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# ON THE EXISTENCE OF BOUNDED CONTINUOUS SOLUTION OF HAMMERSTEIN INTEGRAL EQUATION

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**Abstract.** In this paper, we establish the existence of bounded continuous solutions over any measurable subset of $\mathbf{R}^n$ of some nonlinear integral equations. Our method is based on fixed point theorems.

## 1. Introduction

Nonlinear integral equations (NIE) have been studied by many authors in the literature; see [1–3]. In this paper, we are interested in the study of the existence of continuous solutions of the following Hammerstein integral equation,

$$y(t) = u(t, y(t)) + \int_\Omega k(t,s) F(s, y(s))\, ds, \quad t \in \Omega \tag{1.1}$$

where, $u(.,.), F(.,.) : \Omega \times \mathbf{R} \to$, $k(.,.) : \Omega \times \Omega \to \mathbf{R}$ are given functions, and $\Omega$ is a measurable set in $\mathbf{R}^n$, $n \geq 1$, and $y(.) : \Omega \to \mathbf{R}$ is an unknown function on $\Omega$. We use the following notations:

$$C_a(\Omega) = \{f \in (\Omega), \text{ such that } \|f\|_\infty \leq a\},$$

we consider the space $L^1(\Omega)$ with the norm $\|f(.)\|_1 = \int_\Omega |f(t)|\, dt < \infty$, and the space $L^\infty(\Omega)$ with the norm $\|f\|_\infty = \text{ess } \sup_{t\in\Omega} |f(t)| < \infty$. $\lambda(\Omega)$ is the Lebesgue measure of $\Omega$.

In the first part of this work, we use some generalized Lipschitzian conditions on the functions $u(.,.)$, $F(.,.)$, and we require that $k(.,.)$ be bounded by a measurable function in $L_1$-space. Then, we use Banach's fixed point theorem and prove the existence as well as the uniqueness of a bounded solution belonging to $C(\Omega)$.

In the second part, we change the conditions on $u(.,.)$, $k(.,.)$, $F(.,.)$, we assume that $u(t,x)$ is independent of $x$, and we consider two cases of $\Omega$; in both cases, we prove the existence of a bounded solution belonging to $C(\Omega)$. Moreover, in Case 2, $\Omega$ is compact, and we use Schauder's fixed point theorem.

In the third part, we give an application to a two point boundary value problem.

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## 2. Existence and uniqueness of bounded continuous solution

THEOREM 1. *Suppose that the functions $u(.,.)$, $k(.,.)$, $F(.,.)$ satisfy the following generalized Lipschitzian conditions:*

*1. $u(.,.)$ is continuous on $\Omega \times \mathbf{R}$, $u(t,0)$ is bounded on $\Omega$, and $u(.,.)$ satisfies*

$$|u(t,x) - u(t,y)| \leq b_1(t)\,|x-y|\,,$$

*where, $b_1 \in L^{\infty}(\Omega)$.*

*2. $F(.,.)$ is measurable on $\Omega \times \mathbf{R}$, $F(.,0) \in L^{\infty}(\Omega)$, and $F(.,.)$ satisfies*

$$|F(t,x) - F(t,y)| \leq b_2(t)\,|x-y|\,,$$

*where, $b_2 : \Omega \to \mathbf{R}^+$ is a measurable function.*

*3. $k(.,.)$ is continuous at the first variable, and there exists $g \in L^1(\Omega)$ such that for all $t \in \Omega$, $|k(t,s)| \leq g(s)$ a.e.*

*4. $b = \|b_1\|_\infty + \int_\Omega g(s)b_2(s)\,ds < 1$.*

*Then the Hammerstein integral equation (1.1) has a unique bounded solution in $C(\Omega)$.*

*Proof.* Let $a = \frac{\|u(t,0)\|_\infty + \int_\Omega g(s)F(s,0)ds}{1-b}$, and define the operator $T$ from $C_a(\Omega)$ into itself as follows:

$$Ty(t) = u(t,y(t)) + \int_\Omega k(t,s)F(s,y(s))\,ds, \quad t \in \Omega$$

Claim 1: The operator $T$ is well defined. Let $y \in C_a(\Omega)$; then we have that $u(t,y(t))$ is continuous on $\Omega$. Next, let $(t_n)$ be a sequence in $\Omega$ converging to $t$. Since,

$$\left|\int_\Omega k(t_n,s)F(s,y(s))\,ds - \int_\Omega k(t,s)F(s,y(s))\,ds\right| \leq \int_\Omega |k(t_n,s) - k(t,s)|\,|ab_2(s) + F(s,0)|\,ds,$$

and by Lebesgue's Dominated Convergence Theorem, we have,

$$\lim_{t_n \to t} \int_\Omega |k(t_n,s) - k(t,s)|\,|ab_2(s) + F(s,0)|\,ds = 0.$$

Then the function $\int_\Omega k(.,s)F(s,y(s))ds$ is continuous on $\Omega$, and so $Ty(.)$ is continuous on $\Omega$. Moreover, for $y(.) \in C_a(\Omega)$, we have for all $t \in \Omega$,

$$\begin{aligned} |Ty(t)| &\leq |u(t,y(t))| + \left|\int_\Omega k(t,s)F(s,y(s))\,ds\right| \\ &\leq |u(t,0)| + b_1(t)\,|y(t)| + \int_\Omega g(s)\,|F(s,0)|\,ds + \int_\Omega g(s)b_2(s)\,|y(s)|\,ds \leq a. \end{aligned}$$

Then, $T$ is well defined.

Claim 2: $T$ is a contraction mapping on the Banach space $(C_a(\Omega), \|.\|_\infty)$. Let, $x(.)$, $y(.) \in C_a(\Omega)$. We have,

$$|Tx(t) - Ty(t)| \leq |u(t, x(t)) - u(t, y(t))| + \left| \int_\Omega k(t,s)(F(s, x(s)) - F(s, y(s)))\, ds \right|$$
$$\leq b_1(t) \|x - y\|_\infty + \|x - y\|_\infty \int_\Omega g(s) b_2(s)\, ds \leq b \|x - y\|_\infty .$$

Then by Banach's fixed point theorem, the integral equation (1.1) has a unique bounded solution $y(.) \in C(\Omega)$. ■

Example 1. Consider the following Hammerstein integral equation:

$$y(t) = h(t) + \int_0^\infty \frac{\ln(1 + y^2(s))}{(1 + s^2)(\alpha + t)}\, ds,\ t \in [0, \infty) \tag{2.1}$$

where, $h(.)$ is a bounded continuous function on $[0, \infty)$, and $\alpha$ is a positive number. Let $k(t, s) = \frac{1}{(1+s^2)(\alpha+t)}$, $F(t, s) = \ln(1 + y^2(s))$, hence by using the notations of Theorem 1, we have $b_1(t) = 0,\ b_2(t) = 1,\ g(s) = \frac{1}{\alpha(1+s^2)}$. Then by Theorem 1, we conclude that the Hammerstein integral equation (2.1) has a unique bounded solution $y_\alpha(.) \in C([0, \infty))$ if $\alpha > \frac{\pi}{2}$.

## 3. Existence of bounded continuous solution

In the following, we assume that $u(t, x) = v(t)$ in Equation (1.1).

Theorem 2. *Suppose that the functions $v(.)$, $k(.,.)$, $F(.,.)$ satisfy the following conditions:*

*1. $v(.)$ is bounded and continuous on $\Omega$,*

*2. $F(.,.)$ is a measurable function, continuous at the second variable, and satisfies one of the following two conditions:*

*(i) F(., .) is nonincreasing at the second variable, or*

*(ii) F(., .) is nondecreasing at the second variable, and for all $t \in \Omega$,*

$$v(t) + \int_\Omega k(t,s) F(s, 0)\, ds \geq 0.$$

*Moreover F satisfies*

$$|F(t, x)| \leq b_1(t) b_2(x),\ \textit{fo all } (t, x) \in \Omega \times \mathbf{R},$$

*where, $b_1 : \Omega \to \mathbf{R}$ and $b_2 : \mathbf{R} \to \mathbf{R}_+$ are measurable functions.*

*3. $k(.,.)$ is a nonnegative measurable function and continuous at the first variable; moreover there exists $g \in L^1(\Omega)$ such that for all $t \in \Omega$, $k(t, s) b_1(s) \leq g(s)$ a.e.,*

*4. there exists $a > 0$ satisfying $\|v\|_\infty + \|g\|_1 \sup_{t\in[0,a]} b_2(t) \leq a$.*

*Then the Hammerstein integral equation (1.1) has a bounded solution in $C(\Omega)$.*

*Proof.* Define inductively the sequence $y_{n+1}(t) = Ty_n(t)$, $n \in \mathbf{N}$, $t \in \Omega$ such that

$$y_0(t) = \begin{cases} a, & \text{if } F(.,.) \text{ is nonincreasing at the second variable,} \\ 0, & \text{if } F(.,.) \text{ is nondecreasing at the second variable,} \end{cases}$$

where the operator $T$ is defined from $C_a(\Omega)$ into itself as follows:

$$Ty(t) = v(t) + \int_\Omega k(t,s)F(s,y(s))\,ds.$$

We have, similar to Theorem 1, that operator $T$ is well defined, hence the sequence $\{y_n(t)\}$ is well defined, and by induction, the sequence $\{y_n(t)\}$ is either nonincreasing for all $t \in \Omega$, or nondecreasing for all $t \in \Omega$, so, it converges to some $y(t) \in \mathbf{R}$ for all $t \in \Omega$.

Then, by using Lebesgue's Dominated Convergence Theorem, we get,

$$y(t) = v(t) + \lim_{n\to\infty} \int_\Omega k(t,s)F(s,y_n(s))\,ds = v(t) + \int_\Omega k(t,s)F(s,y(s))\,ds,$$

Now, to show that $y(.) \in C_a(\Omega)$, let $(t_n)$ be a sequence converging to $t$. Then,

$$|y(t_n) - y(t)| \le |v(t_n) - v(t)| + b_2(a) \int_\Omega |k(t_n,s) - k(t,s)|\, b_1(s)\,ds,$$

hence by Lebesgue's Dominated Convergence Theorem, we get $y(.) \in C_a(\Omega)$. Then, (1.1) has a bounded solution $y(.) \in C(\Omega)$. ∎

Example 2. Consider the Hammerstein integral equation (2.1) in Example 1, such that the function $h(.)$ is nonnegative on $[0,\infty)$, hence for $b_1(t) = 1$, $b_2(t) = \ln(1+t^2)$, $g(t) = \frac{1}{\alpha(1+t^2)}$ in Theorem 2. It can be shown easily that for all $\alpha > 0$ there exists $a > 0$ such that

$$\|h\|_\infty + \frac{\pi}{2\alpha} \sup_{t\in[0,a]} b_2(t) \le a,$$

then by Theorem 2, we conclude that (2.1) has a bounded solution $y_\alpha(.) \in C([0,\infty))$ for all $\alpha > 0$.

In the following, we assume that $\Omega$ is compact, and $u(t,s) = v(t)$, for all $(t,x) \in \Omega \times \mathbf{R}$.

In Theorem 3, the main tool in the existence proof of a solution of (1.1) is Schauder's fixed point theorem.

Theorem 3. *Suppose that the functions $v(.)$, $k(.,.)$, $F(.,.)$ satisfy the following conditions:*

*1. $v(.)$ is continuous on $\Omega$,*

*2. $F(.,.)$ is continuous on $\Omega \times \mathbf{R}$, and satisfies:*

$$|F(t,x)| \le b_1(t)b_2(x), \text{ for all } (t,x) \in \Omega \times \mathbf{R},$$

*where, $b_1 \in L^1(\Omega)$, and $b_2 : \mathbf{R} \to \mathbf{R}_+$ is a measurable function,*

*3. $k(.,.)$ is bounded on $\Omega \times \Omega$ and continuous at the first variable,*

*4. there exists $a > 0$ satisfying $\|v\|_\infty + b\sup_{t\in[0,a]} b_2(t) \le a$, where*

$$b = \sup_{t\in\Omega} \int_\Omega k(t,s)b_1(s)\,ds.$$

*Then the Hammerstein integral equation (1.1) has a bounded solution in $C(\Omega)$.*

*Proof.* Define the operator $T$ from $C_a(\Omega)$ into itself as follows:

$$y(t) = u(t, y(t)) + \int_\Omega k(t,s)F(s, y(s))\, ds, \quad t \in \Omega,$$

then, similar to Theorem 1, the operator $T$ is well defined.

The proof is divided into two steps:

Step 1: The operator $T$ is continuous on $(C_a(\Omega), \|.\|_\infty)$. Let $\{y_n(.)\} \subset C_a(\Omega)$ be a sequence converging to $y(.) \in C_a(\Omega)$. Let $\epsilon > 0$, then, from the uniform continuity of $F$ on $\Omega \times [-a, a]$, there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$,

$$\sup_{t\in\Omega} |F(t, y_n(t)) - F(t, y(t))| \leq \frac{\epsilon}{\left(1 + \lambda(\Omega) \sup_{t,s\in\Omega} |k(t,s)|\right)}.$$

Hence, for all $n \geq n_0$, and for all $t \in \Omega$, we have:

$$|Ty_n(t) - Ty(t)| \leq \sup_{t,s\in\Omega} |k(t,s)| \int_\Omega |F(s, y_n(s)) - F(s, y(s))|\, ds \leq \epsilon,$$

and $Ty_n(.)$ converges to $Ty(.)$ in $(C_a(\Omega), \|.\|_\infty)$, and the operator $T$ is continuous.

Step 2: T is totally bounded, by Ascoli-Arzelà's theorem, and we need only to prove that $F = \{T\ y;\ y \in C_a(\Omega)\}$ is equicontinuous. Let $y \in C_a(\Omega)$, and let $t, l \in \Omega$. We have

$$|Ty(t) - Ty(l)| \leq |v(t) - v(l)| + \sup_{x\in[0,\ a]} b_2(x) \int_\Omega |k(t,s) - k(l,s)|\, b_1(s)\, ds,$$

so, for each $\epsilon > 0$, there exists $\delta > 0$ such that if $t, l \in \Omega$ with $|t - l| < \delta$, then $|Ty(t) - Ty(l)| < \epsilon$ for all $y \in C_a(\Omega)$. Then $F$ is equicontinuous, and the proof of Theorem 5 follows from Schauder's fixed point theorem. ■

Remark 1. It is obvious that if $b_2$ is bounded in Theorem 2 (Theorem 3), then the condition 4 in Theorem 2 (Theorem 3) holds.

## 4. Application

Theorem 1, and Theorem 3 immediately yield existence results for two point boundary values problem:

$$\begin{cases} -y''(t) = F(t, y(t)) \text{ on } [0, T] \\ y(0) = \alpha,\ y(T) = \beta \end{cases},\ y \in C^2([0, T]). \tag{4.1}$$

This problem can be written as a Hammerstein integral equation:

$$y(t) = h(t) + \int_0^T k(t,s)F(s, y(s))\, ds, \quad y \in C([0, T]),$$

where $h(t) = \alpha + \frac{(\beta-\alpha)}{T}t$, and $k(t,s) = \begin{cases} \frac{(T-t)}{T}s, & 0 \leq s \leq t \leq T \\ \frac{(T-s)}{T}t, & 0 \leq t \leq s \leq T. \end{cases}$

The following result is directly yielded by applying Theorem 1.

THEOREM 4. *Suppose that $F$ is measurable on $[0,T]\times\mathbf{R}$, $F(t,0)$ is bounded on $[0,T]$ and $F$ satisfies,*

$$|F(t,x)-F(t,y)| \le b_2(t)\,|x-y|\,,$$

*where $b_2:[0,T]\to\mathbf{R}^+$ is a measurable function, and satisfies:*

$$\int_0^T (T-s)s\ b_2(s)\,ds \le T$$

*Then (4.1) has a unique solution in $C^2([0,T])$.*

Also, by applying Theorem 3, the following result takes place:

THEOREM 5. *Suppose that*

*(i) $F$ is continuous on $[0,T]\times\mathbf{R}$, such that*

$$|F(t,x)| \le b_1(t)b_2(x),\ \textit{for all}\ (t,x)\in[0,T]\times\mathbf{R}$$

*where $b_1:[0,T]\to\mathbf{R}_+$ and $b_2:\mathbf{R}\to\mathbf{R}_+$ are measurable functions such that $\|b_1\|_{,\infty}<\infty$*

*(ii) there exists $c>0$ such that* $\max\{|\alpha|,|\beta|\}+\frac{T^2\|b_1\|_\infty}{8}\sup_{t\in[a,c]} b_2(t)\le c$.

*Then (4.1) has a solution in $C^2([0,T])$.*

As a special case, if $F(t,x)=f(t)$ for all $(t,x)\in[0,T]\times\mathbf{R}$, then, we have the following result:

COROLLARY 1. *Suppose that*

*(i) $f:[0,T]\to\mathbf{R}$ is continuous,*

$$|F(t,x)| \le b_1(t)b_2(x),\ \textit{for all}\ (t,x)\in[0,T]\times\mathbf{R}$$

*where $b_1:[0,T]\to\mathbf{R}_+$ and $b_2:\mathbf{R}\to\mathbf{R}_+$ are measurable functions such that $\|b_1\|_\infty<\infty$,*

*(ii) there exists $c>0$ such that* $\max\{|\alpha|,|\beta|\}+\frac{T^2}{8}\sup_{t\in[a,c]} f(t)\le c$.

*Then (4.1) has a solution in $C^2([0,T])$.*

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