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A NOTE ON THE EXPONENTIAL CONVERGENCE RATE FOR PRODUCTS OF SUMS

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Abstract. In this paper, we establish an exponential convergence theorem for products of sums of independent identically distributed positive random variables.

1. Introduction

Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed (i.i.d.) positive random variables and define the partial sums $S_n = \sum_{i=1}^n X_i$ and the product of sums $T_n = \sum_{k=1}^n S_k$ for $n \geq 1$. In the past decade, there have been many studies about the asymptotic properties for the products of partial sums T_n.

The study for the product of partial sums was initiated by Arnold and Villaseñr [1] who considered the limiting properties of the sums of records. In their paper, the authors obtained the following version of the central limit theorem (CLT) for a sequence of i.i.d. exponential r.v.'s $(X_n)_{n\geq 1}$ with the mean equal to one:

$$\frac{\sum_{k=1}^n \log S_k - n\log n + n}{\sqrt{2n}} \xrightarrow{\mathcal{L}} N, \text{ as } n \to \infty,$$

where N is a standard normal r.v. Here we think that it is interesting to recall that the products of i.i.d. positive, square integrable random variables are asymptotically log-normal. This fact is an immediate consequence of the classical CLT. In [11], Rempala and Wesolowski have noted that this limit behavior of the product of partial sums has a universal character and holds for any sequence of square integrable, positive i.i.d. random variables. Namely, they have proved the following result.

THEOREM RW. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu$, $VarX_1 = \sigma^2 > 0$ and the coefficient of variation*

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$\gamma = \sigma/\mu$. *Then*

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\,\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{\mathcal{L}} e^{\sqrt{2}N}.$$

Recently, Gonchigdanzan and Rempala [3] obtained the first almost sure central limit theorem (ASCLT) for the product of the partial sums of i.i.d. positive random variables as follows.

Theorem GR. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu > 0$, $VarX_1 = \sigma^2$. Denote by $\gamma = \sigma/\mu$ the coefficient of variation. Then for any real x,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \leq x\right) = F(x), \quad a.s.$$

where $F(\cdot)$ is the distribution function of the r.v. $e^{\sqrt{2}N}$.

For the further discussions of the CLT, the author refers to [5, 10]. In [4], Huang and Zhang obtained the invariance principle of the product of sums of random variables. It is perhaps worth to notice that by the strong law of large numbers and the property of the geometric mean it follows directly that

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!}\right)^{1/n} \xrightarrow{a.s.} \mu \tag{1.1}$$

if only existence of the first moment is assumed. Very recently the first author [6, 7] obtained CLT and ASCLT for the product of some general partial sums.

The studies on the products of partial sums are usually concentrated on the classic limiting theory, such as, CLT, ASCLT, LIL. The main purpose of this short note is to establish a exponential convergence theorem for the product of sums of i.i.d. positive random variables.

2. Main results

2.1. A moderate deviation principle for the weighted sums

In this subsection, we establish a moderate deviation principle for the weighted sums which will play a key role in proving our main result.

Lemma 2.1. *Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. positive random variables with $\mathbb{E}Y_1 = 0$ and $\mathbb{E}(Y_1^2) = 1$. Assume that the sequence of positive numbers (b_n) is the moderate deviation scale satisfying*

$$b_n \to \infty, \qquad \frac{b_n \log n}{\sqrt{n}} \to 0, \quad \text{as} \quad n \to \infty.$$

If we suppose that the following exponential integrability condition holds: there exists a positive number δ such that

$$\mathbb{E}\exp\left(\delta|Y_1|\right) < \infty, \tag{2.1}$$

then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{1}{b_n\sqrt{2n}} \left| \sum_{i=1}^{n} b_{i,n} Y_i \right| \geq r \right) = -\frac{r^2}{2},$$

where $b_{i,n} = \sum_{k=i}^{n} \frac{1}{k}$, $1 \leq i \leq n$.

Proof. For any $\lambda \in \mathbb{R}$, considering the following logarithmic moment generating function

$$\Lambda_n(\lambda) := \log \mathbb{E} \exp\left(\frac{\lambda}{b_n\sqrt{2n}} \sum_{i=1}^{n} b_{i,n} Y_i \right)$$

by the Gärtner-Ellis theorem [2, 12], we need to calculate the following limit,

$$\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{b_n^2} \Lambda_n(b_n^2 \lambda).$$

From the independence, Taylor formula and the condition (2.1), for all n large enough, we obtain

$$\begin{aligned}
\frac{1}{b_n^2}\Lambda_n(b_n^2\lambda) =& \frac{1}{b_n^2} \log \mathbb{E} \exp\left(\frac{\lambda b_n}{\sqrt{2n}} \sum_{i=1}^{n} b_{i,n} Y_i \right) \\
=& \frac{1}{b_n^2} \sum_{i=1}^{n} \log \mathbb{E} \exp\left(\frac{\lambda b_n}{\sqrt{2n}} b_{i,n} Y_i \right) \\
=& \frac{1}{b_n^2} \sum_{i=1}^{n} \log\left(1 + \frac{\lambda b_n}{\sqrt{2n}} b_{i,n} \mathbb{E} Y_i + \frac{\lambda^2 b_n^2}{4n} b_{i,n}^2 \mathbb{E} Y_i^2 + o\left(\frac{b_n^2 b_{i,n}^2}{n} \right) \right).
\end{aligned}$$

Since $\mathbb{E}Y_i = 0$, $\mathbb{E}Y_i^2 = 1$, and the following fact

$$\sum_{i=1}^{n} b_{i,n}^2 = b_{1,n} + 2\sum_{k=2}^{n}\sum_{i=1}^{k-1} \frac{1}{k} = b_{1,n} + 2\sum_{k=2}^{n} \frac{k-1}{k} = 2n - b_{1,n} = 2n - \sum_{i=1}^{n} \frac{1}{i},$$

we have

$$\lim_{n\to\infty} \frac{1}{b_n^2} \Lambda_n(b_n^2\lambda) = \lim_{n\to\infty} \frac{\lambda^2}{4n} \sum_{i=1}^{n} b_{i,n}^2 = \lim_{n\to\infty} \frac{\lambda^2}{4n} \left(2n - \sum_{i=1}^{n} \frac{1}{i} \right) = \frac{\lambda^2}{2}.$$

By the Gärtner-Ellis theorem, the desired result can be obtained. ■

2.2. Moderate deviation principle for the product of sums

THEOREM 2.2. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive random variables. Denote $\mu = \mathbb{E}(X_1) > 0$, the coefficient of variation $\gamma = \sigma/\mu$, where $\sigma^2 = Var(X_1)$, and $S_k = X_1 + \cdots + X_k$, $k = 1, 2, \ldots$ In addition, assume that the sequence of positive numbers (b_n) is the moderate deviation scale satisfying*

$$b_n \to \infty, \qquad \frac{b_n \log n}{\sqrt{n}} \to 0, \qquad \text{as} \quad n \to \infty.$$

If there exists a sequence of positive real numbers (α_n) *such that*

$$\alpha_n \to \infty, \qquad \frac{\alpha_n}{b_n\sqrt{n}} \to \infty, \qquad \frac{\alpha_n}{n} \to 0, \qquad \frac{\sqrt{n}\log n}{\alpha_n b_n} \to 0 \tag{2.2}$$

and for all $t > 0$

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{1}{b_n\sqrt{n}} \left| \sum_{i=1}^{\alpha_n} \log \frac{S_k}{k} \right| \geq t \right) = -\infty, \tag{2.3}$$

then we have for any $r \geq 1$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\left(\frac{\prod_{k=1}^n S_k}{n!\mu^n} \right)^{\frac{1}{\gamma b_n \sqrt{2n}}} \geq r \right) = -\frac{(\log r)^2}{2}; \tag{2.4}$$

and for any $0 < r < 1$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\left(\frac{\prod_{k=1}^n S_k}{n!\mu^n} \right)^{\frac{1}{\gamma b_n \sqrt{2n}}} \leq r \right) = -\frac{(\log r)^2}{2}. \tag{2.5}$$

Proof. Without loss of generality, let $\mu = 1, \sigma^2 = 1$, then $\gamma = 1$. Let $C_k = S_k/k$, $k = 1, 2\cdots$. For any $r > 0$, $0 < \varepsilon < 1/2$, it follows that

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{b_n\sqrt{2n}} \sum_{k=1}^n \log(C_k) \geq r \right) \\
&\qquad = \mathbb{P}\left(\frac{1}{b_n\sqrt{2n}} \sum_{k=1}^n \log(C_k) \geq r, \max_{\alpha_n \leq k \leq n} |C_k - 1| \geq \varepsilon \right) \\
&\qquad\quad + \mathbb{P}\left(\frac{1}{b_n\sqrt{2n}} \sum_{k=1}^n \log(C_k) \geq r, \max_{\alpha_n \leq k \leq n} |C_k - 1| < \varepsilon \right) \\
&\qquad =: A_n + B_n.
\end{aligned} \tag{2.6}$$

By the comparison inequality in [8, Corollary 4], for any $\varepsilon > 0$, it is obvious that

$$\begin{aligned}
\mathbb{P}\left(\max_{\alpha_n \leq k \leq n} \left| \frac{S_k}{k} - 1 \right| \geq \varepsilon \right) &\leq \mathbb{P}\left(\max_{\alpha_n \leq k \leq n} \left| \sum_{i=1}^k (X_i - 1) \right| \geq \alpha_n \varepsilon \right) \\
&\leq c\mathbb{P}\left(\left| \sum_{i=1}^n (X_i - 1) \right| \geq \alpha_n \varepsilon / c \right) \\
&= c\mathbb{P}\left(\frac{1}{b_n\sqrt{n}} \left| \sum_{i=1}^n (X_i - 1) \right| \geq \frac{\alpha_n \varepsilon}{b_n \sqrt{n} c} \right),
\end{aligned} \tag{2.7}$$

where $c > 0$ is a constant. From the assumption $\frac{\alpha_n}{b_n\sqrt{n}} \to \infty$ $(n \to \infty)$ and the classic moderate deviation principle (cf. [2, 12]), it follows that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\max_{\alpha_n \leq k \leq n} \left| \frac{S_k}{k} - 1 \right| \geq \varepsilon \right) = -\infty. \tag{2.8}$$

From this we know that the term A_n in (2.6) is negligible in the sense of the moderate deviation principle. To estimate the term B_n, we will expand the logarithm: $\log(1+x) = x - \frac{x^2}{(1+\theta x)^2}$, where $\theta \in (0,1)$ depends on $x \in (-1/2, 1/2)$. Let E_n denote the event $\{\max_{\alpha_n \le k \le n} |C_k - 1| < \varepsilon\}$, thus

$$\begin{aligned}
B_n &= \mathbb{P}\Big(\frac{1}{b_n\sqrt{2n}}\sum_{k=1}^{n}\log(C_k) \ge r, \max_{\alpha_n \le k \le n}|C_k - 1| < \varepsilon\Big)\\
&= \mathbb{P}\Big(\frac{1}{b_n\sqrt{2n}}\Big(\sum_{k=1}^{\alpha_n}\log(C_k) + \sum_{k=\alpha_n+1}^{n}(C_k - 1) - \sum_{k=\alpha_n+1}^{n}\frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2}\Big) \ge r, E_n\Big)\\
&= \mathbb{P}\Big(\frac{1}{b_n\sqrt{2n}}\Big(\sum_{k=1}^{\alpha_n}\log(C_k) + \sum_{k=\alpha_n+1}^{n}(C_k - 1) - \Big[\sum_{k=\alpha_n+1}^{n}\frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2}\Big]I_{E_n}\Big) \ge r\Big)\\
&\quad - \mathbb{P}\Big(\frac{1}{b_n\sqrt{2n}}\Big(\sum_{k=1}^{\alpha_n}\log\Big(C_k\Big) + \sum_{k=\alpha_n+1}^{n}(C_k - 1)\Big) \ge r, E_n^c\Big)\\
&=: D_n - F_n.
\end{aligned}$$

By the same reason as for the term A_n, we know that the term F_n is also negligible in the sense of the moderate deviation principle. Furthermore, by the condition (2.3), the term $\frac{1}{b_n\sqrt{2n}}\big|\sum_{k=1}^{\alpha_n}\log\big(C_k\big)\big|$ is negligible with respect to the moderate deviation principle. Similarly as for (2.7), we know that $\frac{1}{b_n\sqrt{2n}}\big|\sum_{k=1}^{\alpha_n}(C_k-1)\big|$ can be neglected in the sense of the moderate deviation principle, so, from Lemma 2.1, we have

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}\left(\frac{1}{b_n\sqrt{2n}}\sum_{k=\alpha_n+1}^{n}(C_k-1) \ge r\right) = -\frac{r^2}{2}.$$

Next if we can prove the claim: for any $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}\left(\frac{1}{b_n\sqrt{n}}\left[\sum_{k=\alpha_n+1}^{n}\frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2}\right]I_{E_n} \ge \varepsilon\right) = -\infty, \tag{2.9}$$

then the desired results can be obtained. Note that for $|x| < 1/2$ and any $\theta_k \in (0,1)$, it follows that $\frac{x^2}{(1+\theta_k x)^2} \le 4x^2$. Therefore we have

$$\mathbb{P}\left(\frac{1}{b_n\sqrt{n}}\left[\sum_{k=\alpha_n+1}^{n}\frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2}\right]I_{E_n} \ge 4\varepsilon\right) \le \mathbb{P}\left(\frac{1}{b_n\sqrt{n}}\sum_{k=\alpha_n+1}^{n}(C_k-1)^2 \ge \varepsilon\right).$$

By Theorem 15 and Lemma 5 in [9, Chapter III], for all n sufficiently large, it follows that

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{b_n\sqrt{n}}\sum_{k=\alpha_n+1}^{n}(C_k-1)^2 \ge \varepsilon\right) &\le \sum_{k=\alpha_n+1}^{n}\mathbb{P}\left(\left|\frac{1}{k}\sum_{i=1}^{k}X_i - 1\right| \ge \sqrt{\frac{\varepsilon b_n}{\sqrt{n}}}\right)\\
&\le 2\sum_{k=\alpha_n}^{n}\exp\left(-c\frac{kb_n}{\sqrt{n}}\right) \le \left(1 - e^{-c\frac{b_n}{\sqrt{n}}}\right)^{-1}\exp\left\{-c\frac{\alpha_n b_n}{\sqrt{n}}\right\}\\
&\le \frac{4}{c\frac{b_n}{\sqrt{n}} - \frac{1}{2}\big(c\frac{b_n}{\sqrt{n}}\big)^2}\exp\left\{-c\frac{\alpha_n b_n}{\sqrt{n}}\right\} \le \frac{8n}{cb_n}\exp\left\{-c\frac{\alpha_n b_n}{\sqrt{n}}\right\},
\end{aligned}$$

where c is a positive constant. Therefore we have

$$\frac{1}{b_n^2}\log\mathbb{P}\bigg(\frac{1}{b_n\sqrt{n}}\sum_{k=\alpha_n+1}^{n}(C_k-1)^2\geq\varepsilon\bigg)\leq -c\frac{\alpha_n}{b_n\sqrt{n}}+\frac{1}{b_n^2}\log\bigg(\frac{4n}{cb_n}\bigg)\to-\infty.$$

Thus the claim (2.9) holds. ∎

REMARK 2.3. If the sequence (b_n) satisfies $\frac{b_n^2}{\log n}\to\infty$, then there exists affirmatively a sequence (α_n) with the properties (2.2).

REMARK 2.4. By the Jensen's inequality, we have

$$\frac{1}{\alpha_n}\sum_{k=1}^{\alpha_n}\log\frac{S_k}{k}\leq\log\bigg(\frac{1}{\alpha_n}\sum_{k=1}^{\alpha_n}\frac{S_k}{k}\bigg),\qquad \log\frac{S_k}{k}\geq\frac{1}{k}\sum_{i=1}^{k}\log X_i,\quad a.e.$$

Hence, in order to make the condition (2.3) hold, it is sufficient to show the following relations: for any $t>0$,

$$\frac{1}{b_n^2}\log\mathbb{P}\bigg(\frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n}\frac{S_k}{k}\geq e^{\frac{b_n\sqrt{n}t}{\alpha_n}}\bigg)=\frac{1}{b_n^2}\log\mathbb{P}\bigg(\frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n}b_{i,\alpha_n}X_i\geq e^{\frac{b_n\sqrt{n}t}{\alpha_n}}\bigg)\to-\infty,$$

and

$$\frac{1}{b_n^2}\log\mathbb{P}\bigg(-\sum_{k=1}^{\alpha_n}\frac{1}{k}\sum_{i=1}^{k}(\log X_i)\geq tb_n\sqrt{n}\bigg)$$
$$=\frac{1}{b_n^2}\log\mathbb{P}\bigg(-\sum_{i=1}^{\alpha_n}b_{i,\alpha_n}(\log X_i)\geq tb_n\sqrt{n}\bigg)\to-\infty,$$

where $b_{i,\alpha_n}=\sum_{k=i}^{\alpha_n}\frac{1}{k}$.

EXAMPLE 2.5. (Bounded random variables) Let (X_n) be a sequence of i.i.d. bounded random variables with $a<X_1<b$, where $0<a<b<\infty$. If $\frac{\log n}{b_n^2}\to 0$ then the assumption (2.3) holds.

Noting $\mathbb{E}\log X_1\leq\log\mathbb{E}X_1=0$, for any $t>0$, let $t_n=tb_n\sqrt{n}/(4|\mathbb{E}\log X_1|)$, then by the Hoeffding's inequality and the facts $\sum_{i=1}^{t_n}b_{i,t_n}=t_n,\sum_{i=1}^{t_n}b_{i,t_n}^2\sim 2t_n$, there exists a constant $c>0$ such that

$$\mathbb{P}\bigg(\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{t_n}\log\frac{S_k}{k}\leq -t\bigg)\leq\mathbb{P}\bigg(-\sum_{k=1}^{t_n}\frac{1}{k}\sum_{i=1}^{k}(\log X_i)\geq\frac{tb_n\sqrt{n}}{2}\bigg)$$
$$=\mathbb{P}\bigg(-\sum_{i=1}^{t_n}b_{i,t_n}(\log X_i-\mathbb{E}\log X_i)\geq\frac{tb_n\sqrt{n}}{4}\bigg)\leq e^{-cb_n\sqrt{n}}\qquad(2.10)$$

and, by the inequality $e^{-x}\leq 1-x+\frac{1}{2}x^2, x>0$, for all n large enough

$$\mathbb{P}\bigg(\frac{1}{b_n\sqrt{n}}\sum_{i=t_n+1}^{\alpha_n}\log\frac{S_k}{k}\leq -t\bigg)\leq\sum_{k=t_n+1}^{\alpha_n}\mathbb{P}\bigg(-\log\frac{S_k}{k}\geq\frac{tb_n\sqrt{n}}{2\alpha_n}\bigg)$$

$$= \sum_{k=t_n+1}^{\alpha_n} \mathbb{P}\Big(1-\frac{S_k}{k} \geq 1-e^{-\frac{tb_n\sqrt{n}}{2\alpha_n}}\Big)$$
$$\leq \sum_{k=t_n+1}^{\alpha_n} \mathbb{P}\Big(1-\frac{S_k}{k} \geq \frac{tb_n\sqrt{n}}{4\alpha_n}\Big) \leq \alpha_n e^{-ct_n\frac{b_n^2 n}{\alpha_n^2}}. \tag{2.11}$$

Hence if we take $\alpha_n = b_n\sqrt{n}(\log n)^{1/2}$, then

$$\lim_{n\to\infty} \frac{1}{b_n^2}\log \mathbb{P}\Big(\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{t_n}\log\frac{S_k}{k} \leq -t\Big) = -\infty. \tag{2.12}$$

Moreover, by the inequality $1+x \leq e^x, x \geq 0$, then

$$\frac{1}{b_n^2}\log \mathbb{P}\Big(\frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n} b_{i,\alpha_n}X_i \geq e^{\frac{b_n\sqrt{n}t}{\alpha_n}}\Big)$$
$$\leq \frac{1}{b_n^2}\log \mathbb{P}\Big(\frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n} b_{i,\alpha_n}(X_i-1) \geq \frac{b_n\sqrt{n}t}{\alpha_n}\Big) \to -\infty$$

by the Hoeffiding's inequality again. So from the above discussions, the condition (2.3) holds.

Example 2.6. (Exponential random variable) Let (X_n) be a sequence of i.i.d. exponential random variables with density function $f(x) = e^{-x}, x > 0$. If $\frac{\log n}{b_n^2} \to 0$, by using the exponential inequalities in [9, Chapter III], it is not difficult to get the inequalities (2.10)-(2.12), which yields the condition (2.3). So we omit these proofs.

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