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A NOTE ON STAR COMPACT SPACES WITH A G_δ-DIAGONAL

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Abstract. In this note we give an example of a Hausdorff, star compact space with a G_δ-diagonal which is not metrizable, which answers negatively a question of van Mill, Tkachuk and Wilson (Problem 4.8 in [J. van Mill, V.V. Tkachuk, R.G. Wilson, *Classes defined by stars and neighbourhood assignments*, Topology Appl. 154 (2007), 2127–2134]).

1. Introduction

By a space we mean a topological space. Let A be a subset of a space X and $\mathcal{U}$ be a family of subsets of X. The star of the set A with respect to $\mathcal{U}$, denoted by $St(A,\mathcal{U})$, is the set $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

Definition 1.1. [2] Let $\mathcal{P}$ be a class (or a property) of a space X. X is said to be *star* $\mathcal{P}$ (*or star-determined by* $\mathcal{P}$) if for every open cover $\mathcal{U}$ of X there exists a subspace Y of X such that $Y \in \mathcal{P}$ and $St(Y,\mathcal{U}) = X$.

By the above definition, a space X is called *star compact* if for every open cover $\mathcal{U}$ of X there exists a compact subset K of X such that $St(K,\mathcal{U}) = X$. In [3], a star compact space is called K-starcompact. It is not difficult to see that every countably compact space is star compact (see [3]). Since every countably compact space with a G_δ-diagonal is metrizable, van Mill, Tkachuk and Wilson asked the following question:

Problem 1.2. [2, Problem 4.8] *Is a star compact space metrizable if it has a point-countable base?*

If a space X has a G_δ-diagonal, then X has a point-countable base. The purpose of this note is to construct an example stated in the abstract which gives a negative answer to the above question in the class of Hausdorff spaces.

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Throughout the paper, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal. Other terms and symbols that we do not define will be used as in [1].

2. An example of a Hausdorff, star compact space with a G_δ-diagonal

EXAMPLE 2.1. There exists a Hausdorff, star compact space with a G_δ-diagonal which is not metrizable.

Proof. Let

$$A = \{a_n : n \in \omega\} \text{ and } B = \{b_m : m \in \omega\},$$
$$Y = \{\langle a_n, b_m\rangle : n \in \omega, m \in \omega\},$$

and let

$$X = Y \cup A \cup \{a\} \text{ where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighborhood of a point $a_n \in A$ for each $n \in \omega$ takes the form

$$U_{a_n}(m) = \{a_n\} \cup \{\langle a_n, b_i\rangle : i > m\} \text{ for } m \in \omega$$

and a basic neighborhood of a takes the form

$$U_a(n) = \{a\} \cup \bigcup\{\langle a_i, b_m\rangle : i > n, m \in \omega\}.$$

Clearly, X is a Hausdorff space by the construction of the topology of X. However, X is not regular since the point a can not be separated from the closed subset A by disjoint open subsets of X. Thus, X is not metrizable. But X has a G_δ-diagonal. In fact,

$$U_m = \bigcup_{n\in\omega}(U_{a_n}(m) \times U_{a_n}(m)) \cup (U_a(m) \times U_a(m))$$
$$\cup \bigcup\{\langle\langle a_n, b_m\rangle, \langle a_n, b_m\rangle\rangle : n \in \omega, m \in \omega\} \text{ for each } m \in \omega.$$

Then U_m is an open subset of $X \times X$ and $\triangle = \bigcap_{m\in\omega} U_m$, which shows that X has a G_δ-diagonal.

We show that X is star compact. Let $\mathcal{U}$ be an open cover of X. For each $n \in \omega$, there exists $U_n \in \mathcal{U}$ such that $a_n \in U_n$, so there exists $m_n \in \omega$ such that $\langle a_n, b_{m_n}\rangle \in U_n$. If we put $S_1 = \{\langle a_n, b_{m_n}\rangle : n \in \omega\} \cup \{a\}$, then S_1 is a convergent sequence with the limit point $\{a\}$, hence S_1 is compact and

$$\{a_n : n \in \omega\} \subseteq St(S_1, \mathcal{U}).$$

On the other hand, choose $U_a \in \mathcal{U}$ such that $a \in U_a$, Then there exists $n \in \omega$ such that $U_a(n) \subseteq U_a$, hence

$$U_a(n) \subseteq St(S_1, \mathcal{U}),$$

since $U_a \cap S_1 \neq \emptyset$. Finally, for $i \leq n$, $\{a_i\} \cup \{\langle a_i, b_m \rangle : m \in \omega\}$ is compact, so there exists a finite subset $F_i \subseteq \{a_i\} \cup \{\langle a_i, b_m \rangle : m \in \omega\}$ such that

$$\{a_i\} \cup \{\langle a_i, b_m \rangle : m \in \omega\} \subseteq St(F_i, \mathcal{U}).$$

Put $F = S_1 \cup \cup\{F_i : i \leq n\}$. Then F is a compact subset of X and $X = St(F, \mathcal{U})$, which completes the proof. ∎

REMARK 2.2. The author does not know if there exists a regular star compact space with a point-countable base, which is not metrizable.

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