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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Abstract. Let $\mathcal{Q}_b(\Phi, \Psi; \alpha)$ be the class of normalized analytic functions defined in the open unit disk and satisfying

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right)\right\} > \alpha$$

for nonzero complex number b and for $0 \le \alpha < 1$. Sufficient condition, involving coefficient inequalities, for $f(z)$ to be in the class $\mathcal{Q}_b(\Phi, \Psi; \alpha)$ is obtained. Our main result contains some interesting corollaries as special cases.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbf{C} \text{ and } |z| < 1\}$. Furthermore, let $\mathcal{P}$ be the class of functions $p(z)$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, which are analytic in $\mathcal{U}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of complex order b ($b \in \mathbf{C}^* := \mathbf{C}\backslash\{0\}$) and type α ($0 \le \alpha < 1$), that is $f(z) \in \mathcal{S}_b^*(\alpha)$, if and only if

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha \qquad (z \in \mathcal{U};\ b \in \mathbf{C}^*),$$

and is said to be convex of complex order b ($b \in \mathbf{C}^*$) and type α ($0 \le \alpha < 1$), denoted by $\mathcal{C}_b(\alpha)$ if and only if

$$\mathrm{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{U};\ b \in \mathbf{C}^*).$$

Note that $\mathcal{S}_b^*(0) = \mathcal{S}_b^*$ and $\mathcal{C}_b(0) = \mathcal{C}_b$ are the classes considered earlier by Nasr and Auof [6] and Wiatrowski [10]. Also, $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}_1(\alpha) = \mathcal{C}(\alpha)$ which are,

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respectively, the familiar classes of starlike functions of order α $(0 \leq \alpha < 1)$ and convex functions of order α $(0 \leq \alpha < 1)$.

Further, let $\mathcal{P}_b(\alpha)$ denote the class of functions $f(z) \in \mathcal{A}$ such that

$$\operatorname{Re}\left\{1 + \frac{1}{b}(f'(z) - 1)\right\} > \alpha \qquad (0 \leq \alpha < 1,\ z \in \mathcal{U};\ b \in \mathbf{C}^*).$$

When $b = 1$, the class $\mathcal{P}_1(\alpha)$ reduces to the class $\mathcal{P}(\alpha)$ of analytic functions studied in [5, 7, 9].

Given two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n \qquad (z \in \mathcal{U}).$$

By using this product we introduce the class of prestarlike functions of complex order b $(b \in \mathbf{C}^*)$ and type α $(0 \leq \alpha < 1)$, which is denoted by $\mathcal{R}_b(\alpha)$. Thus $f(z) \in \mathcal{A}$ is said to be prestarlike function of complex order b ($b \in \mathbf{C}^*$ and type α $(0 \leq \alpha < 1)$, if and only if $f(z) * s_\alpha(z) \in \mathcal{S}_b^*(\alpha)$ where $s_\alpha(z) = z(1-z)^{2\alpha-2} = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n$; $C(\alpha, n) = \prod_{j=2}^{n} \frac{j-2\alpha}{(n-1)!}$ $(n \geq 2)$. It may be noted that $\mathcal{R}_b(0) = \mathcal{C}_b(0)$ and $\mathcal{R}_b(1/2) = \mathcal{S}_b^*(1/2)$. When $b = 1$, the class $\mathcal{R}_1(\alpha)$ reduces to the class $\mathcal{R}(\alpha)$ of prestarlike functions of order α $(0 \leq \alpha < 1)$ (see [8]).

Making use of the Hadamard product, Frasin [1] introduced and studied the following class of analytic functions:

Definition 1.1. Let b $(b \in \mathbf{C}^*)$ and α $(0 \leq \alpha < 1)$ be given. Let the functions

$$\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \quad \text{and} \quad \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$$

be analytic in $\mathcal{U}$, such that $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$ for $n \geq 2$, we say that $f(z) \in \mathcal{A}$ is in $\mathcal{Q}_b(\Phi, \Psi; \alpha)$ if $f(z) * \Psi(z) \neq 0$ and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right)\right\} > \alpha \qquad (z \in \mathcal{U}).$$

We note that, by suitably choosing of $\Phi(z)$ and $\Psi(z)$ we obtain the above subclasses of $\mathcal{A}$ of complex order b and type α : $\mathcal{Q}_b\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha\right) = \mathcal{S}_b^*(\alpha)$; $\mathcal{Q}_b\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha\right) = \mathcal{C}_b(\alpha)$; $\mathcal{Q}_b\left(\frac{z}{(1-z)^2}, z; \alpha\right) = \mathcal{P}_b(\alpha)$ and
$\mathcal{Q}_b\left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha\right) = \mathcal{R}_b(\alpha)$.

In fact many new subclasses of $\mathcal{A}$ of complex order b and type α can be defined and studied by suitably choosing $\Phi(z)$ and $\Psi(z)$. For example,

$$\mathcal{Q}_b\left(\frac{z}{1-z}, z; \alpha\right) := \mathcal{T}_b(\alpha) = \left\{f(z) \in \mathcal{A} : \operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{f(z)}{z} - 1\right)\right\} > \alpha\right\},$$

and

$$\mathcal{Q}_b\left(\frac{z+z^2}{(1-z)^3}, z; \alpha\right) : \mathcal{M}_b(\alpha) = \left\{f(z) \in \mathcal{A} : \operatorname{Re}\left\{1 + \frac{1}{b}\left((zf'(z))' - 1\right)\right\} > \alpha\right\}$$

and so on.

In this paper, we obtain sufficient condition, involving coefficient inequalities, for $f(z)$ to be in the class $\mathcal{Q}_b(\Phi,\Psi;\alpha)$. Several special cases and consequences of these coefficient inequalities are also pointed out.

In order to derive our main results, we have to recall here the following lemmas:

LEMMA 1.1. [4] *A function* $p(z)\in\mathcal{P}$ *satisfies* $\operatorname{Re}p(z)>0$ $(z\in\mathcal{U})$ *if and only if* $p(z)\neq\frac{x-1}{x+1}$ $(z\in\mathcal{U})$ *for all* $|x|=1$.

LEMMA 1.2. *A function* $f(z)\in\mathcal{A}$ *is in* $\mathcal{Q}_b(\Phi,\Psi;\alpha)$ *if and only if* $1+\sum_{n=2}^{\infty}A_nz^{n-1}\neq 0$ *where*

$$A_n=\frac{\lambda_n+(2b-1-2\alpha b)\mu_n+x(\lambda_n-\mu_n)}{2b(1-\alpha)}a_n$$

and $\lambda_1=\mu_1=1$.

Proof. Applying Lemma 1.1, we have

$$\frac{1+\frac{1}{b}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right)-\alpha}{1-\alpha}\neq\frac{x-1}{x+1}\quad(z\in\mathcal{U};\ x\in\mathbf{C};\ \ |x|=1).\tag{1.1}$$

Then, we need not consider Lemma 1.1 for $z=0$, because it follows that $p(0)=1\neq\frac{x-1}{x+1}$ for all $|x|=1$. From (1.1), it follows that

$$(x+1)(f(z)*\Phi(z))+(2b-1-2\alpha b-x)(f(z)*\Psi(z))\neq 0.$$

Thus, we have $2b(1-\alpha)z+\sum_{n=2}^{\infty}\left[\lambda_n+(2b-1-2\alpha b)\mu_n+x(\lambda_n-\mu_n)\right]a_nz^n\neq 0$ $(z\in\mathcal{U};\ x\in\mathbf{C};\ |x|=1)$, or, equivalently,

$$2b(1-\alpha)z\left(1+\sum_{n=2}^{\infty}\frac{\lambda_n+(2b-1-2\alpha b)\mu_n+x(\lambda_n-\mu_n)}{2b(1-\alpha)}a_nz^{n-1}\right)\neq 0\tag{1.2}$$

$(z\in\mathcal{U};\ x\in\mathbf{C};\ |x|=1)$. Now, dividing both sides of (1.2) by $2b(1-\alpha)z$ $(z\neq 0)$, we obtain

$$1+\sum_{n=2}^{\infty}\frac{\lambda_n+(2b-1-2\alpha b)\mu_n+x(\lambda_n-\mu_n)}{2b(1-\alpha)}a_nz^{n-1}\neq 0$$

$(z\in\mathcal{U};\ x\in\mathbf{C};\ |x|=1)$, which completes the proof of Lemma 1.2. ■

2. Coefficient conditions for functions in the class $\mathcal{Q}_b(\Phi,\Psi;\alpha)$

With the help of Lemma 1.2, we have

THEOREM 2.1. *If* $f(z)\in\mathcal{A}$ *satisfies the following condition:*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(\lambda_k+(2b-1-2\alpha b)\mu_k)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$
$$\left.+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(\lambda_k-\mu_k)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right)\leq 2\,|b|\,(1-\alpha)$$

$(0\leq\alpha<1;\ b\in\mathbf{C}^*;\ \gamma,\delta\in\mathbf{R})$, *then* $f(z)\in\mathcal{Q}_b(\Phi,\Psi;\alpha)$.

Proof. Note that $(1-z)^{\gamma} \neq 0$, $(1+z)^{\delta} \neq 0$ $(\gamma, \delta \in \mathbf{R};\ z \in \mathcal{U})$. Thus to prove that $1+\sum_{n=2}^{\infty} A_n z^{n-1} \neq 0$, it is sufficient that

$$\left(1+\sum_{n=2}^{\infty} A_n z^{n-1}\right)(1-z)^{\gamma}(1+z)^{\delta}$$
$$= 1+\sum_{n=2}^{\infty}\left[\sum_{l=1}^{n}\left\{\sum_{k=1}^{l} A_k(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right] z^{n-1} \neq 0,$$

where $A_0 = 0$ and $A_1 = 1$. Therefore, if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l} A_k(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right| \leq 1,$$

that is, if

$$\frac{1}{2\,|b|\,(1-\alpha)}\sum_{n=2}^{\infty}\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(\lambda_k+(2b-1-2\alpha b)\mu_k\right.\right.$$
$$\left.\left. +x[\lambda_k-\mu_k])(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|$$
$$\leq \frac{1}{2\,|b|\,(1-\alpha)}\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(\lambda_k+(2b-1-2\alpha b)\mu_k)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\times\right.\right.$$
$$\left.\left.\times\binom{\delta}{n-l}\right| + |x|\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(\lambda_k-\mu_k)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right)$$
$$\leq 1 \qquad (0 \leq \alpha < 1;\ b \in \mathbf{C}^*;\ x \in \mathbf{C};\ |x| = 1;\ \gamma, \delta \in \mathbf{R}),$$

then $f(z) \in \mathcal{Q}_b(\Phi, \Psi; \alpha)$ and so the proof is completed. ■

3. Particular cases

By considering some special cases of the analytic functions $\Phi(z)$ and $\Psi(z)$, we deduce the following coefficient conditions for functions $f(z)$ to be in the subclasses $\mathcal{S}_{\alpha}^*(b)$, $\mathcal{C}_{\alpha}(b)$, $\mathcal{P}_{\alpha}(b)$ and $\mathcal{R}_{\alpha}(b)$ of analytic functions of complex order b $(b \in \mathbf{C}^*)$ and type α $(0 \leq \alpha < 1)$ as defined in Section 1.

Letting $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z/(1-z)$ in Theorem 2.1, we have

Corollary 3.1 *If $f(z) \in \mathcal{A}$ satisfies the following condition:*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k+2b-1-2\alpha b)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$
$$\left. +\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-1)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right) \leq 2\,|b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$, $b \in \mathbf{C}^*$ and $\gamma, \delta \in \mathbf{R}$, then $f(z) \in \mathcal{S}_{\alpha}^*(b)$. In particular, for $\gamma = \delta = 0$, if $f(z) \in \mathcal{A}$ satisfies the following condition:

$$\sum_{n=2}^{\infty}\{|n+2b-1-2\alpha b| + (n-1)\}\,|a_n| \leq 2\,|b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$ and $b \in \mathbf{C}^*$, then $f(z) \in \mathcal{S}_{\alpha}^*(b)$.

Letting $\Phi(z) = (z + z^2)/(1 - z)^3$ and $\Psi(z) = z/(1 - z)^2$ in Theorem 2.1, we have

Corollary 3.2. *If $f(z) \in \mathcal{A}$ satisfies the following condition:*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^2+(2b-1-2\alpha b)k)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right| + \left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^2-k)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right) \leq 2\,|b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$, $b \in \mathbf{C}^*$ and $\gamma, \delta \in \mathbf{R}$, then $f(z) \in \mathcal{C}_b(\alpha)$. In particular, for $\gamma = \delta = 0$, if $f(z) \in \mathcal{A}$ satisfies the following condition:

$$\sum_{n=2}^{\infty} n\{|n-1+2b| + (n-1)\}\,|a_n| \leq 2\,|b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$ and $b \in \mathbf{C}^*$, then $f(z) \in \mathcal{C}_b(\alpha)$.

Letting $\Phi(z) = (z + (1-2\alpha)z^2)/(1 - z)^{3-2\alpha}$ and $\Psi(z) = z/(1 - z)^{2-2\alpha}$ in Theorem 2.1, we have

Corollary 3.3. *If $f(z) \in \mathcal{A}$ satisfies the following condition:*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(C(\alpha,k)(k+(2b-1-2\alpha b))(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right| + \left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}C(\alpha,k)(k-1)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right) \leq 2\,|b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$, $b \in \mathbf{C}^*$ and $\gamma, \delta \in \mathbf{R}$, then $f(z) \in \mathcal{R}_b(\alpha)$. In particular, for $\gamma = \delta = 0$, if $f(z) \in \mathcal{A}$ satisfies the following condition:

$$\sum_{n=2}^{\infty} C(\alpha,n)[|n+2b-1-2\alpha b| + (n-1)]\,|a_n| \leq |b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$ and $b \in \mathbf{C}^*$, then $f(z) \in \mathcal{R}_b(\alpha)$.

Letting $\Phi(z) = z/(1 - z)^2$ and $\Psi(z) = z$ in Theorem 2.1, we have

Corollary 3.4. *If $f(z) \in \mathcal{A}$ satisfies the following condition:*

$$\sum_{n=2}^{\infty}\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right| \leq |b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$, $b \in \mathbf{C}^*$ and $\gamma, \delta \in \mathbf{R}$, then $f(z) \in \mathcal{P}_b(\alpha)$. In particular, for $\gamma = \delta = 0$, if $f(z) \in \mathcal{A}$ satisfies the following condition $\sum_{n=2}^{\infty} n\,|a_n| \leq |b|\,(1-\alpha)$ for some α $(0 \leq \alpha < 1)$ and $b \in \mathbf{C}^*$, then $f(z) \in \mathcal{P}_b(\alpha)$.

Letting $\Phi(z) = z/(1 - z)$ and $\Psi(z) = z$ in Theorem 2.1, we have

Corollary 3.5. *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right| \leq |b|\,(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$, $b \in \mathbf{C}^*$ *and* $\gamma, \delta \in \mathbf{R}$, *then* $f(z) \in \mathcal{T}_b(\alpha)$. *In particular, for* $\gamma = \delta = 0$, *if* $f(z) \in \mathcal{A}$ *satisfies the following condition* $\sum_{n=2}^{\infty} |a_n| \leq |b|(1-\alpha)$ *for some* α $(0 \leq \alpha < 1)$ *and* $b \in \mathbf{C}^*$, *then* $f(z) \in \mathcal{T}_b(\alpha)$.

Remark 3.6. (i) If we set $\alpha = 0$ in Corollary 3.1 and Corollary 3.2, we have sufficient conditions for functions $f(z)$ to be in the classes $\mathcal{S}^*(b)$ and $\mathcal{C}(b)$ obtained by Hayami and Owa in [2].

(ii) If we set $b = 1$ in Corollary 3.1 and Corollary 3.2, we have sufficient conditions for functions $f(z)$ to be in the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ obtained by Hayami et al. in [4].

(iii) If we set $b = 1$ in Corollary 3.4 and Corollary 3.5, we have sufficient conditions for functions $f(z)$ to be in the classes $\mathcal{P}(\alpha)$ and $\mathcal{T}(\alpha)$ obtained by Hayami and Owa in [3].

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