset V_\alpha \subset U_\alpha$. If U_α is $(\mathcal{P}_i)$-open, then by definition of parallel barycentric refinement, $x \in U$ for some $U \in \mathcal{U}^i_{n+2}$. So $x \in St(B^n_\alpha, \mathcal{U}^i_{n+2}) = G^n_\alpha$. Therefore $\mathcal{G} = \{G^n_\alpha \mid n \in N, \alpha \in A\}$ is a cover of X and hence a parallel refinement of $\mathcal{U}$. We now show that there exists no $U \in \mathcal{U}_{n+2}$ intersecting both G^n_α and G^n_β for $\alpha \neq \beta$, whenever both U_α and U_β are $(\mathcal{P}_i)$-open. Suppose if possible, $U \in \mathcal{U}_{n+2}$ intersects both G^n_α and G^n_β for $\alpha \neq \beta$ with $U_\alpha, U_\beta \in \mathcal{P}_i$. Then there exist $H_1, H_2 \in \mathcal{U}^i_{n+2}$ such that H_1 intersects both B^n_α and U, and H_2 intersects both B^n_β and U. Hence $St(U, \mathcal{U}^i_{n+2})$ intersects both B^n_α and B^n_β. Since $\mathcal{U}_{n+2}$ is a star refinement of $\mathcal{U}_n$, it follows that some $W \in \mathcal{U}_n$ intersects both B^n_α and B^n_β. Therefore $St(B^n_\alpha, \mathcal{U}_n)$ intersects B^n_β which contradicts (4).

Since $\mathcal{U}_{n+2}$ is a parallel refinement of $\mathcal{U}$, it thus follows that for each $n \in N$, $\mathcal{G}_n = \{G^n_\alpha \mid \alpha \in A\}$ is $\mathcal{U}$-locally finite. Also we have $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}^n_\alpha$.

Since X is pairwise Hausdorff, any singleton subset of X is $(\mathcal{P}_i)$-closed for $i = 1$ and 2. Therefore by Theorem 3.2, X is pairwise regular. Next we show that both $(X, \mathcal{P}_1)$ and $(X, \mathcal{P}_2)$ are regular topological spaces. Let F be a $(\mathcal{P}_i)$-closed subset of X with $x \notin F, i = 1, 2$. Considering $\{x\}$ as a $(\mathcal{P}_i)$-closed set, we get a parallel star refinement $\mathcal{V}$ of $\{X - \{x\}, X - F\}$. Then $G = St(\{x\}, \mathcal{V})$ and $H = St(F, \mathcal{V})$ are $(\mathcal{P}_i)$-open sets with $x \in G, F \subset H$ and $G \cap H = \emptyset$. So $(X, \mathcal{P}_i)$ is regular. Hence X is strongly pairwise regular. Therefore by Theorem 3.7, X is pairwise paracompact. ■

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