МАТЕМАТИЧКИ ВЕСНИК
62, 4 (2010), 259–264
December 2010

оригинални научни рад
research paper

BERNSTEIN-DURRMEYER TYPE OPERATORS PRESERVING LINEAR FUNCTIONS

Vijay Gupta and Oktay Duman

Abstract. In this paper, in order to converge faster to a function being approximated we modify two different Bernstein-Durrmeyer type operators introduced in [5] and [7] such that linear functions are preserved.

1. Introduction

Many well-known approximating operators, L_n, preserve the linear functions, i.e., $L_n(e_0;x) = e_0(x)$ and $L_n(e_1;x) = e_1(x)$ for $e_i(x) = x^i$ $(i = 0,1)$. These conditions hold, specifically, for the Bernstein polynomials, the Szász-Mirakjan operators, the Baskakov operators, and so on. However, in recent years, the operators introduced by Gupta [5], Gupta and Maheshwari [7], respectively, do not preserve the test function e_1. In this case a natural question arises: can we modify these operators such that the linear functions are preserved? In this paper we mainly focus on this problem and find affirmative answers. Actually, the basic reason of this idea is to converge faster to the function being approximated. Really, we demonstrate that our modified operators have better error estimations on some appropriate subintervals of $[0,1]$ than the ones used in [5, 7].

This paper is organized as follows: In the first section we recall some basic definitions and results obtained in [5, 7]. In Section 2, we construct our operators preserving the linear functions and give their Korovkin-type approximation theorems. In Section 3, we show that these modified operators have a better error estimation on the interval $\left[\frac{1}{2}, \frac{3}{5}\right]$ (resp. $\left[\frac{2}{5}, \frac{1}{2}\right]$) than the operators introduced in [5] (resp. [7]). Finally, the last section of the paper is devoted to the remarks and discussion.

Gupta [5] has introduced the following positive linear operators, which give the modification of Bernstein-Durrmeyer operators,

$$P_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 b_{n,k}(t) f(t)\,dt, \tag{1}$$

2010 AMS Subject Classification: 41A35, 47G10.

Keywords and phrases: Positive linear operators; Korovkin-type approximation theorem; Bernstein-Durrmeyer operators.

where $x \in [0,1]$, $n \in \mathbb{N}$ and, for $\phi_n(x) = (1-x)^n$,

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \text{ and } b_{n,k}(t) = (-1)^{k+1} \frac{x^k}{k!} \phi_n^{(k+1)}(t). \tag{2}$$

Very recently, Gupta and Maheshwari [7] have introduced another modification of Bernstein-Durrmeyer operators and investigated their approximation properties for functions of bounded variation :

$$R_n(f;x) = n \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t) f(t) dt + (1-x)^n f(0), \tag{3}$$

where $x \in [0,1]$, $n \in \mathbb{N}$ and the term $p_{n,k}(x)$ is given above.

We should also note that various approximation results of these operators and related topics may be found in the papers [1, 4, 6, 8, 9].

Throughout the paper we use the test functions e_i as

$$e_i(x) = x^i, \quad i = 0, 1, 2,$$

and the moment function φ_x as $\varphi_x(t) = t - x$.

Now following the results obtained in [6, 9] we may write the next lemmas, immediately.

Lemma 1. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

(i) $P_n(e_0;x) = 1$,

(ii) $P_n(e_1;x) = \dfrac{nx+1}{n+1}$,

(iii) $P_n(e_2;x) = \dfrac{n^2x^2 - n(x-4) + 2}{(n+1)(n+2)}$.

Lemma 2. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

(i) $P_n(\varphi_x;x) = \dfrac{1-x}{n+1}$,

(ii) $P_n(\varphi_x^2;x) = \dfrac{-2x^2(n-1) + 2x(n-2)x + 2}{(n+1)(n+2)}$.

Lemma 3. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

(i) $R_n(e_0;x) = 1$,

(ii) $R_n(e_1;x) = \dfrac{nx}{n+1}$,

(iii) $R_n(e_2;x) = \dfrac{nx(x(n-1)+2)}{(n+1)(n+2)}$.

Lemma 4. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

(i) $R_n(\varphi_x;x) = \dfrac{-x}{n+1}$,

(ii) $P_n(\varphi_x^2;x) = \dfrac{x(1-x)(2n+1) - (1-3x)x}{(n+1)(n+2)}$.

2. Construction of the operators

In this section, we modify the operators given by (1) and (3) such that the linear functions are preserved.

We start with the operator P_n. Then, by defining

$$r_n(x) = \frac{(n+1)x - 1}{n}, \tag{4}$$

we replace x in the definition of $P_n(f;x)$ by $r_n(x)$. So, to get $r_n(x) \in [0,1]$ for any $n \in \mathbb{N}$ we have to use the restriction $x \in \left[\frac{1}{2}, 1\right]$ by (4). Then we give the following modification of the operators $P_n(f;x)$ defined by (1):

$$D_n(f;x) = \sum_{k=0}^{n} d_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) dt, \tag{5}$$

where $x \in \left[\frac{1}{2}, 1\right]$, $n \in \mathbb{N}$, the term $b_{n,k}(t)$ is given in (2), and

$$d_{n,k}(x) = \binom{n}{k} \frac{(n+1)^{n-k}((n+1)x-1)^k (1-x)^{n-k}}{n^n}.$$

In a similar manner, defining $q_n(x) = \frac{(n+1)x}{n}$, from the definition of $R_n(f;x)$ given by (3) and using the restriction $x \in \left[0, \frac{1}{2}\right]$, we have the following positive linear operators at once:

$$T_n(f;x) = n \sum_{k=0}^{n} t_{n,k}(x) \int_0^1 p_{n-1,k-1}(t) f(t) dt + \frac{(n-(n+1)x)^n}{n^n} f(0), \tag{6}$$

where $x \in \left[0, \frac{1}{2}\right]$, $n \in \mathbb{N}$, the term $p_{n-1,k-1}(t)$ is given in (2), and

$$t_{n,k}(x) = \binom{n}{k} \frac{(n+1)^k x^k (n-(n+1)x)^{n-k}}{n^n}.$$

Now, the next results follow from Lemmas 1–4.

Lemma 5. *For $x \in \left[\frac{1}{2}, 1\right]$ and $n \in \mathbb{N}$, we have*
(i) $D_n(e_0;x) = 1$,
(ii) $D_n(e_1;x) = x$,
(iii) $D_n(e_2;x) = \dfrac{(n^2-1)x^2 + 2(n+1)x - 1}{n(n+2)}$.

Lemma 6. *For $x \in \left[\frac{1}{2}, 1\right]$ and $n \in \mathbb{N}$, we have*
(i) $D_n(\varphi_x;x) = 0$,
(ii) $D_n(\varphi_x^2;x) = \dfrac{(1-x)(2nx + x - 1)}{n(n+2)}$.

Lemma 7. *For $x \in \left[0, \frac{1}{2}\right]$ and $n \in \mathbb{N}$, we have*
(i) $T_n(e_0;x) = 1$,
(ii) $T_n(e_1;x) = x$,
(iii) $T_n(e_2;x) = \dfrac{x((n^2-1)x + 2n)}{n(n+2)}$.

Lemma 8. *For $x \in \left[0, \frac{1}{2}\right]$ and $n \in \mathbb{N}$, we have*

(i) $T_n(\varphi_x; x) = 0$,

(ii) $T_n(\varphi_x^2; x) = \dfrac{x(2n(1-x)-x)}{n(n+2)}$.

Lemmas 5 and 7 easily show that our operators D_n and T_n preserve the linear functions, that is, for $h(t) = at + b$, where a, b any real constants, we obtain

$$D_n(h; x) = T_n(h; x) = h(x).$$

On the other hand, the above lemmas guarantee that the following Korovkin-type approximation results hold.

Theorem 1. *For all $f \in C[0,1]$, the sequence $\{D_n(f;x)\}_{n\in\mathbb{N}}$ is uniformly convergent to $f(x)$ with respect to $x \in [1/2, 1]$.*

Theorem 2. *For all $f \in C[0,1]$, the sequence $\{T_n(f;x)\}_{n\in\mathbb{N}}$ is uniformly convergent to $f(x)$ with respect to $x \in [0, 1/2]$.*

3. Better error estimation

In this section we compute the rates of convergence of the operators D_n and T_n given by (5) and (6), respectively. Then, we will show that our operators have better error estimations than that of the operators P_n and R_n. To achieve this we use the modulus of continuity.

Recall that, for $f \in C[0,1]$ and $x \in \left[\frac{1}{2}, 1\right]$ (or, $x \in \left[0, \frac{1}{2}\right]$), the modulus of continuity of f denoted by $\omega(f, \delta_x)$ is defined to be

$$\omega(f, \delta_x) = \sup_{x-\delta \le t \le x+\delta,;\, t\in[0,1]} |f(t) - f(x)|\,.$$

Then we obtain the following result.

Theorem 3. *For every $f \in C[0,1]$, $x \in \left[\frac{1}{2}, 1\right]$ and $n \in \mathbb{N}$, we have*

$$|D_n(f;x) - f(x)| \le 2\omega(f, \delta_x),$$

where $\delta_x := \sqrt{\dfrac{(1-x)(2nx+x-1)}{n(n+2)}}$.

Proof. Now, let $f \in C[0,1]$ and $x \in \left[\frac{1}{2}, 1\right]$. Using linearity and monotonicity of D_n we easily get, for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$|D_n(f;x) - f(x)| \le \omega(f, \delta) \left\{1 + \frac{1}{\delta}\sqrt{D_n\left(\varphi_x^2; x\right)}\right\},$$

Now applying Lemma 6*(ii)* and choosing $\delta = \delta_x$ the proof is completed. ■

Remark. For the operator P_n given by (1) we may write that, for every $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$,

$$|P_n(f;x) - f(x)| \le 2\omega(f, \alpha_x), \tag{7}$$

where $\alpha_x := \sqrt{\dfrac{-2x^2(n-1) + 2x(n-2)x + 2}{(n+1)(n+2)}}$.

Now we claim that the error estimation in Theorem 3 is better than that of (7) provided $f \in C[0,1]$ and $x \in \left[\frac{1}{2}, \frac{3}{5}\right]$. Indeed, in order to get this better estimation we must show that $\delta_x \le \alpha_x$ for appropriate x's. Using also the restriction $x \in \left[\frac{1}{2}, 1\right]$, one can obtain that

$$\delta_x \le \alpha_x \Leftrightarrow \frac{(1-x)(2nx+x-1)}{n(n+2)} \le \frac{-2x^2(n-1)+2x(n-2)x+2}{(n+1)(n+2)}$$
$$\Leftrightarrow \frac{(1-x)((5n+1)x-(3n+1))}{n(n+1)(n+2)} \le 0 \Leftrightarrow x \le \frac{3n+1}{5n+1}.$$

Observe now that

$$\frac{3n+1}{5n+1} > \frac{3}{5} \quad \text{for any } n \in \mathbb{N}.$$

Thus, considering the above inequalities we can say that if $x \in \left[\frac{1}{2}, \frac{3}{5}\right]$, then we have $\delta_x \le \alpha_x$, which corrects our claim.

A similar idea as in Theorem 3 lead us the following result at once.

Theorem 4. *For every $f \in C[0,1]$, $x \in \left[0, \frac{1}{2}\right]$ and $n \in \mathbb{N}$, we have*

$$|T_n(f;x) - f(x)| \le 2\omega(f, u_x),$$

where $u_x := \sqrt{\dfrac{x(2n(1-x)-x)}{n(n+2)}}$.

Furthermore, for the operator R_n given by (3) we get, for every $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, that

$$|R_n(f;x) - f(x)| \le 2\omega(f, v_x),$$

where $v_x := \sqrt{\dfrac{x(1-x)(2n+1)-(1-3x)x}{(n+1)(n+2)}}$.

Now considering the above remark the similar claim is valid for the operators T_n on the interval $\left[\frac{2}{5}, \frac{1}{2}\right]$. Indeed, in order to get a better estimation we must show that $u_x \le v_x$ for appropriate x's. So, we may write that

$$u_x \le v_x \Leftrightarrow \frac{x(2n(1-x)-x)}{n(n+2)} \le \frac{x(1-x)(2n+1)-(1-3x)x}{(n+1)(n+2)}$$
$$\Leftrightarrow -\frac{x(x+(5x-2)n)}{n(n+1)(n+2)} \le 0 \Leftrightarrow x \ge \frac{2n}{5n+1}.$$

However, since

$$\frac{2n}{5n+1} < \frac{2}{5} \quad \text{for any } n \in \mathbb{N},$$

the above inequalities yield that if $x \in \left[\frac{2}{5}, \frac{1}{2}\right]$, then we have $u_x \le v_x$, which corrects our claim again.

4. Concluding remarks

Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in the approximation theory, which gains us to approximate much faster to the function being approximated. In this paper, such a study was accomplished for two different modifications of Bernstein-Durrmeyer operators.

Observe that the positive linear operators P_n and R_n do not preserve neither the linear functions nor the test function $e_2(x) = x^2$. In this note, we present the modifications of P_n and R_n so that they preserve the linear functions. In this case we demonstrate that our modified operators have better error estimations on some appropriate intervals than the operators P_n and R_n. However, one can ask that is it possible to modify the operators such that the test function e_2 is preserved? Although such investigations was accomplished for Bernstein polynomials by King [10], for Szasz-Mirakjan operators by Duman and Özarslan [3], for Meyer-König and Zeller operators by Özarslan and Duman [11], and for some summation-type positive linear operators by Agratini [2], unfortunately, it seems to be too hard for the operators P_n and R_n due to more complicated calculations. Maybe it will be done with the help of a good algorithm in computer programming. Thus, for now, these structures are open problems in the approximation theory.

REFERENCES

[**1**] O. Agratini, *On the rate of convergence of some integral operators for functions of bounded variation*, Studia Sci. Math. Hungar. **42** (2005), 235–252.

[**2**] O. Agratini, *Linear operators that preserve some test functions*, Int. J. Math. Math. Sci. Art ID. 94136 (2006), 11 pp.

[**3**] O. Duman, M.A. Özarslan, *Szász-Mirakjan type operators providing a better error estimation*, Appl. Math. Lett. **20** (2007), 1184–1188.

[**4**] S.S. Guo, *On the rate of convergence of the Durrmeyer operator for functions of bounded variation*, J. Approx. Theory **51** (1987), 183–192.

[**5**] V. Gupta, *A note on the rate of convergence of Durrmeyer type operators for function of bounded variation*, Soochow J. Math. **23** (1997), 115–118.

[**6**] V. Gupta, N. Ispir, *On simultaneous approximation for some modified Bernstein-type operators*, Int. J. Math. Math. Sci. **2004** (2004), 3951–3958.

[**7**] V. Gupta, P. Maheshwari, *Bezier variant of a new Durrmeyer type operators*, Riv. Math. Univ. Parma **2** (2003), 9–21.

[**8**] V. Gupta, T. Shervashidze, M. Craciun, *Rate of approximation for certain Durrmeyer operators*, Georgian Math. J. **13** (2006), 277–284.

[**9**] V. Gupta, G.S. Srivastava, *Approximation by Durrmeyer type operators*, Ann. Polon. Math. **64** (1996), 153–159.

[**10**] J.P. King, *Positive linear operators which preserve x^2*, Acta. Math. Hungar. **99** (2003), 203–208.

[**11**] M.A. Özarslan, O. Duman, *MKZ type operators providing a better estimation on* $[1/2, 1)$, Canad. Math. Bull. **50** (2007), 434–439.

(received 15.07.2009)

Vijay Gupta, School of Applied Sciences, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi-110078, India

E-mail: vijay@nsit.ac.in

Oktay Duman, TOBB Economics and Technology University, Faculty of Arts and Sciences, Department of Mathematics, Söğütözü 06530, Ankara, Turkey

E-mail: oduman@etu.edu.tr