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CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS AND n-STARLIKE WITH RESPECT TO CERTAIN POINTS

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Abstract. In this paper we introduce three subclasses of T; $S^{\star}_{s,n}T(\alpha,\beta)$, $S^{\star}_{c,n}T(\alpha,\beta)$ and $S^{\star}_{sc,n}T(\alpha,\beta)$ consisting of analytic functions with negative coefficients defined by using Salagean operator and are, respectively, n-starlike with respect to symmetric points, n-starlike with respect to conjugate points and n-starlike with respect to symmetric conjugate points. Several properties like, coefficient bounds, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are investigated.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. For a function $f(z) \in S$, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z) \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1, 2, \dots\}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [8]. Let $S^\star$ be the subclass of S consisting of starlike functions in U. It is well known that

$$f \in S^\star \quad \text{if and only if} \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in U). \tag{1.5}$$

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Let $S_s^\star$ be the subclass of S consisting of functions of the form (1.1) satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0 \quad (z\in U). \tag{1.6}$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [7] (see also Robertson [6], Stankiewicz [10], Wu [12] and Owa et al. [5]). In [2], El-Ashwah and Thomas, introduced and studied two other classes namely the class $S_c^\star$ consisting of functions starlike with respect to conjugate points and $S_{sc}^\star$ consisting of functions starlike with respect to symmetric conjugate points.

In [11], Sudharsan et al. introduced the class $S_s^\star(\alpha,\beta)$ of functions $f(z)\in S$ and satisfying the following condition (see also [9]):

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-1\right|<\beta\left|\alpha\frac{zf'(z)}{f(z)-f(-z)}+1\right| \tag{1.7}$$

for some $0\le\alpha\le1$, $0<\beta\le1$ and $z\in U$.

Let T denote the subclass of S consisting of functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k \quad (a_k\ge0). \tag{1.8}$$

Definition 1. Let the function $f(z)$ be defined by (1.8). Then $f(z)$ is said to be n-starlike with respect to symmetric points if it satisfies the following condition:

$$\left|\frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}-1\right|<\beta\left|\alpha\frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}+1\right|, \tag{1.9}$$

where $n\in N_0=N\cup\{0\}$, $0\le\alpha\le1$, $0<\beta\le1$, $0\le\frac{2(1-\beta)}{1+\alpha\beta}<1$ and $z\in U$. We denote the class of n-starlike with respect to symmetric points by $S_{s,n}^\star T(\alpha,\beta)$.

Definition 2. Let the function $f(z)$ be defined by (1.8). Then $f(z)$ is said to be n-starlike with respect to conjugate points if it satisfies the following condition:

$$\left|\frac{D^{n+1}f(z)}{D^nf(z)+\overline{D^nf(\overline{z})}}-1\right|<\beta\left|\alpha\frac{D^{n+1}f(z)}{D^nf(z)+\overline{D^nf(\overline{z})}}+1\right|, \tag{1.10}$$

where $n\in N_0$, $0\le\alpha\le1$, $0<\beta\le1$, $0\le\frac{2(1-\beta)}{1+\alpha\beta}<1$ and $z\in U$. We denote the class of n-starlike with respect to conjugate points by $S_{c,n}^\star T(\alpha,\beta)$.

Definition 3. Let the function $f(z)$ be defined by (1.8). Then $f(z)$ is said to be n-starlike with respect to symmetric conjugate points if it satisfies the following condition:

$$\left|\frac{D^{n+1}f(z)}{D^nf(z)-\overline{D^nf(-\overline{z})}}-1\right|<\beta\left|\alpha\frac{D^{n+1}f(z)}{D^nf(z)-\overline{D^nf(-\overline{z})}}+1\right|, \tag{1.11}$$

where $n \in N_0$, $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$. We denote the class of n-starlike with respect to symmetric conjugate points by $S^{\star}_{sc,n}T\,(\alpha,\beta)$.

We note that the classes $S^{\star}_{s,0}T(\alpha,\beta) = S^{\star}_{s}T(\alpha,\beta)$, $S^{\star}_{c,0}T(\alpha,\beta) = S^{\star}_{c}T(\alpha,\beta)$ and $S^{\star}_{sc,0}T(\alpha,\beta) = S^{\star}_{sc}T(\alpha,\beta)$ were studied by Halim et al. [4] with $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$ and $0 \leq \frac{2(1-\beta)}{1+\alpha\beta} < 1$. Also Halim et al. [3] studied these mentioned classes.

2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $n \in N_0$, $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$. We shall use the technique of Dziok [1] to prove the following theorems.

Theorem 1. *Let the function $f(z)$ be defined by* (1.8) *and $D^n f(z) - D^n f(-z) \neq 0$ for $z \neq 0$. Then $f(z) \in S^{\star}_{s,n}T(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty} k^n\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}a_k \leq \beta(2+\alpha) - 1. \tag{2.1}$$

Proof. Let $|z| = 1$. Then we have

$$\begin{aligned}
&|D^{n+1}f(z) - D^n f(z) + D^n f(-z)| - \beta|\alpha D^{n+1}f(z) + D^n f(z) - D^n f(-z)| \\
&= \Big|z + \sum_{k=2}^{\infty} k^n[k-1+(-1)^k]a_k z^k\Big| - \beta\Big|(\alpha+2)z - \sum_{k=2}^{\infty} k^n[\alpha k + 1 - (-1)^k]a_k z^k\Big| \\
&\qquad \leq \sum_{k=2}^{\infty} k^n\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}a_k - [\beta(\alpha+2)-1] \leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f \in S^{\star}_{s,n}T(\alpha,\beta)$.

For the converse, assume that

$$\left|\frac{\dfrac{D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} - 1}{\alpha\dfrac{D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} + 1}\right| = \left|\frac{-z - \sum_{k=2}^{\infty} k^n[k-1+(-1)^k]a_k z^k}{(\alpha+2)z - \sum_{k=2}^{\infty} k^n[\alpha k + 1 - (-1)^k]a_k z^k}\right| < \beta.$$

Since $|\operatorname{Re} z| \leq |z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{z + \sum_{k=2}^{\infty} k^n[k-1+(-1)^k]a_k z^k}{(\alpha+2)z - \sum_{k=2}^{\infty} k^n[\alpha k + 1 - (-1)^k]a_k z^k}\right\} < \beta. \tag{2.2}$$

Choose values of z on the real axis so that $\frac{D^{n+1}f(z)}{D^n f(z) - D^n f(-z)}$ is real and $D^n f(z) - D^n f(-z) \neq 0$ for $z \neq 0$. Upon clearing the denominator in (2.2) and letting $z \longrightarrow 1^-$ through real values, we obtain

$$1 + \sum_{k=2}^{\infty} k^n[k-1+(-1)^k]a_k \leq \beta(\alpha+2) - \beta\sum_{k=2}^{\infty} k^n[\alpha k + 1 - (-1)^k]a_k.$$

This gives the required condition. ∎

Corollary 1. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $S^{\star}_{s,n}T(\alpha,\beta)$. *Then we have*

$$a_k \leq \frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}} \quad (k \geq 2;\ n \in N_0). \tag{2.3}$$

The equality in (2.3) *is attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}z^k \quad (k \geq 2;\ n \in N_0). \tag{2.4}$$

Theorem 2. *Let the function* $f(z)$ *be defined by* (1.8). *Then* $f(z) \in S^{\star}_{c,n}T(\alpha,\beta)$ *if and only if*

$$\sum_{k=2}^{\infty} k^n\{(1+\alpha\beta)k+2(\beta-1)\}a_k \leq \beta(2+\alpha)-1. \tag{2.5}$$

Corollary 2. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $S^{\star}_{c,n}T(\alpha,\beta)$. *Then we have*

$$a_k \leq \frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+2(\beta-1)\}} \quad (k \geq 2; n \in N_0). \tag{2.6}$$

The equality in (2.6) *is attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+2(\beta-1)\}}z^k \quad (k \geq 2; n \in N_0). \tag{2.7}$$

Theorem 3. *Let the function* $f(z)$ *be defined by* (1.8). *Then* $f(z) \in S^{\star}_{sc,n}T(\alpha,\beta)$ *if and only if*

$$\sum_{k=2}^{\infty} k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}a_k \leq \beta(2+\alpha)-1. \tag{2.8}$$

Corollary 3. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $S^{\star}_{sc,n}T(\alpha,\beta)$. *Then we have*

$$a_k \leq \frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}} \quad (k \geq 2;\ n \in N_0). \tag{2.9}$$

The equality in (2.9) *is attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}z^k \quad (k \geq 2;\ n \in N_0). \tag{2.10}$$

3. Distortion theorems

THEOREM 4. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $S^{\star}_{s,n}T(\alpha,\beta)$. *Then we have*

$$|z|-\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}|z|^2\le|D^i f(z)|\le|z|+\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}|z|^2 \tag{3.1}$$

for $z\in U$, *where* $0\le i\le n$. *The result is sharp.*

Proof. Note that $f(z)\in S^{\star}_{s,n}T(\alpha,\beta)$ if and only if $D^i f(z)\in S^{\star}_{s,n-i}T(\alpha,\beta)$, and that

$$D^i f(z)=z-\sum_{k=2}^{\infty}k^i a_k z^k. \tag{3.2}$$

Using Theorem 1, we know that

$$\begin{aligned}2^{n+1-i}(1+\alpha\beta)\sum_{k=2}^{\infty}k^i a_k &\le \sum_{k=2}^{\infty}k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}a_k\\ &\le \beta(2+\alpha)-1\end{aligned} \tag{3.3}$$

that is, that

$$\sum_{k=2}^{\infty}k^i a_k\le\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}. \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$|D^i f(z)|\ge|z|-|z|^2\sum_{k=2}^{\infty}k^i a_k\ge|z|-\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}|z|^2 \tag{3.5}$$

and

$$|D^i f(z)|\le|z|+|z|^2\sum_{k=2}^{\infty}k^i a_k\le|z|+\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}|z|^2. \tag{3.6}$$

Finally, we note that the equality in (3.1) is attained by the function

$$D^i f(z)=z-\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}z^2 \tag{3.7}$$

or by

$$f(z)=z-\frac{\beta(2+\alpha)-1}{2^{n+1}(1+\alpha\beta)}z^2.\quad\blacksquare \tag{3.8}$$

COROLLARY 4. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $S^{\star}_{s,n}T(\alpha,\beta)$. *Then we have*

$$|z|-\frac{\beta(2+\alpha)-1}{2^{n+1}(1+\alpha\beta)}|z|^2\le|f(z)|\le|z|+\frac{\beta(2+\alpha)-1}{2^{n+1}(1+\alpha\beta)}|z|^2 \tag{3.9}$$

for $z\in U$. *The result is sharp for the function* $f(z)$ *given by* (3.8).

Proof. Taking $i=0$ in Theorem 4, we can easily show (3.9). ■

Corollary 5. *Let the function $f(z)$ defined by (1.8) be in the class $S^{\star}_{s,n}T(\alpha,\beta)$. Then we have*

$$1-\frac{\beta(2+\alpha)-1}{2^n(1+\alpha\beta)}|z|\leq|f^{'}(z)|\leq 1+\frac{\beta(2+\alpha)-1}{2^n(1+\alpha\beta)}|z| \tag{3.10}$$

for $z\in U$. The result is sharp for the function $f(z)$ given by (3.8).

Similarly we can prove the following result.

Theorem 5. *Let the function $f(z)$ be defined by (1.8) be in the class $S^{\star}_{c,n}T(\alpha,\beta)$. Then we have*

$$|z|-\frac{\beta(2+\alpha)-1}{2^{n+1-i}\beta(1+\alpha)}|z|^2\leq|D^if(z)|\leq|z|+\frac{\beta(2+\alpha)-1}{2^{n+1-i}\beta(1+\alpha)}|z|^2 \tag{3.11}$$

for $z\in U$, where $0\leq i\leq n$. The result is sharp, for the function $f(z)$ given by

$$D^if(z)=z-\frac{\beta(2+\alpha)-1}{2^{n+1-i}\beta(1+\alpha)}z^2 \tag{3.12}$$

or by

$$f(z)=z-\frac{\beta(2+\alpha)-1}{2^{n+1}\beta(1+\alpha)}z^2. \tag{3.13}$$

Corollary 6. *Let the function $f(z)$ defined by (1.8) be in the class $S^{\star}_{c,n}T(\alpha,\beta)$. Then we have*

$$|z|-\frac{\beta(2+\alpha)-1}{2^{n+1}\beta(1+\alpha)}|z|^2\leq|f(z)|\leq|z|+\frac{\beta(2+\alpha)-1}{2^{n+1}\beta(1+\alpha)}|z|^2 \tag{3.14}$$

for $z\in U$. The result is sharp for the function $f(z)$ given by (3.13).

Corollary 7. *Let the function $f(z)$ defined by (1.8) be in the class $S^{\star}_{c,n}T(\alpha,\beta)$. Then we have*

$$1-\frac{\beta(2+\alpha)-1}{2^n\beta(1+\alpha)}|z|\leq|f^{'}(z)|\leq 1+\frac{\beta(2+\alpha)-1}{2^n\beta(1+\alpha)}|z| \tag{3.15}$$

for $z\in U$. The result is sharp for the function $f(z)$ given by (3.13).

Theorem 6. *Let the function $f(z)$ be defined by (1.8) be in the class $S^{\star}_{sc,n}T(\alpha,\beta)$. Then we have*

$$|z|-\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}|z|^2\leq|D^if(z)|\leq|z|+\frac{\beta(2+\alpha)-1}{2^{n+1-i}(1+\alpha\beta)}|z|^2 \tag{3.16}$$

for $z\in U$, where $0\leq i\leq n$. The result is sharp.

4. Extreme points

Theorem 7. *The class $S^{\star}_{s,n}T(\alpha,\beta)$is closed under convex linear combination.*

Proof. Let the functions $f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}z^k$ $(a_{k,j}\ge 0;\ j=1,2)$ be in the class $S^{\star}_{s,n}T(\alpha,\beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=\lambda f_1(z)+(1-\lambda)f_2(z)\quad(0\le\lambda\le 1)\tag{4.1}$$

is in the class $S^{\star}_{s,n}T(\alpha,\beta)$. Since, for $0\le\lambda\le 1$,

$$h(z)=z-\sum_{k=2}^{\infty}[\lambda a_{k,1}+(1-\lambda)a_{k,2}]z^k,$$

with the aid of Theorem 1, we have

$$\sum_{k=2}^{\infty}k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}[\lambda a_{k,1}+(1-\lambda)a_{k,2}]\le[\beta(2+\alpha)-1],$$

which implies that $h(z)\in S^{\star}_{s,n}T(\alpha,\beta)$. ∎

As a consequence of Theorem 1, there exist extreme points of the class $S^{\star}_{s,n}T(\alpha,\beta)$.

Theorem 8. *Let $f_1(z)=z$ and*

$$f_k(z)=z-\frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}z^k\quad(k\ge 2)\tag{4.2}$$

for $0\le\alpha\le 1$, $0<\beta\le 1$ and $n\in N_0$. Then $f(z)$ is in the class $S^{\star}_{s,n}T(\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z),\tag{4.3}$$

where $\lambda_k\ge 0$ $(k\ge 1)$ and $\sum_{k=1}^{\infty}\lambda_k=1$.

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)=z-\sum_{k=2}^{\infty}\frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}\lambda_k z^k.\tag{4.4}$$

Then we get

$$\sum_{k=2}^{\infty}\frac{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{\beta(2+\alpha)-1}\,\frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}\lambda_k=\sum_{k=2}^{\infty}\lambda_k=1-\lambda_1\le 1.\tag{4.5}$$

By virtue of Theorem 1, this shows that $f(z)\in S^{\star}_{s,n}T(\alpha,\beta)$.

On the other hand, suppose that the function $f(z)$ defined by (1.8) is in the class $S^{\star}_{s,n}T(\alpha,\beta)$. Again, by using Theorem 1, we can show that

$$a_k \le \frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}} \quad (k\ge 2;\ n\in N_0). \tag{4.6}$$

Setting

$$\lambda_k = \frac{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{\beta(2+\alpha)-1} \quad (k\ge 2;\ n\in N_0), \tag{4.7}$$

and

$$\lambda_1 = 1-\sum_{k=2}^{\infty}\lambda_k, \tag{4.8}$$

we can see that $f(z)$ can be expressed in the form (4.3). This completes the proof of Theorem 8. ∎

Corollary 8. *The extreme points of the class $S^{\star}_{s,n}T(\alpha,\beta)$ are the functions $f_k(z)$ $(k\ge 1)$ given by Theorem 8.*

Similarly we can prove the following results.

Theorem 9. *Let $f_1(z)=z$ and*

$$f_k(z) = z-\frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+2(\beta-1)\}}z^k \quad (k\ge 2)$$

for $0\le\alpha\le 1$, $0<\beta\le 1$ and $n\in N_0$. Then $f(z)$ is in the class $S^{\star}_{c,n}T(\alpha,\beta)$ if and only if it can be expressed in the form $f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)$, where $\lambda_k\ge 0$ $(k\ge 1)$ and $\sum_{k=1}^{\infty}\lambda_k=1$.

Corollary 9. *The extreme points of the class $S^{\star}_{c,n}T(\alpha,\beta)$ are the functions $f_k(z)$ $(k\ge 1)$ given by Theorem 9.*

Theorem 10. *Let $f_1(z)=z$ and*

$$f_k(z) = z-\frac{\beta(2+\alpha)-1}{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}z^k \quad (k\ge 2)$$

for $0\le\alpha\le 1$, $0<\beta\le 1$ and $n\in N_0$. Then $f(z)$ is in the class $S^{\star}_{sc,n}T(\alpha,\beta)$ if and only if it can be expressed in the form $f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)$, where $\lambda_k\ge 0$ $(k\ge 1)$ and $\sum_{k=1}^{\infty}\lambda_k=1$.

Corollary 10. *The extreme points of the class $S^{\star}_{sc,n}T(\alpha,\beta)$ are the functions $f_k(z)$ $(k\ge 1)$ given by Theorem 10.*

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 11. *Let the function $f(z)$ defined by (1.8) be in the class $S^{\star}_{s,n}T(\alpha,\beta)$, then $f(z)$ is close-to-convex of order δ $(0\le\delta<1)$ in $|z|<r_1$, where*

$$r_1 = \inf_k\left\{\frac{(1-\delta)k^{n-1}\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{\beta(2+\alpha)-1}\right\}^{\frac{1}{k-1}} \quad (k\ge 2). \tag{5.1}$$

The result is sharp with the extremal function given by (2.4).

Proof. For close-to-convexity it is sufficient to show that $|f^{'}(z)-1|\le 1-\delta$ for $|z|<r_1$. We have

$$|f^{'}(z)-1|\le \sum_{k=2}^{\infty} ka_k|z|^{k-1}.$$

Thus $|f^{'}(z)-1|\le 1-\delta$ if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\delta}\right)a_k|z|^{k-1}\le 1. \tag{5.2}$$

According to Theorem 1, we have

$$\sum_{k=2}^{\infty}\frac{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{\beta(2+\alpha)-1}a_k\le 1. \tag{5.3}$$

Hence (5.2) will be true if

$$\left(\frac{k}{1-\delta}\right)|z|^{k-1}\le\frac{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{\beta(2+\alpha)-1}$$

or if

$$|z|\le\left\{\frac{(1-\delta)k^{n-1}\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{\beta(2+\alpha)-1}\right\}^{\frac{1}{k-1}}\quad(k\ge 2). \tag{5.4}$$

The theorem follows from (5.4). ∎

THEOREM 12. *Let the function $f(z)$ defined by* (1.8) *be in the class $S^{\star}_{s,n}T(\alpha,\beta)$, then $f(z)$ is starlike of order δ $(0\le\delta<1)$ in $|z|<r_2$, where*

$$r_2=\inf_k\left\{\frac{(1-\delta)k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{(k-\delta)[\beta(2+\alpha)-1]}\right\}^{\frac{1}{k-1}}\qquad(k\ge 2). \tag{5.5}$$

The result is sharp with the extremal function given by (2.4) *and r_2 attains its infimum for $k=2$.*

Proof. It is sufficient to show that $|\frac{zf^{'}(z)}{f(z)}-1|\le 1-\delta$ for $|z|<r_2$. We have

$$\left|\frac{zf^{'}(z)}{f(z)}-1\right|\le\frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus $|\frac{zf^{'}(z)}{f(z)}-1|\le 1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{(k-\delta)a_k|z|^{k-1}}{(1-\delta)}\le 1. \tag{5.6}$$

Hence, by using (5.3), (5.6) will be true if

$$\frac{(k-\delta)|z|^{k-1}}{(1-\delta)}\le\frac{k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{\beta(2+\alpha)-1}$$

or if

$$|z| \leq \left\{ \frac{(1-\delta)k^n\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{(k-\delta)[\beta(2+\alpha)-1]} \right\}^{\frac{1}{k-1}} \qquad (k \geq 2). \tag{5.7}$$

The theorem follows easily from (5.7). ■

REMARK. It is clear that r_2 attains its infimum at $k=2$ for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(2+\alpha)-1}{2^{n+1}(1+\alpha\beta)} z^2.$$

Also, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = |z| \left| \frac{\beta(2+\alpha)-1}{2^{n+1}(1+\alpha\beta) - [\beta(2+\alpha)-1]z} \right|.$$

Then

$$\frac{[\beta(2+\alpha)-1]|z|}{2^{n+1}(1+\alpha\beta) - [\beta(2+\alpha)-1]|z|} < 1-\delta,$$

that is, we have

$$(2-\delta)[\beta(2+\alpha)-1]|z| < (1-\delta)[2^{n+1}(1+\alpha\beta)].$$

Then, we have

$$|z| \leq \frac{(1-\delta)[2^{n+1}(1+\alpha\beta)]}{(2-\delta)[\beta(2+\alpha)-1]}.$$

COROLLARY 11. *Let the function $f(z)$ defined by (1.8) be in the class $S^{\star}_{s,n}T(\alpha,\beta)$, then $f(z)$ is convex of order δ $(0 \leq \delta < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_k \left\{ \frac{(1-\delta)k^{n-1}\{(1+\alpha\beta)k+(\beta-1)[1-(-1)^k]\}}{(k-\delta)[\beta(2+\alpha)-1]} \right\}^{\frac{1}{k-1}} \qquad (k \geq 2). \tag{5.8}$$

The result is sharp with the extremal function given by (2.4).

6. Integral operators

THEOREM 13. *Let the function $f(z)$ defined by (1.8) be in the class $S^{\star}_{s,n}T(\alpha,\beta)$ and c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{6.1}$$

also belongs to the class $S^{\star}_{s,n}T(\alpha,\beta)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{6.2}$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k. \tag{6.3}$$

Therefore

$$\begin{aligned}\sum_{k=2}^{\infty} & k^n\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}b_k \\ &= \sum_{k=2}^{\infty} k^n\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}(\tfrac{c+1}{c+k})a_k \\ &\le \sum_{k=2}^{\infty} k^n\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}a_k \le \beta(2+\alpha) - 1, \end{aligned} \tag{6.4}$$

since $f(z) \in S^{\star}_{s,n}T(\alpha,\beta)$. Hence, by Theorem 1, $F(z) \in S^{\star}_{s,n}T(\alpha,\beta)$. ■

THEOREM 14. *Let c be a real number such that $c > -1$. If $F(z) \in S^{\star}_{s,n}T(\alpha,\beta)$. Then the function $F(z)$ defined by* (6.1) *is univalent in $|z| < r^{\star}$, where*

$$r^{\star} = \inf_k \left\{ \frac{k^n\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}(c+1)}{[\beta(2+\alpha)-1](c+k)} \right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.5}$$

The result is sharp.

Proof. Let $F(z) = z - \sum_{k=2}^{\infty} a_k z^k (a_k \ge 0)$. It follows from (6.1) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{(c+1)} = z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k. \quad (c > -1) \tag{6.6}$$

In order to obtain the required result it suffices to show that $|f^{'}(z) - 1| < 1$ in $|z| < r^{\star}$. Now

$$|f^{'}(z) - 1| \le \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1}.$$

Thus $|f^{'}(z) - 1| < 1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1} < 1. \tag{6.7}$$

Hence by using (5.3), (6.7) will be satisfied if

$$\frac{k(c+k)}{(c+1)}|z|^{k-1} < \frac{k^n\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}}{[\beta(2+\alpha)-1]},$$

i.e., if

$$|z| < \left[\frac{k^{n-1}\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}(c+1)}{(c+k)[\beta(2+\alpha)-1]} \right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.8}$$

Therefore $F(z)$ is univalent in $|z| < r^{\star}$. Sharpness follows if we take

$$f(z) = z - \frac{(c+k)[\beta(2+\alpha)-1]}{k^{n-1}\{(1+\alpha\beta)k + (\beta-1)[1-(-1)^k]\}(c+1)} z^k \tag{6.9}$$

$(k \ge 2;\ n \in N_0;\ c > -1)$.

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