

WAVELETS AND THE COMPLETE INVARIANCE PROPERTY

Rajeshwari Dubey and Aparna Vyas

Abstract. In this paper, we obtain that the space \mathcal{W} of orthonormal wavelets enjoys the complete invariance property with respect to homeomorphisms. Further, it is obtained that the cylinder, the cone and the suspension of \mathcal{W} possess the complete invariance property. Certain subspaces of \mathcal{W} are also considered in this connection.

1. Introduction

The set \mathcal{W} of all one-dimensional orthonormal wavelets on \mathbb{R} forms a subset of the unit ball of the space $L^2(\mathbb{R})$. The topological property of path-connectivity of \mathcal{W} and certain subsets of \mathcal{W} have drawn attention of several workers in the field of wavelets during the past one decade [6, 7, 11, 14]. Such a study has also been carried over to higher dimensional orthonormal wavelets.

In the same spirit, in this paper, we initiate the study of the topological notion of the complete invariance property in respect of such sets. A topological space X is said to possess the complete invariance property (CIP) if each of its non-empty closed sets is the fixed point set, for some self-continuous map f on X [12]. In case f can be found to be a homeomorphism, we say that the space enjoys the complete invariance property with respect to a homeomorphism (CIPH) [8]. These notions have been extensively studied in [1, 8, 9, 10, 12] and other references therein.

In the following Section 2, we describe necessary pre-requisites in two parts, one for wavelets and the other for the notions of CIP and CIPH. In Section 3, by introducing an action of the unit circle S^1 on \mathcal{W} , we obtain that the space \mathcal{W} of orthonormal wavelets and many of its subspaces such as the subspace of MRA wavelets, the subspace of MSF wavelets and the subspace of MRA-MSF wavelets are blessed with CIPH. Finally, we obtain that cylinders, cones and suspensions over such spaces enjoy CIP.

2010 AMS Subject Classification: 42C15, 42C40, 54H25.

Keywords and phrases: Wavelets; MSF Wavelets; MRA Wavelets; CIP; CIPH; dimension function.

2. Pre-requisites

A. Wavelets

Let $L^1(\mathbb{R})$ be the collection of all Lebesgue integrable functions on \mathbb{R} and $L^2(\mathbb{R})$ be that of all Lebesgue square integrable functions on \mathbb{R} . With the usual addition and scalar multiplication of functions together with the inner-product $\langle f, g \rangle$ of $f, g \in L^2(\mathbb{R})$ defined by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx,$$

$L^2(\mathbb{R})$ becomes a Hilbert space. The Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-i\xi t} dt,$$

where $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This extends uniquely to an operator on $L^2(\mathbb{R})$.

An *orthonormal wavelet* or simply a *wavelet* of $L^2(\mathbb{R})$ is a function $\psi \in L^2(\mathbb{R})$ such that $\{2^{j/2}\psi(2^j t - k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. The set of all wavelets is denoted by \mathcal{W} .

RESULT 2.1. [5] A function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet iff

(i) $\|\psi\|_2 = 1$,

(ii) $\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = \chi_{\mathbb{R}}$, for a.e., $\xi \in \mathbb{R}$,

(iii) $\sum_{j \geq 0} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0$, for a.e., $\xi \in \mathbb{R}$ and for $q \in 2\mathbb{Z} + 1$.

One of the methods of constructing orthonormal wavelets is based on the existence of a family of closed subspaces of $L^2(\mathbb{R})$ satisfying certain properties. Such a family is called a *multiresolution analysis*, or, simply an MRA.

DEFINITION 2.2. [5] A pair $(\{V_j\}_{j \in \mathbb{Z}}, \varphi)$ consisting of a family $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ together with a function $\varphi \in V_0$ is called a *multiresolution analysis* (MRA) if it satisfies the following conditions:

(a) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,

(b) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$, for all $j \in \mathbb{Z}$,

(c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,

(d) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(e) $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0 .

The function φ is called a *scaling function* for the given MRA. An MRA determines a function ψ lying in the orthogonal complement of V_0 in V_1 which is an orthonormal wavelet for $L^2(\mathbb{R})$. Such a ψ is called an *MRA wavelet* arising through the MRA $(\{V_j\}_{j \in \mathbb{Z}}, \varphi)$. The set of all MRA wavelets is denoted by \mathcal{W}^M .

For a wavelet ψ , the following

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \left| \hat{\psi}(2^j(\xi + 2k\pi)) \right|^2$$

describes the *dimension function* D_ψ for ψ . We use the following characterization of an MRA wavelet.

RESULT 2.3. [5] *A wavelet $\psi \in L^2(\mathbb{R})$ is an MRA wavelet iff $D_\psi(\xi) = 1$, for almost every $\xi \in \mathbb{R}$.*

Dai and Larson [3] called a measurable set W of the real line to be a *wavelet set* if the characteristic function χ_W on W is equal to $\sqrt{2\pi}$ times the modulus of the Fourier transform $\hat{\psi}$ for some wavelet ψ on $L^2(\mathbb{R})$. A wavelet whose Fourier transform has the support to be of smallest possible measure is called a *minimally supported frequency (MSF) wavelet* [4]. In fact, an MSF wavelet ψ is a wavelet which is associated with a wavelet set W in the sense that the support of $\hat{\psi}$ is W . The set of all MSF wavelets is denoted by \mathcal{W}^S .

B. CIP and CIPH

For a self-continuous map f on a topological space X , $\text{Fix } f$ denotes the set of all fixed points of f . A point $x \in X$ is called fixed point of f if $f(x) = x$. While dealing with the converse problem of Brouwer's fixed point theorem, Robbins [10] sowed the germ of the topological notion of the complete invariance property which got the formal definition by Ward as follows:

DEFINITION 2.4. [12] A topological space X is said to possess the *complete invariance property (CIP)* if each of its non-empty closed sets is $\text{Fix } f$, for some self-continuous map on X .

DEFINITION 2.5. [8] A topological space X is said to possess the *complete invariance property with respect to homeomorphism (CIPH)* if each of its non-empty closed sets is the fixed point set, $\text{Fix } f$, for some self-homeomorphism on X .

To examine whether the property CIPH is possessed by a space, Ward, in his paper [12], introduce the following notions.

DEFINITION 2.6. (1) A space X has property Q if for every non-empty closed subset K of X there is a point $p \in K$, a retract R of X containing K and a deformation $H : R \times I \rightarrow R$ such that $H(x, t) \neq x$ if $x \neq p$ and $t > 0$.

(2) If in (1) we omit p and stipulate that $H(x, t) \neq x$ if $x \notin K$ and $t > 0$, then we say that X has property $Q(\text{weak})$.

(3) A space X has property W if for every point $p \in X$, there is a deformation $H : X \times I \rightarrow X$ such that $H(x, t) \neq x$ if $x \neq p$ and $t > 0$.

(4) If in (3) $H(x, t) \neq x$, whenever $t > 0$, we say that X has property $W(\text{strong})$.

REMARK 2.7. $W(\text{strong}) \Rightarrow W \Rightarrow Q \Rightarrow Q(\text{weak})$.

Below we quote results which we use in the sequel.

RESULT 2.8. [9] *A metric space (X, δ) having property W has CIP.*

RESULT 2.9. [8] *A metric space X possessing a bounded metric on which the unit circle S^1 acts freely such that each orbit is isometric to S^1 , has CIPH.*

For common notions and terminologies in wavelets and topology, we refer to [2, 5, 13]. The unit closed interval $[0, 1]$ of the real line is denoted by I .

3. Wavelet spaces and CIP

In this Section, we obtain a natural action of the multiplicative group of the unit circle S^1 on \mathcal{W} which is a free action and provides orbits each of which is isometric to S^1 . We further observe that the subsets \mathcal{W}^M , \mathcal{W}^S and $\mathcal{W}^M \cap \mathcal{W}^S$ are invariant subsets of the topological transformation group \mathcal{W} . Employing Results 2.8 and 2.9, we obtain the following:

THEOREM 3.1. *The spaces \mathcal{W} , \mathcal{W}^M , \mathcal{W}^S and $\mathcal{W}^M \cap \mathcal{W}^S$ possess CIPH.*

THEOREM 3.2. *The cylinder $\mathcal{W} \times I$ possesses CIP. Also, cylinders $\mathcal{W}^M \times I$, $\mathcal{W}^S \times I$ and $(\mathcal{W}^M \cap \mathcal{W}^S) \times I$ possess CIP.*

THEOREM 3.3. *The product space $X \times Y$, where $X, Y \in \{\mathcal{W}, \mathcal{W}^M, \mathcal{W}^S, \mathcal{W}^M \cap \mathcal{W}^S\}$, has CIP.*

THEOREM 3.4. *The cone $C(\mathcal{W})$ of \mathcal{W} possesses CIP. Also, cones $C(\mathcal{W}^M)$, $C(\mathcal{W}^S)$ and $C(\mathcal{W}^M \cap \mathcal{W}^S)$ possess CIP.*

THEOREM 3.5. *The suspension $S(\mathcal{W})$ of \mathcal{W} possesses CIP. Also, suspensions $S(\mathcal{W}^M)$, $S(\mathcal{W}^S)$ and $S(\mathcal{W}^M \cap \mathcal{W}^S)$ possess CIP.*

On account of Result 2.1, it is straightforward to see that for $e^{i\theta} \in S^1$ and $\psi \in \mathcal{W}$, $e^{i\theta} \cdot \psi \in \mathcal{W}$. Thus, the map

$$\eta : S^1 \times \mathcal{W} \rightarrow \mathcal{W}$$

defined by $\eta(e^{i\theta}, \psi) = e^{i\theta} \cdot \psi$, describes a free action of S^1 on \mathcal{W} . The following

- (i) $\eta(1, \psi) = \psi$, and
- (ii) $\eta(e^{i\theta_1}, \eta(e^{i\theta_2}, \psi)) = \eta(e^{i(\theta_1 + \theta_2)}, \psi)$

are easily checked. For the continuity of η at $(e^{i\theta}, \psi)$, we simply observe that

$$\|e^{i\theta} \cdot \psi - e^{i\theta_1} \cdot \psi_1\|_2 \leq |e^{i\theta} - e^{i\theta_1}| + \|\psi - \psi_1\|_2,$$

where $(e^{i\theta_1}, \psi_1) \in S^1 \times \mathcal{W}$.

The orbit of ψ is given by

$$\eta(S^1 \times \{\psi\}) \equiv \{\eta(e^{i\theta}, \psi) : \theta \in [0, 2\pi)\}$$

which is isometric to S^1 via the map $\varphi : \eta(S^1 \times \{\psi\}) \rightarrow S^1$ which sends $e^{i\theta} \cdot \psi$ to $e^{i\theta}$.

Next, since the dimension function for ψ coincides with that of $\eta(e^{i\theta}, \psi)$, it follows from Result 2.3 that ψ is an MRA wavelet iff $\eta(e^{i\theta}, \psi)$ is an MRA wavelet.

Also, we note that ψ is an MSF wavelet iff $\eta(e^{i\theta}, \psi)$ is an MSF wavelet. Thus \mathcal{W}^M , \mathcal{W}^S and $\mathcal{W}^M \cap \mathcal{W}^S$ are invariant sets in the topological transformation group \mathcal{W} with respect to the induced action. The orbits of these invariant sets remain isometric to S^1 . Thus Theorem 3.1, follows from Result 2.9. ■

Proof of Theorem 3.2. In view of Result 2.8, it is sufficient to show that $\mathcal{W} \times I$ has the property W . The following shows that this property is enjoyed by both \mathcal{W} as well as by I .

(a) The map $H_1 : \mathcal{W} \times I \rightarrow \mathcal{W}$ defined by

$$H_1(\psi, t) = e^{i\pi t} \cdot \psi,$$

where $(\psi, t) \in \mathcal{W} \times I$ is a deformation such that for $t > 0$, $H_1(\psi, t) \neq \psi$. Therefore, \mathcal{W} enjoys the property $W(\text{strong})$ and hence W .

(b) For each $s \in I$, the map $H_2 : I \times I \rightarrow I$ defined by

$$H_2(x, t) = (1 - t)x + ts,$$

where $(x, t) \in I \times I$, is a deformation such that for $t > 0$ and $x \neq s$, $H_2(x, t) \neq x$. Thus, I has the property W .

Define $H : (\mathcal{W} \times I) \times I \rightarrow \mathcal{W} \times I$ by

$$H((\psi, t_1), t_2) = (H_1(\psi, t_2), H_2(t_1, t_2)).$$

It is easy to check that H is a deformation such that for $t_2 > 0$ and $(\psi, t_1) \neq (\psi_1, s)$, $H((\psi, t_1), t_2) \neq (\psi, t_1)$. Thus $\mathcal{W} \times I$ has the property W . ■

Proof of Theorem 3.3. It is similar to that of Theorem 3.2, by observing that \mathcal{W}^M , \mathcal{W}^S and $\mathcal{W}^M \cap \mathcal{W}^S$ possess the property W . ■

Proof of Theorem 3.4. It is sufficient to obtain that the cone $C(\mathcal{W})$ has the property $W(\text{strong})$. Since \mathcal{W} has the property $W(\text{strong})$, there is a deformation

$$H : \mathcal{W} \times I \rightarrow \mathcal{W}$$

such that $H(x, t) \neq x$, if $t > 0$. Now, the map

$$H' : C(\mathcal{W}) \times I \rightarrow C(\mathcal{W})$$

defined by $H'([x, t], t') = [H(x, t'), |t - t'|]$, where $[x, t] \in C(\mathcal{W})$ and $t' \in I$, shows that $C(\mathcal{W})$ has the property $W(\text{strong})$. ■

Proof of Theorem 3.5. It is similar to that of Theorem 3.4. ■

REMARK 3.6. Analogous results to those in Theorems 3.1, 3.2, 3.3, 3.4 and 3.5 hold in case of a higher dimension as well.

REMARK 3.7. [8] The complete invariance property is not necessarily preserved under the product of spaces and by cones.

ACKNOWLEDGEMENTS. The authors thank the anonymous referees for reading the manuscript carefully and to their supervisor Professor K. K. Azad for his guidance and encouragement.

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(received 11.05.2009; in revised form 29.10.2009)

Rajeshwari Dubey, Department of Mathematics, University of Allahabad, Allahabad-211 002, India

E-mail: rajeshwari_uni@yahoo.com

Aparna Vyas, Department of Mathematics, University of Allahabad, Allahabad-211 002, India

Mailing Address: Vijaya Niwas, 198, Mumfordganj, Allahabad-211 002, India

E-mail: aparna.vyaas@gmail.com, dpub@pphmj.com