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ON REPRESENTATION OF DERIVATIVES OF FUNCTIONS IN L_p

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This paper is dedicated to professor Veselin Perić on the ocassion of his 80^{th} birthday

Abstract. A theorem on the expansion of the derivative $f^{(r_1,r_2,\dots,r_n)}$, where $f \in L_p$, and the derivatives of singular integrals into the series of band-limited functions (entire functions of exponential type), which converges in L_p for $1 \le p \le q < \infty$, is proved. The norms of their items are estimated by best approximations by "an angle".

1. Introduction and preliminaries

Theorems which refer to an approximation by an angle from trigonometric polynomials of 2π-periodic functions are proved in the paper [6]. The main results of that paper is the converse theorem of approximation by which the modulus of smoothness $\omega_k(f^{(r)})_q$ of the derivative $f^{(r)}$ is estimated by the best approximation by the angle $Y(f)_p$ of the function f in the norm of the L_p space, $1 \le p \le q < \infty$.

The proof of the converse theorem of approximation is based on the theory of representation of a derivative of a function. Therefore, the complete proof of the corresponding theorem of representation of the derivative $f^{(r_1,r_2,\dots,r_n)}$ into a series whose terms are entire functions of the exponential type is given in this paper. This theorem is mentioned in the paper [7] with a short instruction for its proof. Since the proof of this theorem is complex and long and the theorem has significant uses in approximation theory, the complete proof is given in this paper.

We also expand into a series the derivatives of singular integrals of a function, which are formed by the general Fejér's kernel. This theorem enables us to get new results which are related to the approximation by an angle and the mixed modulus of smoothness of the derivative of the function $f(x_1,\dots,x_n) \in L_p(R^n)$. Therefore, this theorem is important for obtaining new results.

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Approximation by an angle of functions of several variables is a good tool for examination of classes (spaces) of functions with a dominant mixed modulus of smoothness, (see [3], [6]).

Results concerning these classes (spaces) have been obtained by M.K. Potapov in [3] and his other related papers. Book [4] deals with several classes of Besov-Hardy-Sobolev function spaces on the Euclidean n-space. It also covers spaces in which properties of dominating mixed smoothness is predominate.

For simplicity, the theorem of representation will be proved for the case $n = 2$, i.e. for functions of two variables $f(x, y) \in L_p(R^2)$, $1 \le p < \infty$. As usual, we say that the function $f = f(x_1, \dots, x_n) \in L_p(R^n)$, $1 \le p < \infty$, if it is measurable on R^n and if

$$\|f\|_p = \left(\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |f(x_1, \dots . x_n)|^p \, dx_1 \, \dots \, dx_n \right)^{\frac{1}{p}} < \infty.$$

Let $g_{\nu_i}(x_1, \dots, x_n) \in L_p(R^n)$ be an entire function of exponential type $\nu_i \ge 0$ with respect to the variable x_i, $i = 1, \dots, n$, and, in general, it is an ordinary function with respect to other variables.

In particular, if $g_{\nu_i} \in L_p$, $1 \le p < \infty$ and $\nu_i = 0$, then $g_{\nu_i} \equiv 0$, (see [2]).

The quantity

$$Y_{\nu_{i_1}, \dots, \nu_{i_m}}(f)_p = \inf_{g_{\nu_{i_j}} \in L_p} \left\| f - \sum_{j=1}^{m} g_{\nu_{i_j}} \right\|_p, \quad (\nu_{i_j} \ge 0), \tag{1.1}$$

is called the best approximation by the m-dimensional angle of a function f with respect to the variables $x_{i_1}, \dots, x_{i_m}$, $(1 \le i_j \le n,\ 1 \le j \le m \le n)$.

We will use the general Fejér integral, which is, for a function f of one variable, defined by the following equality (see [1])

$$K_\lambda f = K_\lambda f(x) = \frac{\lambda}{2} \int_{-\infty}^{\infty} f(x - t) \Phi \left(\frac{\lambda t}{2} \right) dt, \quad \lambda > 0, \tag{1.2}$$

where

$$\Phi(u) = \frac{\cos u - \cos 2u}{\pi u^2}, \quad (\|\Phi\|_1 < \infty). \tag{1.3}$$

It was proved in [1] that $K_\lambda f$ is an entire function of type λ if $\frac{f(x)}{1+|x|} \in L(R)$ or $\frac{f(x)}{1+|x|} \in L_2(R)$. Also, it was proved in [1] that $K_\lambda f = f$ if f is an entire function of type $\tau \le \frac{\lambda}{2}$, under the condition $\frac{f(x)}{1+|x|} \in L_2(R)$.

For a function $f(x, y) \in L_p(R^2)$, $1 \le p < \infty$, we form the following functions (see [5]):

$$K_{\mu\infty} f = K_{\mu\infty} f(x, y) = \frac{\mu}{2} \int_{-\infty}^{\infty} f(t, y) \Phi \left[\frac{\mu}{2}(x - t) \right] dt, \quad \mu > 0, \tag{1.4}$$

$$K_{\infty\nu} f = K_{\infty\nu} f(x, y) = \frac{\nu}{2} \int_{-\infty}^{\infty} f(x, u) \Phi \left[\frac{\nu}{2}(y - u) \right] du, \quad \nu > 0, \tag{1.5}$$

$$K_{\mu\nu} f = K_{\mu\nu} f(x,y) = K_{\mu\infty} K_{\infty\nu} f, \quad \nu,\mu > 0. \tag{1.6}$$

The function $K_{\mu\infty} f$ is entire of exponential type μ with respect to x, and $K_{\infty\nu} f$ is entire of type ν with respect to y, if $f(x,y) \in L_p(R^2)$. The function $K_{\mu\nu} f$ is entire of type μ with respect to x and of type ν with respect to y.

For $\nu = 0$, $\mu = 0$ we put $K_{0\infty} f = 0$, $K_{\infty 0} f = 0$, $K_{00} f = 0$, and $K_{\mu 0} f = 0$, $K_{0\nu} f = 0$, $\mu > 0$, $\nu > 0$.

Denote

$$\chi_{\mu\nu} f = K_{2\mu\,\infty} f + K_{\infty\,2\nu} f - K_{2\mu\,2\nu} f. \tag{1.7}$$

Then (see [5, Lemma 1]):

$$\|f - \chi_{\mu\nu} f\|_p \le C Y_{\mu\nu}(f)_p, \tag{1.8}$$

$$\|f - K_{2\mu\,\infty} f\|_p \le C Y_{\mu}(f)_p, \quad Y_\mu = Y_{\mu\infty} \tag{1.9}$$

$$\|f - K_{\infty\,2\nu} f\|_p \le C Y_{\nu}(f)_p, \quad Y_\nu = Y_{\infty\nu} \tag{1.10}$$

for $\mu \ge 0$, $\nu \ge 0$, $1 \le p < \infty$, where C is an absolute constant.

From the entire functions $K_{\mu\nu} f$ and $\chi_{\mu\nu} f$ we form the following entire functions

$$\begin{aligned} \xi_{ij} = \xi_{ij} f &= K_{2^{i+1} 2^{j+1}} f - K_{2^{i+1} 2^{j}} f - K_{2^{i} 2^{j+1}} f + K_{2^{i} 2^{j}} f \\ &= -\{\chi_{2^i 2^j} f - \chi_{2^i [2^{j-1}]} f - \chi_{[2^{i-1}] 2^j} f + \chi_{[2^{i-1}][2^{j-1}]} f\}, \end{aligned} \tag{1.11}$$

where $i,j = 0,1,2,\ldots,n$ and $[2^{i-1}] = 2^{i-1}$ for $i \ge 1$, $[2^{0-1}] = 0$.

Functions $\xi_{ij} = \xi_{ij} f$ are entire of type 2^{i+1} with respect to x and of type 2^{j+1} with respect to y. In view of (1.11) and (1.8) we conclude that

$$\|\xi_{ij}\|_p \le C Y_{[2^{i-1}][2^{j-1}]}(f)_p \tag{1.12}$$

for $1 \le p < \infty$, $i,j = 0,1,2,\ldots$.

We note that $Y_0(f)_p = Y_{00}(f)_p = \|f\|_p$ for $f \in L_p$, $1 \le p < \infty$.

The symbol $a \ll b$, $a > 0$, $b > 0$, denotes that $a \le Cb$, where C is a positive constant.

As usual, the derivative $f^{(r_1,r_2)}$ of a function $f(x,y)$ is

$$f^{(r_1,r_2)} = \frac{\partial^{r_1+r_2} f}{\partial x^{r_1} \partial y^{r_2}}, \quad r_i = 0,1,2,\ldots$$

As a consequence of the theorem of representation [5, Theorem 2] we get

Lemma 1. *If $f(x,y) \in L_p(R^2)$, $1 \le p < \infty$, then the following equality holds in L_p*

$$f(x,y) = K_{22} f + \sum_{j=2}^{\infty} T_{2j} + \sum_{i=2}^{\infty} U_{i2} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \xi_{ij} \tag{1.13}$$

where ξ_{ij} are entire functions of type 2^{i+1} with respect to x, and of type 2^{j+1} with respect to y, given in (1.11); T_{2j} are entire functions of type 2 with respect to x, and of type 2^j with respect to y; U_{i2} are entire functions of type 2^i with respect to x and of type 2 with respect to y. We define

$$\begin{aligned} T_{21} = K_{22} f, \quad T_{2j} &= K_{2\,2^j} f - K_{2\,2^{j-1}} f, \quad j = 2,3,\ldots, \\ U_{i2} &= K_{2^i\,2} f - K_{2^{i-1}\,2} f, \quad i = 2,3,\ldots \end{aligned} \tag{1.14}$$

Proof. For fixed numbers μ and N denote the sum

$$S_{\mu N}=\sum_{i=1}^{\mu}\sum_{j=1}^{N}\xi_{ij}=\sum_{i=1}^{\mu}\sum_{j=1}^{N}[\chi_{2^{i-1}2^j}-\chi_{2^{i-1}2^{j-1}}-(\chi_{2^i2^j}-\chi_{2^i2^{j-1}})]$$
$$=\sum_{j=1}^{\mu}[\chi_{2^{i-1}2^N}-\chi_{2^{i-1}1}-(\chi_{2^i2^N}-\chi_{2^i1})]$$
$$=\sum_{j=1}^{\mu}[(\chi_{2^{i-1}2^N}-\chi_{2^i2^N})+(\chi_{2^i1}-\chi_{2^{i-1}1})]=-(\chi_{2^\mu 2^N}-\chi_{1\,2^N})+\chi_{2^\mu 1}-\chi_{11}.$$

Therefore

$$S_{\mu N}=S_{\mu N}f=-\chi_{2^\mu 2^N}f+\chi_{12^N}f+\chi_{2^\mu 1}f-\chi_{11}f, \tag{1.15}$$
$$\chi_{11}f+S_{\mu N}f=-\chi_{2^\mu 2^N}f+\chi_{12^N}f+\chi_{2^\mu 1}f. \tag{1.16}$$

We get

$$f-(\chi_{11}f+S_{\mu N}f)=f-\chi_{1\,2^N}f+f-\chi_{2^\mu 1}f+\chi_{2^\mu 2^N}f-f \tag{1.17}$$

and then

$$\|f-(\chi_{11}f+S_{\mu N}f)\|_p\ll Y_{1\,2^N}(f)_p+Y_{2^\mu 1}(f)_p+Y_{2^\mu 2^N}(f)_p. \tag{1.18}$$

Since $Y_{1\,2^N}\to 0$, $Y_{2^\mu 1}\to 0$, $Y_{2^\mu 2^N}\to 0$ as $\mu, N\to\infty$, then from (1.18) we get

$$f\overset{(p)}{=}\chi_{11}f+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\xi_{ij}f. \tag{1.19}$$

We need to represent $\chi_{11}f$ into a series of entire functions whose norms (up to a constant factor) are smaller than the best approximation by the angle Y. From the equality $\chi_{11}f=K_{2\infty}f+K_{\infty 2}f-K_{22}$ we represent into a series the functions $K_{2\infty}f$ and $K_{\infty 2}f$. Denote the sum

$$S_N=S_Nf=\sum_{j=1}^{N}T_{2j}f, \tag{1.20}$$
$$T_{21}f=K_{22}f,\quad T_{2j}f=K_{2\,2^j}f-K_{2\,2^{j-1}}f,\quad j=2,3,\dots$$

It holds that

$$S_N=K_{22}f+\sum_{j=2}^{N}(K_{2\,2^j}-K_{2\,2^{j-1}})=K_{2\,2^N}f. \tag{1.21}$$

Therefore

$$K_{2\infty}f-S_N=K_{2\infty}f-K_{2\,2^N}f=K_{2\infty}(f-K_{\infty 2^N}f), \tag{1.22}$$

from which we get

$$\|K_{2\infty}f-S_N\|_p\ll Y_{\infty 2^N}(f)_p=Y_{2^N}(f)_p. \tag{1.23}$$

Since $Y_{2^N}\to 0$ as $N\to\infty$, we conclude that in L_p the following equality holds

$$K_{2\infty}f\overset{(p)}{=}\sum_{j=1}^{\infty}T_{2j}f=K_{22}f+\sum_{j=2}^{\infty}T_{2j}. \tag{1.24}$$

Then

$$\|T_{21}f\| = \|K_{22}f\| \ll \|f\|. \tag{1.25}$$

For $T_{2j}f$ the following holds

$$\begin{aligned} T_{2j}f &= K_{2\,2^j}f - K_{2\,2^{j-1}}f = K_{2\infty}(K_{\infty 2^j}f) - K_{2\infty}(K_{\infty 2^{j-1}}f) \\ &= K_{2\infty}(K_{\infty 2^j}f) - K_{2\infty}f + K_{2\infty}f - K_{2\infty}(K_{\infty 2^{j-1}}f) \\ &= K_{2\infty}(K_{\infty 2^j}f - f) + K_{2\infty}(f - K_{\infty 2^{j-1}}f). \end{aligned}$$

From this equality we get

$$\|T_{2j}f\| \ll Y_{\infty 2^{j-1}}(f)_p + Y_{\infty 2^{j-2}}(f)_p \ll Y_{\infty 2^{j-2}}(f)_p = Y_{2^{j-2}}(f)_p, \quad j = 2, 3, \ldots \tag{1.26}$$

To represent functions $K_{\infty 2}f$ note the sum

$$S_\mu = \sum_{i=2}^{\mu} U_{i2}f, \quad U_{i2}f = K_{2^i 2}f - K_{2^{i-1}2}f. \tag{1.27}$$

The following holds

$$S_\mu = \sum_{i=2}^{\mu} K_{2^i 2}f - K_{2^{i-1}2}f = K_{2^\mu 2}f - K_{22}f. \tag{1.28}$$

Hence

$$K_{\infty 2}f - K_{22}f - S_\mu = K_{\infty 2}f - K_{2^\mu 2}f \tag{1.29}$$

and $\|K_{\infty 2}f - K_{22}f - S_\mu\|_p \ll \|K_{\infty 2}(f - K_{2^\mu \infty}f)\|$ and then

$$\|K_{\infty 2}f - K_{22}f - S_\mu\|_p \ll Y_{2^\mu \infty}(f)_p = Y_{2^\mu}(f)_p. \tag{1.30}$$

When $\mu \to \infty$ then $Y_{2^\mu} \to 0$, which means that

$$K_{\infty 2}f - K_{22}f \overset{(p)}{=} \sum_{i=2}^{\infty} U_{i2}. \tag{1.31}$$

From (1.19) and in view of (1.24) and (1.31) we get (1.13). Lemma 1 has been proved. ■

Remark 1. Let us emphasize that $\|U_{i2}\|$ is also estimated by the best approximation by one-dimensional angle. We have

$$\begin{aligned} U_{i2}f &= K_{2^i 2}f - K_{2^{i-1}2}f = K_{\infty 2}(K_{2^i \infty}f) - K_{\infty 2}(K_{2^{i-1}\infty}f) \\ &= K_{\infty 2}(K_{2^i \infty}f) - K_{\infty 2}f + K_{\infty 2}f - K_{\infty 2}(K_{2^{i-1}\infty}f) \\ &= K_{\infty 2}(K_{2^i \infty}f - f) + K_{\infty 2}(f - K_{2^{i-1}\infty}f). \end{aligned}$$

From this equality we get

$$\|U_{i2}f\|_p \ll \|K_{2^i \infty}f - f\| + \|f - K_{2^{i-1}\infty}f\| \ll Y_{2^{i-1}\infty}(f)_p + Y_{2^{i-2}\infty}(f)_p.$$

Hence

$$\|U_{i2}f\| \ll Y_{2^{i-2}\infty}(f)_p = Y_{2^{i-2}}(f)_p, \quad i = 2, 3, \ldots \tag{1.32}$$

Remark 2. For the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{ij} = g_{00} + \sum_{j=1}^{\infty} g_{0j} + \sum_{j=1}^{\infty} g_{i0} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g_{ij} \tag{1.33}$$

denote

$$g_{00} = K_{22}f, \quad g_{0j} = T_{2(j+1)}f, \quad j = 1, 2, \dots$$
$$g_{i0} = U_{(i+1)2}f, \quad i = 1, 2, \dots, \qquad g_{ij} = \xi_{ij}, \quad i, j = 1, 2, \dots \tag{1.34}$$

Then

$$\sum_{j=1}^{\infty} g_{0j} = \sum_{j=1}^{\infty} T_{2(j+1)}f = \sum_{j=2}^{\infty} T_{2j}f, \quad \sum_{i=1}^{\infty} g_{i0} = \sum_{i=1}^{\infty} U_{(i+1)2}f = \sum_{i=2}^{\infty} U_{i2}f \tag{1.35}$$

and the equality (1.13) from Lemma 1 becomes

$$f(x, y) \overset{(p)}{=} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g_{ij}, \quad g_{ij} = g_{ij}f. \tag{1.36}$$

Therefore, for norms of the terms of this series the following holds

$$\|g_{00}\| \ll \|f\|, \quad \|g_{0j}\| \ll Y_{2^{j-1}}(f)_p, \quad j = 1, 2, \dots$$
$$\|g_{i0}\| \ll Y_{2^{i-1}}(f)_p, \quad i = 1, 2, \dots, \qquad \|g_{ij}\| \ll Y_{2^{i-1}2^{j-1}}(f)_p, \quad i, j = 1, 2, \dots$$

LEMMA 2. *For the function $f(x, y) \in L_p(R^2)$, $1 \le p < \infty$, in the sense of L_p the following equalities hold*

$$K_{2^{\mu+1}\infty}f \overset{(p)}{=} \sum_{i=0}^{\mu} \sum_{j=0}^{\infty} g_{ij}f, \quad \mu = 1, 2, \dots \tag{1.37}$$

$$K_{\infty 2^{\nu+1}}f \overset{(p)}{=} \sum_{i=0}^{\infty} \sum_{j=0}^{\nu} g_{ij}f, \quad \nu = 1, 2, \dots \tag{1.38}$$

Proof. For a fixed number μ denote the partial sums of the series (1.36) by

$$G_{\mu N} = G_{\mu N}f = \sum_{i=0}^{\mu} \sum_{j=0}^{N} g_{ij}f = g_{00} + \sum_{j=1}^{N} g_{0j} + \sum_{i=1}^{\mu} g_{i0} + \sum_{i=1}^{\mu} \sum_{j=1}^{N} g_{ij}. \tag{1.39}$$

Using the equality (1.34) for $G_{\mu N}$ we have

$$\begin{aligned} G_{\mu N} &= K_{22} + \sum_{j=1}^{N} T_{2(j+1)} + \sum_{i=1}^{\mu} U_{(i+1)2} + \sum_{i=1}^{\mu} \sum_{j=1}^{N} \xi_{ij} \\ &= K_{22} + \sum_{j=1}^{N} K_{22^{j+1}} - K_{22^j} + \sum_{i=1}^{\mu} K_{2^{i+1}2} - K_{2^i 2} + \sum_{i=1}^{\mu} \sum_{j=1}^{N} \xi_{ij} \\ &= K_{22} + K_{22^{N+1}} - K_{22} + K_{2^{\mu+1}2} - K_{22} + \sum_{i=1}^{\mu} \sum_{j=1}^{N} \xi_{ij}. \end{aligned}$$

Therefore

$$G_{\mu N} = K_{2^{\mu+1}2} + K_{22^{N+1}} - K_{22} + \sum_{i=1}^{\mu} \sum_{j=1}^{N} \xi_{ij}. \tag{1.40}$$

Expressing ξ by χ, and then by K, we get (see (1.15))

$$\sum_{i=1}^{\mu} \sum_{j=1}^{N} \xi_{ij} = K_{2^{\mu+1}2^{N+1}} - K_{22^{N+1}} - K_{2^{\mu+1}2} + K_{22}. \tag{1.41}$$

From (1.40), using (1.41), it follows

$$G_{\mu N} f = \sum_{i=0}^{\mu} \sum_{j=0}^{N} g_{ij} = K_{2^{\mu+1} 2^{N+1}} f. \tag{1.42}$$

Now in view of (1.42) we get

$$K_{2^{\mu+1}\infty} f - G_{\mu N} f = K_{2^{\mu+1}\infty} f - K_{2^{\mu+1} 2^{N+1}} f = K_{2^{\mu+1}\infty}(f - K_{\infty 2^{N+1}} f) \tag{1.43}$$

and then

$$\|K_{2^{\mu+1}\infty} f - G_{\mu N} f\|_p \ll Y_{\infty 2^N}(f)_p = Y_{2^N}(f)_p. \tag{1.44}$$

Since $Y_{2^N} \to 0$ as $N \to \infty$, then, based on (1.44), we conclude that (1.37) holds. Equality (1.38) can be proved in the same way. Lemma 2 has been proved. ■

Remark 3. By definition of the best approximation by an angle we have

$$Y_{0j}(f)_p = \inf_{g \in L_p} \|f - (g_{0\infty} + g_{\infty j})\|_p = \inf_{g \in L_p} \|f - g_{\infty j}\|_p$$

because $g_{0\infty} = 0$ (due to the assumption that $g \in L_p$, $1 \le p < \infty$). Therefore

$$Y_{0j}(f)_p = Y_{\infty j}(f)_p = Y_j(f)_p, \quad j = 1, 2, \ldots$$

In the same way

$$Y_{i0}(f)_p = Y_{i\infty}(f)_p = Y_i(f)_p, \quad i = 1, 2, \ldots, \qquad Y_{00}(f)_p = \|f\|_p.$$

2. Representation of the derivative of a function

In this paragraph we will prove a theorem about the representation into a series of the derivative of singular integrals (1.4) and (1.5) and the derivative of a function. The terms of the series are entire functions whose norm is estimated using the best approximation by an angle.

Theorem 2.1. *Let $f(x, y) \in L_p(R^2)$, and let for non-negative integers r_i and numbers*

$$\sigma_i = r_i + \frac{1}{p} + \frac{1}{q}, \quad i = 1, 2, \quad 1 \le p \le q < \infty,$$

the following inequalities hold

$$\begin{gathered} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (i+1)^{\sigma_1 q - 1} (j+1)^{\sigma_2 q - 1} Y_{ij}^q(f)_p < \infty \\ \sum_{i=1}^{\infty} (i+1)^{\sigma_1 q - 1} Y_i^q(f)_p < \infty, \quad \sum_{j=1}^{\infty} (j+1)^{\sigma_2 q - 1} Y_j^q(f)_p < \infty. \end{gathered} \tag{2.1}$$

Then the functions $K_{2^{\mu+1}\infty} f$, $K_{\infty 2^{\nu+1}} f$, $f(x, y)$ have derivatives which belong to the space L_q and in the sense of L_q the following equalities hold

$$(K_{2^{\mu+1}\infty} f)^{(r_1, r_2)} \overset{(q)}{=} \sum_{i=0}^{\mu} \sum_{j=0}^{\infty} g_{ij}^{(r_1, r_2)}, \quad \mu = 1, 2, \ldots, \tag{2.2}$$

$$(K_{\infty 2^{\nu+1}} f)^{(r_1, r_2)} \overset{(q)}{=} \sum_{i=0}^{\infty} \sum_{j=0}^{\nu} g_{ij}^{(r_1, r_2)}, \quad \nu = 1, 2, \ldots, \tag{2.3}$$

$$f^{(r_1, r_2)} \overset{(q)}{=} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g_{ij}^{(r_1, r_2)}, \tag{2.4}$$

where the entire functions g_{ij} are given by equalities (1.34), (1,14) *and* (1.11).

Proof. We will prove that the equality (1.37) holds in the sense of L_q. Denote

$$G_{\mu N}^{P} = G_{\mu N}^{P} f = \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} g_{ij} f = G_{\mu P} - G_{\mu N}, \quad P > N+1, \tag{2.5}$$

$$A = \|G_{\mu N}^{P}\|_q^q = \Big\| \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} g_{ij} \Big\|_q^q. \tag{2.6}$$

In the proof of this theorem we will follow the pattern of the proofs of the corresponding theorem in paper [6] which corresponds to the periodic functions.

For a given number q denote $[q]+1 = m$. This means that $m \in \{2,3,\dots\}$ and that $\frac{q}{m} < 1$. Therefore, it follows from (2.6) that

$$A \le \iint \Big(\sum_{i=0}^{\mu} \sum_{j=N+1}^{P} |g_{ij}|^{q/m} \Big)^m \, dx\, dy, \quad \int = \int_{-\infty}^{\infty}. \tag{2.7}$$

Denote

$$\delta_{ij} = |g_{ij}|^{q/m}. \tag{2.8}$$

Now we have

$$A \le \iint \Big(\sum_{i=0}^{\mu} \sum_{j=N+1}^{P} \delta_{ij} \Big)^m \, dx\, dy. \tag{2.9}$$

Since m is a natural number, it is

$$\Big(\sum_{i=0}^{\mu} \sum_{j=N+1}^{P} \delta_{ij} \Big)^m = \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} \prod_{k=1}^{m} \delta_{i_k j_k}. \tag{2.10}$$

Now from (2.7), in view of (2.8), (2.9) and (2.10), we get

$$A \le \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} \iint \prod_{k=1}^{m} \delta_{i_k j_k} \, dx\, dy. \tag{2.11}$$

From the equality

$$\prod_{k=1}^{m} D_k^{m-1} = \prod_{\substack{r,s=1\\ r<s}}^{m} D_r D_s \tag{2.12}$$

we get

$$\prod_{k=1}^{m} D_k = \Big(\prod_{\substack{r,s=1\\ r<s}}^{m} D_r D_s \Big)^{\frac{1}{m-1}}. \tag{2.13}$$

Denoting $D_k = \delta_{i_k j_k}$, from (2.11), using (2.13), we get

$$A \le \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} \iint \Big(\prod_{\substack{r,s=1\\ r<s}}^{m} \delta_{i_r j_r} \delta_{i_s j_s} \Big)^{\frac{1}{m-1}} dx\, dy. \tag{2.14}$$

We apply the Hölder integral inequality to the product of $\gamma = \frac{m(m-1)}{2}$ factors of power $\frac{1}{\gamma}$, based on which we get

$$\iint \Bigg(\prod_{\substack{r,s=1 \\ r<s}}^{m} \delta_{i_r j_r} \delta_{i_s j_s} \Bigg)^{\frac{1}{m-1}} dx\, dy \le \prod_{\substack{r,s=1 \\ r<s}}^{m} \left[\iint (\delta_{i_r j_r} \delta_{i_s j_s})^{\frac{m}{2}}\, dx\, dy \right]^{\frac{2}{m(m-1)}}. \quad (2.15)$$

Denote

$$\Gamma_{rs} = \iint (\delta_{i_r j_r} \delta_{i_s j_s})^{\frac{m}{2}}\, dx\, dy. \quad (2.16)$$

Now from (2.14), in view of (2.15) and (2.16), it follows that

$$A \le \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} \prod_{\substack{r,s=1 \\ r<s}}^{m} (\Gamma_{rs})^{\frac{2}{m(m-1)}}. \quad (2.17)$$

We will now estimate numbers Γ_{rs}. For numbers $\alpha = \frac{p+q}{p}$, $\alpha' = \frac{p+q}{q}$ the equality $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ holds. Therefore we can apply the Hölder inequality based on which we get

$$\Gamma_{rs} \le (\|g_{i_r j_r}\|_{\alpha q/2})^{\frac{q}{2}} (\|g_{i_s j_s}\|_{\alpha' q/2})^{\frac{q}{2}}. \quad (2.18)$$

Functions $g_{ij} = g_{ij} f$ are entire of exponential type 2^i with respect to x and 2^j with respect to y. Therefore, based on the inequality of S.M. Nikol'skiĭ [2, 3.3.5] we conclude that the following holds

$$(\|g_{i_r j_r}\|_{\alpha q/2})^{\frac{q}{2}} \ll 2^{(i_r + j_r)(\frac{q}{2p} - \frac{1}{\alpha})} (\|g_{i_r j_r}\|_p)^{\frac{q}{2}}, \quad (2.19)$$

$$(\|g_{i_s j_s}\|_{\alpha' q/2})^{\frac{q}{2}} \ll 2^{(i_s + j_s)(\frac{q}{2p} - \frac{1}{\alpha'})} (\|g_{i_s j_s}\|_p)^{\frac{q}{2}}. \quad (2.20)$$

Using the equality

$$\frac{q}{2p} - \frac{1}{\beta} = \frac{q}{2} \left(\frac{1}{p} - \frac{1}{q} \right) + \frac{1}{2} - \frac{1}{\beta}, \quad \beta \in \{\alpha, \alpha'\}, \quad (2.21)$$

from (2.18), based on (2.19) and (2.20), we get

$$\Gamma_{rs} \ll 2^{(i_r + j_r)(\frac{1}{2} - \frac{1}{\alpha})} 2^{(i_s + j_s)(\frac{1}{2} - \frac{1}{\alpha'})} \Bigg\{ 2^{(i_r + j_r) q (\frac{1}{p} - \frac{1}{q})} \times$$

$$\times Y^q_{[2^{i_r-1}][2^{j_r-1}]}(f)_p 2^{(i_s + j_s) q (\frac{1}{p} - \frac{1}{q})} Y^q_{[2^{i_s-1}][2^{j_s-1}]}(f)_p \Bigg\}^{\frac{1}{2}}. \quad (2.22)$$

Denote

$$H_{ij} = 2^{(i + j) q (\frac{1}{p} - \frac{1}{q})} Y^q_{[2^{i-1}][2^{j-1}]}(f)_p, \quad (2.23)$$

Since

$$(i_r + j_r) \left(\frac{1}{2} - \frac{1}{\alpha} \right) + (i_s + j_s) \left(\frac{1}{2} - \frac{1}{\alpha'} \right) = [-(i_s - i_r) - (j_s - j_r)] \left(\frac{1}{2} - \frac{1}{\alpha} \right) \quad (2.24)$$

then from (2.22), using (2.23) and (2.24), we get

$$\Gamma_{rs} \ll 2^{-(i_s - i_r)(\frac{1}{2} - \frac{1}{\alpha})} 2^{-(j_s - j_r)(\frac{1}{2} - \frac{1}{\alpha})} H_{i_r j_r}^{\frac{1}{2}} H_{i_s j_s}^{\frac{1}{2}}. \tag{2.25}$$

If we apply the Hölder inequality (taking exponent α' with respect to the first term, and α to the second), then we can conclude in the same way that the following inequality holds

$$\Gamma_{rs} \ll 2^{-(i_r - i_s)(\frac{1}{2} - \frac{1}{\alpha})} 2^{-(j_r - j_s)(\frac{1}{2} - \frac{1}{\alpha})} H_{i_r j_r}^{\frac{1}{2}} H_{i_s j_s}^{\frac{1}{2}}. \tag{2.26}$$

From inequalities (2.25) and (2.26) it follows that

$$\Gamma_{rs} \ll 2^{-|i_s - i_r|(\frac{1}{2} - \frac{1}{\alpha})} 2^{-|j_s - j_r|(\frac{1}{2} - \frac{1}{\alpha})} H_{i_r j_r}^{\frac{1}{2}} H_{i_s j_s}^{\frac{1}{2}}. \tag{2.27}$$

Denote

$$a(i_s, i_r) = 2^{-|i_s - i_r|(\frac{1}{2} - \frac{1}{\alpha})}, \quad b(j_s, j_r) = 2^{-|j_s - j_r|(\frac{1}{2} - \frac{1}{\alpha})}, \tag{2.28}$$

$$Q = \prod_{\substack{r,s=1 \\ r<s}}^{m} \left\{ a(i_s, i_r) b(j_s, j_r) H_{i_r j_r}^{\frac{1}{2}} H_{i_s j_s}^{\frac{1}{2}} \right\}^{\frac{2}{m(m-1)}}. \tag{2.29}$$

From (2.17), based on (2.27), (2.28), (2.29), we get

$$A \ll \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} Q. \tag{2.30}$$

We will estimate the product Q. Using (2.13) we get

$$\prod_{\substack{r,s=1 \\ r<s}}^{m} \left\{ H_{i_r j_r}^{\frac{1}{2}} H_{i_s j_s}^{\frac{1}{2}} \right\}^{\frac{1}{m-1}} = \prod_{k=1}^{m} H_{i_k j_k}^{\frac{1}{2}}. \tag{2.31}$$

Now from (2.29), in view of (2.31), we get

$$Q = \prod_{k=1}^{m} H_{i_k j_k}^{\frac{1}{m}} \prod_{\substack{r,s=1 \\ r<s}}^{m} \{a(i_s, i_r)\}^{\frac{2}{m(m-1)}} \prod_{\substack{r,s=1 \\ r<s}}^{m} \{b(j_s, j_r)\}^{\frac{2}{m(m-1)}}. \tag{2.32}$$

Since $a(i_s, i_r) = a(i_r, i_s)$ and $a(i_r, i_r) = 1$, then

$$\prod_{\substack{r,s=1 \\ r<s}}^{m} a(i_r, i_s) = \prod_{r=1}^{m} \prod_{s=1}^{m} a^{\frac{1}{2}}(i_r, i_s). \tag{2.33}$$

Also it is

$$\prod_{\substack{r,s=1 \\ r<s}}^{m} b(j_r, j_s) = \prod_{r=1}^{m} \prod_{s=1}^{m} b^{\frac{1}{2}}(j_r, j_s). \tag{2.34}$$

The product Q, in view of (2.32), (2.33) and (2.34), can be written as

$$Q = \prod_{r=1}^{m} H_{i_r j_r}^{\frac{1}{2}} \Big\{ \prod_{s=1}^{m} [a(i_r, i_s) b(j_r, j_s)]^{\frac{1}{m-1}} \Big\}^{\frac{1}{m}}. \tag{2.35}$$

Now from (2.30) based on (2.35) it follows that

$$A \ll \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} \prod_{r=1}^{m} H_{i_r j_r}^{\frac{1}{2}} \Big\{ \prod_{s=1}^{m} [a(i_r, i_s) b(j_r, j_s)]^{\frac{1}{m-1}} \Big\}^{\frac{1}{m}}. \tag{2.36}$$

The terms in the sum (2.36) are products of m factors $L_r^{1/m}$ where

$$L_r = H_{i_r j_r}^{\frac{1}{m}} \prod_{s=1}^{m} [a(i_r, i_s) b(j_r, j_s)]^{\frac{1}{m-1}}, \qquad Q = \prod_{r=1}^{m} L_r^{\frac{1}{m}}.$$

Therefore we can apply Hölder's inequality with the power $\frac{1}{m}$ and get the inequality

$$A \ll \prod_{r=1}^{m} \Big\{ \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} H_{i_r j_r} \prod_{s=1}^{m} [a(i_r, i_s) b(j_r, j_s)]^{\frac{1}{m-1}} \Big\}^{\frac{1}{m}}$$

which can be written as

$$A \ll \prod_{r=1}^{m} \Big\{ \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} H_{i_r j_r} \prod_{s=1}^{m} [a(i_r, i_s)]^{\frac{1}{m-1}} \prod_{t=1}^{m} [b(j_r, j_t)]^{\frac{1}{m-1}} \Big\}^{\frac{1}{m}}. \tag{2.37}$$

Denote

$$M_r = \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} H_{i_r j_r} \prod_{s=1}^{m} [a(i_r, i_s)]^{\frac{1}{m-1}} \prod_{t=1}^{m} [b(j_r, j_t)]^{\frac{1}{m-1}}, \tag{2.38}$$

$r = 1, 2, \ldots, m$. Since $i_r = 0, 1, \ldots, \mu$, $j_r = N+1, N+2, \ldots, P$ for every $r = 1, 2, \ldots, m$, it is

$$M_1 = M_2 = \cdots = M_m = M. \tag{2.39}$$

For example, we will calculate $M = M_1$. Since $a(i_1, i_1) = 1$, $b(j_1, j_1) = 1$, it is

$$M = M_1 = \sum_{i_1=0}^{\mu} \cdots \sum_{i_m=0}^{\mu} \sum_{j_1=N+1}^{P} \cdots \sum_{j_m=N+1}^{P} H_{i_1 j_1} \prod_{s=2}^{m} [a(i_1, i_s)]^{\frac{1}{m-1}} \prod_{t=2}^{m} [b(j_1, j_t)]^{\frac{1}{m-1}}, \tag{2.40}$$

We have

$$M = \sum_{i_1=0}^{\mu} \sum_{j_1=N+1}^{P} H_{i_1 j_1} \sum_{i_2=0}^{\mu} [a(i_1, i_2)]^{\frac{1}{m-1}} \cdots \sum_{i_m=0}^{\mu} [a(i_1, i_m)]^{\frac{1}{m-1}} \times$$
$$\times \sum_{j_2=N+1}^{P} [b(j_1, j_2)]^{\frac{1}{m-1}} \cdots \sum_{j_m=N+1}^{P} [b(j_1, j_m)]^{\frac{1}{m-1}}. \tag{2.41}$$

For the sums $\sum a$ and $\sum b$ from equalities (2.41), based on (2.27) and (2.28), it holds

$$\sum_{i_r=0}^{\mu} [a(i_1, i_r)]^{\frac{1}{m-1}} \leqslant C(p, q), \qquad \sum_{j_t=N+1}^{P} [b(j_1, j_t)]^{\frac{1}{m-1}} \leqslant C(p, q) \tag{2.42}$$

for $r, t = 2, 3, \ldots, m$. The constant C depends only on p and q.

From (2.41), based on (2.39) and (2.40), we get

$$M \ll \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} H_{ij}. \tag{2.43}$$

Now from (2.37), in view of (2.38), (2.39), (2.43) and (2.23), it follows that

$$A \ll \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} 2^{(i+j)q(\frac{1}{p}-\frac{1}{q})} Y^{q}_{[2^{i-1}][2^{j-1}]}(f)_p. \tag{2.44}$$

Using (2.5), (2.6) and (2.44) we deduce that

$$\|G_{\mu P} - G_{\mu N}\|_q \ll \Big\{ \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} 2^{(i+j)q(\frac{1}{p}-\frac{1}{q})} Y^{q}_{[2^{i-1}][2^{j-1}]}(f)_p \Big\}^{\frac{1}{q}}. \tag{2.45}$$

Based on the inequality (2.45) and the condition (2.1) we deduce that the sequence $G_{\mu N}$, $N = 0, 1, 2, \dots$ is a Cauchy sequence in the space L_q. Since L_q is complete, then there exists a function $h(x, y) \in L_q$ such that $\lim_{N\to\infty} \|G_{\mu N} - h(x, y)\|_q = 0$, i.e.

$$h(x, y) \overset{(q)}{=} \sum_{i=0}^{\mu} \sum_{j=0}^{\infty} g_{ij}. \tag{2.46}$$

In view of the equality (1.37), Lemma 2, and equality (2.46) we deduce that (see [2, 1.3.9]) it holds that

$$K_{2^{\mu+1}\infty} f \overset{(q)}{=} \sum_{i=0}^{\mu} \sum_{j=0}^{\infty} g_{ij}. \tag{2.47}$$

Using the equality (2.47) in the next step we will prove equality (2.2).

For the sequence $G_{\mu N}$ the following equality holds

$$G^{(r_1,r_2)}_{\mu N} = \sum_{i=0}^{\mu} \sum_{j=0}^{N} g^{(r_1,r_2)}_{ij}. \tag{2.48}$$

It follows that

$$\|G^{(r_1,r_2)}_{\mu P} - G^{(r_1,r_2)}_{\mu N}\|_q = \Big\| \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} g^{(r_1,r_2)}_{ij} \Big\|_q. \tag{2.49}$$

Denote

$$B = \Big\| \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} g^{(r_1,r_2)}_{ij} \Big\|_q^q = \Big\| \sum_{i=0}^{\mu} \sum_{j=N+1}^{P} \varphi_{ij} \Big\|_q^q \tag{2.50}$$

where $\varphi_{ij} = g^{(r_1,r_2)}_{ij}$. The function φ_{ij} is an entire function of the same type as g_{ij}. Therefore, we can use the same method we used to estimate quantity A.

Denote

$$\delta_{ij}(\varphi) = |\varphi_{ij}|^{\frac{q}{m}}. \tag{2.51}$$

The corresponding quantity Γ is

$$\Gamma_{rs}(B) = \iint [\delta_{i_r j_r}(\varphi)\delta_{i_s j_s}(\varphi)]^{\frac{m}{2}}\, dx\, dy = \iint |g^{(r_1,r_2)}_{i_r j_r}(\varphi)|^{\frac{q}{2}} |g^{(r_1,r_2)}_{i_s j_s}(\varphi)|^{\frac{q}{2}}\, dx\, dy. \tag{2.52}$$

Therefore we get (see (2.18))

$$\Gamma_{rs}(B) \leqslant (\|\varphi_{i_r j_r}\|_{\alpha q/2})^{\frac{q}{2}} (\|\varphi_{i_s j_s}\|_{\alpha' q/2})^{\frac{q}{2}}. \tag{2.53}$$

By applying another metrics inequality of S.M. Nikol'skiĭ [2, 3.3.5] we get

$$(\|\varphi_{i_rj_r}\|_{\alpha q/2})^{\frac{q}{2}} \ll 2^{(i_r+j_r)(\frac{q}{2p}-\frac{1}{\alpha})}(\|\varphi_{i_rj_r}\|_p)^{\frac{q}{2}}, \quad (2.54)$$

$$(\|\varphi_{i_sj_s}\|_{\alpha' q/2})^{\frac{q}{2}} \ll 2^{(i_s+j_s)(\frac{q}{2p}-\frac{1}{\alpha'})}(\|\varphi_{i_sj_s}\|_p)^{\frac{q}{2}}, \quad (2.55)$$

Applying an inequality of Bernstein type [2, 3.2.2] we get

$$\|\varphi_{ij}\|_p = \|g_{ij}^{(r_1,r_2)}\|_p \ll 2^{ir_1+jr_2}\|g_{ij}\|_p. \quad (2.56)$$

From (2.54) based on (2.56) we get

$$(\|\varphi_{i_rj_r}\|_{\alpha q/2})^{\frac{q}{2}} \ll 2^{i_r(r_1\frac{q}{2}+\frac{q}{2p}-\frac{1}{\alpha})}2^{j_r(r_2\frac{q}{2}+\frac{q}{2p}-\frac{1}{\alpha})}Y^{\frac{q}{2}}_{[2^{i_r-1}][2^{j_r-1}]}(f)_p. \quad (2.57)$$

In the same way we deduce

$$(\|\varphi_{i_sj_s}\|_{\alpha' q/2})^{\frac{q}{2}} \ll 2^{i_s(r_1\frac{q}{2}+\frac{q}{2p}-\frac{1}{\alpha'})}2^{j_s(r_2\frac{q}{2}+\frac{q}{2p}-\frac{1}{\alpha'})}Y^{\frac{q}{2}}_{[2^{i_s-1}][2^{j_s-1}]}(f)_p. \quad (2.58)$$

The equality $\sigma_i = r_i + \frac{1}{p} - \frac{1}{q}$ is equivalent to the equality

$$\frac{qr_i}{2} + \frac{q}{2p} - \frac{1}{\beta} = \frac{q\sigma_i}{2} + \frac{1}{2} - \frac{1}{\beta}, \quad \beta \in \{\alpha, \alpha'\}. \quad (2.59)$$

Therefore the inequalities (2.57) and (2.58) can be written as

$$(\|\varphi_{i_rj_r}\|_{\alpha q/2})^{\frac{q}{2}} \ll 2^{i_r(\frac{q}{2}\sigma_1+\frac{1}{2}-\frac{1}{\alpha})}2^{j_r(\frac{q}{2}\sigma_2+\frac{1}{2}-\frac{1}{\alpha})}Y^{\frac{q}{2}}_{[2^{i_r-1}][2^{j_r-1}]}(f)_p, \quad (2.60)$$

$$(\|\varphi_{i_sj_s}\|_{\alpha' q/2})^{\frac{q}{2}} \ll 2^{i_s(\frac{q}{2}\sigma_1+\frac{1}{2}-\frac{1}{\alpha'})}2^{j_s(\frac{q}{2}\sigma_2+\frac{1}{2}-\frac{1}{\alpha'})}Y^{\frac{q}{2}}_{[2^{i_s-1}][2^{j_s-1}]}(f)_p. \quad (2.61)$$

From (2.53), based on (2.60) and (2.61), it follows that

$$\Gamma_{rs} \ll 2^{(i_r+j_r)(\frac{1}{2}-\frac{1}{\alpha})}2^{(i_s+j_s)(\frac{1}{2}-\frac{1}{\alpha'})}\times$$
$$\times\{2^{i_rq\sigma_1}2^{j_rq\sigma_2}Y^q_{[2^{i_r-1}][2^{j_r-1}]}(f)_p 2^{i_sq\sigma_1}2^{j_sq\sigma_2}Y^q_{[2^{i_s-1}][2^{j_s-1}]}(f)_p\}^{\frac{1}{2}}. \quad (2.62)$$

If we denote

$$H_{ij}(B) = 2^{iq\sigma_1}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p \quad (2.63)$$

then it follows from (2.62) that

$$\Gamma_{rs}(B) \ll 2^{(i_r+j_r)(\frac{1}{2}-\frac{1}{\alpha})}2^{(i_s+j_s)(\frac{1}{2}-\frac{1}{\alpha'})}H^{\frac{1}{2}}_{i_rj_r}(B)H^{\frac{1}{2}}_{i_sj_s}(B).$$

Expressing α' using α and applying the same method as for $\Gamma_{rs}(A)$, we deduce

$$\Gamma_{rs}(B) \ll 2^{-|i_s-i_r|(\frac{1}{2}-\frac{1}{\alpha})}2^{-|j_s-j_r|(\frac{1}{2}-\frac{1}{\alpha})}H^{\frac{1}{2}}_{i_rj_r}(B)H^{\frac{1}{2}}_{i_sj_s}(B). \quad (2.64)$$

Repeating the same method for which from (2.27) we obtained (2.44), we deduce that the following holds

$$B \ll \sum_{j=0}^{\mu}\sum_{j=N+1}^{P} 2^{iq\sigma_1}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p. \quad (2.65)$$

Based on (2.49), (2.50) and (2.65) we deduce that

$$\|G_{\mu P}^{(r_1,r_2)} - G_{\mu N}^{(r_1,r_2)}\|_q \ll \Big\{\sum_{j=0}^{\mu}\sum_{j=N+1}^{P} 2^{iq\sigma_1} 2^{jq\sigma_2} Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{\frac{1}{q}}. \tag{2.66}$$

The condition (2.1) of the theorem and the inequality (2.66) mean that the sequence $G_{\mu N}^{(r_1,r_2)}$, $N = 0, 1, 2, \dots$ is a Cauchy sequence in L_q. Since L_q is complete, there exists a function $\psi(x,y) \in L_q$ such that $\lim_{n\to\infty} \|G_{\mu N}^{(r_1,r_2)} - \psi(x,y)\|_q = 0$, i.e.

$$\psi(x,y) \overset{(q)}{=} \sum_{i=0}^{\mu}\sum_{j=0}^{\infty} g_{ij}^{(r_1,r_2)}. \tag{2.67}$$

In view of the equalities (2.47) and (2.67) we deduce that the following holds (see [2, 4.4.7])

$$(K_{2^\mu\infty} f)^{(r_1,r_2)} = \psi(x,y). \tag{2.68}$$

Hence, the equality (2.2) has been proved.

The equality (2.3) follows from the equality (2.2) when x and y are swapped.

The equality (2.4) can essentially be proved in the same way. We use Lemma 1, i.e.. the equality (1.36). In the following step we will prove that the equality (1.36) holds in L_q. The equality (1.36) can be written as

$$f - G_{MN} = \sum_{i=M+1}^{\infty}\sum_{j=0}^{N} g_{ij} + \sum_{i=0}^{M}\sum_{j=N+1}^{\infty} g_{ij} + \sum_{i=M+1}^{\infty}\sum_{j=N+1}^{\infty} g_{ij}$$

and then

$$\begin{aligned} \|f - G_{MN}\|_q &\le \Big\|\sum_{i=M+1}^{\infty}\sum_{j=0}^{N} g_{ij}\Big\|_q + \Big\|\sum_{i=0}^{M}\sum_{j=N+1}^{\infty} g_{ij}\Big\|_q + \Big\|\sum_{i=M+1}^{\infty}\sum_{j=N+1}^{\infty} g_{ij}\Big\|_q \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned} \tag{2.69}$$

For the sums Σ_i it holds that $\Sigma_i \ll \Sigma_i(\sigma)$, $i = 1, 2, 3$, where $\Sigma_i(\sigma)$ is the corresponding remainder of the series (2.1) for $r_1 = 0$, $r_2 = 0$ (the sum of the series (2.1) can be expressed in terms of one series, see Remark 3 from Section 1). Therefore $\|f - G_{MN}\|_q \to 0$ as $M, N \to \infty$. This means that the equality (1.36) holds in L_q, i.e. the following holds

$$f(x,y) \overset{(q)}{=} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{ij}. \tag{2.70}$$

In view of the equality (2.70) and the condition (2.1), and using the method which established the equality (2.2), we deduce that the sequence $\xi_M = \sum_{i=0}^{M}\sum_{j=0}^{\infty} g_{ij}^{(r_1)}$ converges in L_q and that (see [2, 4.4.5 and 4.4.7])

$$f^{(r_1)} \overset{(q)}{=} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{ij}^{(r_1)}. \tag{2.71}$$

Again, in the same way, we deduce that the following sequence converges in L_q

$$\eta_N = \sum_{i=0}^{\infty}\sum_{j=0}^{N} [g_{ij}^{(r_1)}]^{(r_2)} = \sum_{i=0}^{\infty}\sum_{j=0}^{N} g_{ij}^{(r_1,r_2)}$$

and that based on (2.71) the equality (2.4) holds. Theorem 2.1 has been proved. ■

Corollary 1. In view of (2.4), (2.50) and (2.65) we conclude that under the conditions of Theorem 2.1 the following inequality holds

$$\|f^{(r_1,r_2)}\|_q \ll \left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)^{\sigma_1 q-1}(j+1)^{\sigma_2 q-1}Y_{ij}^q(f)_p\right\}^{\frac{1}{q}}. \tag{2.72}$$

If $q=p$ then $\sigma_i=r_i$, $i=1,2$, $1\le p<\infty$ and we get the corresponding inequality for the norm in the L_p space.

Corollary 2. In the same way for $q=p$ the condition (2.1) becomes weaker and the equalities (2.2), (2.3) and (2.4) hold in L_p.

3. The consequences of the theorem of representation

Apart from the above given corollaries of the theorem of representation we also give the following important corollary as a theorem.

Theorem 3.1. *Let the conditions of Theorem 2.1 hold for a function $f(x,y)$. Then*

$$(K_{2^{\mu+1}\infty}f)^{(r_1,r_2)} = K_{2^{\mu+1}\infty}f^{(r_1,r_2)}, \quad (K_{\infty 2^{\nu+1}}f)^{(r_1,r_2)} = K_{\infty 2^{\nu+1}}f^{(r_1,r_2)}, \tag{3.1}$$

$$(\chi_{2^\mu 2^\nu}f)^{(r_1,r_2)} = \chi_{2^\mu 2^\nu}f^{(r_1,r_2)}, \quad \mu,\nu=1,2,\dots \tag{3.2}$$

Proof. Denote $h(x,y)=f^{(r_1,r_2)}$, $h\in L_q$. Then based on Lemma 1 (the equality (1.36)), the following equality holds

$$f^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{ij}f^{(r_1,r_2)}, \tag{3.3}$$

and in view of Lemma 2, the equalities (1.37) and (1.38), and Theorem 2.1 the following equalities hold

$$K_{2^{\mu+1}\infty}f^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{\mu}\sum_{j=0}^{\infty} g_{ij}f^{(r_1,r_2)}, \quad \mu=1,2,\dots, \tag{3.4}$$

$$K_{\infty 2^{\nu+1}}f^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{\infty}\sum_{j=0}^{\nu} g_{ij}f^{(r_1,r_2)}, \quad \nu=1,2,\dots \tag{3.5}$$

Since $g_{ij}f^{(r_1,r_2)}$ is expressed by $Kf^{(r_1,r_2)}$ (see equalities (1.34) and (1.11)), and since for the entire functions $K_{\mu\nu}f$ the equality $(K_{\mu\nu}f)^{(r_1,r_2)}=K_{\mu\nu}f^{(r_1,r_2)}$ holds (see Lemma 1.4 in [7]), it is

$$g_{ij}f^{(r_1,r_2)} = (g_{ij}f)^{(r_1,r_2)} = g_{ij}^{(r_1,r_2)}f. \tag{3.6}$$

Now from equalities (2.2), (2.3), (2.4) and equalities (3.3), (3.4), (3.5) and in view of the equality (3.6), equalities (3.1) follow. The equality (3.2) is a consequence of the equalities (3.1) and (1.7). Theorem 3.1 has been proved.

Remark 4. The theorems we have proved enable us to prove the inequalities by which the best approximation by an angle and the mixed modulus of smoothness in the norm of the space L_q are estimated by the best approximation by an angle in the norm of the space L_p, $1 \leq p \leq q < \infty$ (the converse theorem of approximation by an angle). The results can be used further to examine the space (the classes) of functions which are defined by the mixed modulus of smoothness (spaces SH of Nikol'skiĭ type and spaces SB of Besov type).

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