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SOME REMARKS ON ALMOST LINDELÖF SPACES AND WEAKLY LINDELÖF SPACES

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Abstract. A space X is *almost Lindelöf* (*weakly Lindelöf*) if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$ (respectively, $\overline{\bigcup \mathcal{V}} = X$). In this paper, we investigate the relationships among almost Lindelöf spaces, weakly Lindelöf spaces and Lindelöf spaces, and also study topological properties of almost Lindelöf spaces and weakly Lindelöf spaces.

1. Introduction

By a space we mean a topological space. Let us recall that a space X is *Lindelöf* if every open cover of X has a countable subcover. As a generalization of Lindelöfness, Willard and Mathur [8] defined a space X to be *almost Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$. Frolik [4] defined a space X to be *weakly Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{\bigcup \mathcal{V}} = X$. Clearly, every Lindelöf space is almost Lindelöf and every almost Lindelöf space is weakly Lindelöf, but the converses do not hold (see Examples 2.2 and 2.3). On the study of almost Lindelöf spaces and weakly Lindelöf spaces, the readers can see the references [1, 2, 4, 5, 6].

The purpose of this paper is to investigate the relationships among almost Lindelöf spaces, weakly Lindelöf spaces and Lindelöf spaces, and also to study topological properties of almost Lindelöf spaces and weakly Lindelöf spaces.

Recall that the *extent* $e(X)$ of a space X is the smallest cardinal number κ such that the cardinality of every discrete closed subset of X is not greater than κ. The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals.

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For a cardinal κ, $cf(\kappa)$ denotes the cofinality κ. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [3].

2. Some examples

In this section, we give some examples showing the relationship among almost Lindelöf spaces, weakly Lindelöf spaces and Lindelöf spaces. First, we give a well-known result for the sake of completeness.

PROPOSITION 2.1. *If X is a regular almost Lindelöf space, then X Lindelöf.*

In the following, we give an example showing that Proposition 2.1 is not true for Urysohn spaces.

EXAMPLE 2.2. There exists an Urysohn almost Lindelöf space X which is not Lindelöf.

Proof. Let

$$A = \{a_\alpha : \alpha < \omega_1\}, \quad B = \{b_i : i \in \omega\}, \quad Y = \{\langle a_\alpha, b_i\rangle : \alpha < \omega_1, i \in \omega\}$$

and

$$X = Y \cup A \cup \{a\} \ \text{ where } \ a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighborhood of $a_\alpha \in A$ for each $\alpha < \omega_1$ takes the form

$$U_{a_\alpha}(i) = \{a_\alpha\} \cup \{\langle a_\alpha, b_j\rangle : \alpha < \omega_1, j \geq i\} \text{ where } i \in \omega$$

and a basic neighborhood of a takes the form

$$U_a(\alpha) = \{a\} \cup \bigcup \{\langle a_\beta, b_i\rangle : \beta > \alpha, i \in \omega\}\} \text{ where } \alpha < \omega_1.$$

Clearly, X is a Urysohn space. Moreover X is not regular, since the point a can not be separated from the closed set $\{a_\alpha : \alpha < \omega_1\}$. Since $\{a_\alpha : \alpha < \omega_1\}$ is an uncountable discrete closet set of X, then X is not Lindelöf.

We show that X is almost Lindelöf. Let $\mathcal{U}$ be any open cover of X. Then there exists some $U_a \in \mathcal{U}$ such that $a \in U_a$. By the definition of topology of X, there exists a $\beta < \omega_1$ such that $U_a(\beta) \subseteq U_a$, then

$$\{a_\alpha : \alpha > \beta\} \cup \{a\} \cup \{\langle a_\alpha, b_i\rangle : \alpha > \beta, i \in \omega\} \subseteq \overline{U_a}.$$

It is not difficult to see that $X \setminus \overline{U_a}$ is at most countable, so there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $X \setminus \overline{U_a} \subseteq \bigcup \mathcal{V}$. If we put $\mathcal{V} = \{U_a\} \cup \mathcal{V}$, then $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$ such that $X = \{\overline{V} : V \in \mathcal{V}\}$, which completes the proof. ∎

For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

EXAMPLE 2.3. There exists a Tychonoff weakly Lindelöf space which is not almost Lindelöf.

Proof. Let D be a discrete space of cardinality ω_1, let

$$X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$$

be the subspace of the product of βD and $\omega + 1$.

We show that X is weakly Lindelöf. Let $\mathcal{U}$ be any open cover of X. Since $\beta D \times \omega$ is a σ-compact dense subset of X, then there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\beta D \times \omega \subseteq \bigcup \mathcal{V}$, hence $X = \overline{\bigcup \mathcal{V}}$, since $\beta D \times \omega$ is a dense subset of X, which shows that X is weakly Lindelöf.

Next, we show that X is not almost Lindelöf. Since $|D| = \omega_1$, we can enumerate D as $\{d_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, let $U_\alpha = \{d_\alpha\} \times (\omega + 1)$. For each $n \in \omega$, let $V_n = \beta D \times \{n\}$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_n : n \in \omega\}$$

of X. It is not difficult to see that $\bigcup \mathcal{V} = \bigcup \{\overline{V} : V \in \mathcal{V}\}$ for each a countable subset $\mathcal{V}$ of $\mathcal{U}$. Let $\mathcal{V}$ be any countable subset of $\mathcal{U}$ and let $\alpha_0 = \sup\{\alpha : U_\alpha \in \mathcal{V}\}$. Then $\alpha_0 < \omega_1$,since $\mathcal{V}$ is countable. If we pick $\alpha' > \alpha_0$, then $\langle d_{\alpha'}, \omega\rangle \notin \{\overline{V} : V \in \mathcal{V}\}$, since $U_{\alpha'}$ is the only element of $\mathcal{U}$ containing $\langle d_{\alpha'}, \omega\rangle$ and $\bigcup \mathcal{V} = \bigcup \{\overline{V} : V \in \mathcal{V}\}$, which completes the proof. ∎

If we take D of arbitrarily big cardinality instead of ω_1 in the proof in Example 2.3, we easily get the following result.

PROPOSITION 2.4. *For every infinite cardinal κ, there exists a Tychonoff weakly Lindelöf space X such that $e(X) \geq \kappa$.*

REMARK 2.1. F. Cammaroto and G. Santoro [2] also constructed an example showing that there exists a Tychonoff weakly Lindelöf space that is not almost Lindelöf (see Example 3.11 [2]). Example 2.3 is simpler than their construction.

REMARK 2.2. As one of the referees observed, it is easy to see that every CCC space is weakly Lindelöf, so every CCC, non Lindelöf Tychonoff space (for example, a $\sum$-product in 2^κ) would work as such an example. However we include Example 2.3 here, since we use it later in the text.

It is well known that the extent of a Lindelöf space is countable. However, similar to the argument from Example 2.2, we can prove the following proposition showing that the extent of a Urysohn almost Lindelöf space can be arbitrarily big.

PROPOSITION 2.5. *For every infinite cardinal κ, there exists a Urysohn almost Lindelöf space X such that $e(X) \geq \kappa$.*

3. Behavior with respect to products, images and subspaces

From Example 2.2, it is not difficult to see that the closed subset of a Urysohn almost Lindelöf space need not be almost Lindelöf. The following example shows that a regular closed subspace of a Urysohn almost Lindelöf spaces need not be almost Lindelöf.

EXAMPLE 3.1. There exist a Urysohn almost Lindelöf space X having a regular closed subset which is not almost Lindelöf.

Proof. Let S_1 be the space X from Example 2.2 and let S_2 be the space X from Example 2.3.

We assume that $S_1 \cap S_2 = \emptyset$. Since $|D| = \omega_1$, we can enumerate D as $\{d_\alpha : \alpha < \omega_1\}$. Let $\varphi : D \times \{\omega\} \to A$ be a bijection defined by

$$\varphi(\langle d_\alpha, \omega \rangle) = a_\alpha \text{ for each } \alpha < \omega_1.$$

Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \omega \rangle$ with $\varphi(\langle d_\alpha, \omega \rangle)$ for each $\alpha < \omega_1$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map. Let $Y = \pi(S_2)$. Then, Y is not almost Lindelöf in X since it is homeomorphic to S_2.

Now, we show X is almost Lindelöf. Let $\mathcal{U}$ be an open cover of X. Since $\pi(S_1)$ is almost Lindelöf, then there exists a countable subfamily $\mathcal{V}'$ of $\mathcal{U}$ such that $\pi(S_1) \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}'\}$; on the other hand, for each $n \in \omega$, since $\pi(\beta D \times \{n\})$ is a compact subset of X, there exists a finite subfamily $\mathcal{V}_n$ of $\mathcal{U}$ such that $\pi(\beta D \times \{n\}) \subseteq \bigcup \mathcal{V}_n$. If we put $\mathcal{V} = \mathcal{V}' \cup \bigcup\{\mathcal{V}_n : n \in \omega\}$, then $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$ such that $X = \bigcup\{\overline{V} : V \in \mathcal{V}\}$, which shows that X is almost Lindelöf. ∎

From Example 2.3, it is not difficult to see that the closed subset of a Tychonoff weakly Lindelöf space need not be weakly Lindelöf. However we have the following positive result.

PROPOSITION 3.2. *Every regular closed subset of a weakly Lindelöf space X is weakly Lindelöf.*

Proof. Let X be a weakly Lindelöf space and let F be a regular closed subset of X. Let $\mathcal{U}$ be an open cover of F. For each $U \in \mathcal{U}$, there exists an open subset V_U in X such that $V_U \cap F = U$. Then $\{V_U : U \in \mathcal{U}\} \cup \{X \setminus F\}$ is an open cover of X. Hence there exists a countable subset $\mathcal{V}$ of $\{V_U : U \in \mathcal{U}\} \cup \{X \setminus F\}$ such that $X = \overline{\bigcup \mathcal{V}}$, since X is weakly Lindelöf. Let $\mathcal{W} = \mathcal{V} \setminus \{X \setminus F\}$. Then $IntF \subseteq \overline{\bigcup \mathcal{W}}$. Hence $F = \overline{IntF} \subseteq \overline{\bigcup \mathcal{W}}$, since F is a regular closed subset of X. Thus $F = F \cap \overline{\bigcup \mathcal{W}} = \mathrm{cl}_F(F \cap (\bigcup \mathcal{W})) = \mathrm{cl}_F(\cup\{F \cap W : W \in \mathcal{W}\}$. Since $\{F \cap W : W \in \mathcal{W}\}$ is a countable subset of $\mathcal{U}$ and $F = \mathrm{cl}_F(\bigcup\{F \cap W : W \in \mathcal{W}\})$, then F is weakly Lindelöf, which completes the proof. ∎

The following positive results are obvious.

PROPOSITION 3.3. *If X is an almost Lindelöf space* (*a weakly Lindelöf space*), *then every clopen subset of X is almost Lindelöf* (*respectively, weakly Lindelöf*).

PROPOSITION 3.4. *The sum $\bigoplus_{s \in S} X_s$ is almost Lindelöf* (*weakly Lindelöf*) *if and only if all spaces X_s are almost Lindelöf* (*respectively, weakly Lindelöf*) *and the set S is countable.*

PROPOSITION 3.5. *A continuous image of an almost Lindelöf space* (*a weakly Lindelöf space*) *is almost Lindelöf* (*respectively, weakly Lindelöf*).

Next, we turn to consider preimages. To show that the preimage of an almost Lindelöf space (a weakly Lindelöf space) under a closed 2-to-1 continuous map need not be almost Lindelöf(respectively, weakly Lindelöf) we use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well known that X is Lindelöf if and only if $A(X)$ is Lindelöf. But the statement is not true for almost Lindelöf Urysohn spaces and weakly Lindelöf spaces.

Example 3.6. There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is a Uryshon almost Lindelöf space, but $A(X)$ is not a weakly Lindelöf space (hence is not almost Lindelöf).

Proof. Let X be the space from Example 2.2. Then X is almost Lindelöf and has an infinite discrete closed subset $A = \{a_\alpha : \alpha < \omega_1\}$. Hence the Alexandroff duplicate $A(X)$ of X is not weakly Lindelöf, since $A \times \{1\}$ is an uncountable infinite discrete, open and closed set in $A(X)$. Let $f : A(X) \to X$ be the natural map. Then f is a closed 2-to-1 continuous map, which completes the proof. ∎

If in the previous argument we use Example 2.3 instead of Example 2.2, we get the following:

Example 3.7. There exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is a Tychonoff weakly Lindelöf space, but X is not weakly Lindelöf.

Remark 3.1. The proof of Example 3.6 shows that the Alexandorff duplicate $A(X)$ need not be almost Lindelöf for a Urysohn almost Lindelöf space X and the proof of Example 3.7 shows that the Alexandorff duplicate $A(X)$ need not be weakly Lindelöf for a Tychonoff weakly Lindelöf space X.

Proposition 3.8. *For a space X, the following conditions are equivalent:*

(1) *X is Lindelöf;*
(2) *$A(X)$ is Lindelöf;*
(3) *$A(X)$ is almost Lindelöf;*
(4) *$A(X)$ is weakly Lindelöf.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious. To show that (4) ⇒ (1), suppose that X is not Lindelöf. Let $\mathcal{U}$ be an open cover of X witnessing that X is not Lindelöf. Then $\{U \times \{0,1\} : U \in \mathcal{U}\}$ is an open cover of $A(X)$ that witnesses that $A(X)$ is not weakly Lindelöf, since all points of $X \times \{1\}$ are isolated in $A(X)$, which completes the proof. ∎

Recall from [7] that a mapping f from a space X to a space Y is called *almost open* if $f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)}$ for each open subset U of Y.

Proposition 3.9. *If $f : X \to Y$ is an almost open and perfect continuous mapping and Y is an almost Lindelöf space, then X is almost Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. Then there is a finite subfamily $\mathcal{U}_y$ of $\mathcal{U}$ such that

$$f^{-1}(y) \subseteq \bigcup \mathcal{U}_y \text{ for each } y \in Y.$$

Let $U_y = \bigcup \mathcal{U}_y$. Then $V_y = Y \setminus f(X \setminus U_y)$ is an open neighborhood of y, since f is closed. Let $\mathcal{V} = \{V_y : y \in Y\}$, then $\mathcal{V}$ is an open cover of Y, hence there exists a countable subfamily $\{V_{y_n} : n \in \omega\}$ of $\mathcal{V}$ such that $Y = \bigcup_{n\in\omega} \overline{V_{y_n}}$, since Y is almost Lindelöf. Since f is almost open, then

$$X = f^{-1}(\bigcup_{n\in\omega} \overline{V_{y_n}}) = \bigcup_{n\in\omega} f^{-1}(\overline{V_{y_n}}) \subseteq \bigcup_{n\in\omega} \overline{f^{-1}(V_{y_n})}$$

$$\subseteq \bigcup_{n\in\omega} \overline{U_{y_n}} \subseteq \bigcup_{n\in\omega} \overline{\bigcup \mathcal{U}_{y_n}} \subseteq \bigcup_{n\in\omega} \bigcup \{\overline{U} : U \in \mathcal{U}_{y_n}\},$$

since $\mathcal{U}_{y_n}$ is finite. Hence X is almost Lindelöf , which completes the proof. ∎

Similar to the proof of Proposition 3.9, we can prove the following proposition.

Proposition 3.10. *If $f : X \to Y$ is an almost open and perfect continuous mapping and Y is a weakly Lindelöf spaces, then X is weakly Lindelöf.*

It is well known that the product of two Lindelöf spaces need not be Lindelöf, which shows that the product of two almost Lindelöf need not be almost Lindelöf, since every Lindelöf space is almost Lindelöf and every almost Lindelöf space is Lindelöf for regular spaces. Since the product of a Lindelöf space and a compact space is Lindelöf, then the product of a regular almost Lindelöf space and a compact space is almost Lindelöf. For almost Lindelöf spaces, we have the similar result.

Proposition 3.11. *If X is almost Lindelöf and Y is a compact space, then $X \times Y$ is almost Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. Without loss of generality we can assume that $\mathcal{U}$ consists of basic open sets of $X \times Y$. Since $\{x\} \times Y$ is a compact subset of $X \times Y$ for each $x \in X$, there exists a finite subfamily $\{U_{x_i} \times V_{x_i} : i = 1, 2, ...n_x\}$ of $\mathcal{U}$ such that

$$\{x\} \times Y \subseteq \bigcup \{U_{x_i} \times V_{x_i} : 1 \leq i \leq n_x\}.$$

Let $W_x = \bigcap \{U_{x_i} : 1 \leq i \leq n_x\}$. Then

$$\{x\} \times Y \subseteq \bigcup \{W_x \times V_{x_i} : 1 \leq i \leq n_x\}.$$

Let $\mathcal{W} = \{W_x : x \in X\}$. Then $\mathcal{W}$ is an open cover of X. Since X is almost Lindelöf, there is a countable subfamily $\{W_{x_j} : j \in \omega\}$ of $\mathcal{W}$ such that $X = \bigcup_{j\in\omega} \overline{W_{x_j}}$, since X is almost Lindelöf. Let

$$\mathcal{V} = \{U_{x_{j_i}} \times V_{x_{j_i}} : 1 \leq i \leq n_{x_j}, j \in \omega\}.$$

Then $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$. To show that $X \times Y = \bigcup \{\overline{O} : O \in \mathcal{V}\}$, let $\langle s, t\rangle \in X \times Y$ be fixed. Let $U_s \times V_t$ be any open neighborhoods of $\langle s, t\rangle$ in $X \times Y$ where U_s and V_t are open neighborhood of x and y in X and Y, respectively. Since $X = \bigcup_{j\in\omega} \overline{W_{x_j}}$, then there exists a $j \in \omega$ such that $s \in \overline{W_{x_j}}$. Thus

$$(U_s \times V_t) \cap (\bigcup \{W_{x_j} \times V_{x_{j_i}} : 1 \leq i \leq n_{x_j}\}) \neq \emptyset.$$

Therefore

$$(U_s \times V_t) \cap (\bigcup \{U_{x_{j_i}} \times V_{x_{j_i}} : 1 \le i \le n_{x_j}\}) \neq \emptyset.$$

We have

$$\langle s, t \rangle \in \overline{\bigcup \{U_{x_{j_i}} \times V_{x_{j_i}} : 1 \le i \le n_{x_j}\}} = \bigcup \{\overline{U_{x_{j_i}} \times V_{x_{j_i}}} : 1 \le i \le n_{x_j}\}.$$

This implies $\langle s, t \rangle \in \bigcup \{\overline{O} : O \in \mathcal{V}\}$. Hence $X \times Y = \bigcup \{\overline{O} : O \in \mathcal{V}\}$, which shows that $X \times Y$ is almost Lindelöf. ∎

Similar to the proof of Proposition 3.11, we can prove the following proposition.

Proposition 3.12. *If X is weakly Lindelöf and Y is a compact space, then $X \times Y$ is weakly Lindelöf.*

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