ON SLIGHT HOMOGENEOUS AND COUNTABLE
DENSE HOMOGENEOUS SPACES

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Abstract. As two separated concepts connectedness and homogeneity are generalized by slight homogeneity. Sum theorems and product theorems regarding slight homogeneous spaces are obtained. Many results concerning slight homogeneous components are given. Also, SCDH spaces generalize CDH spaces. It is proved in an extremally disconnected SCDH space that, all slightly homogenous components are clopen and SCDH subspaces. Many other results and many examples and counter-examples concerning slight homogeneity and SCDH spaces are obtained.

1. Introduction

Throughout this paper by a space we mean a topological space. A space (X, τ) is homogeneous if for any two points $x, y \in X$ there exists an autohomeomorphism f on (X, τ) such that $f(x) = y$. Let $\smile$ be a relation defined on X by $x \smile y$ if there is a homeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(x) = y$. This relation turns out to be an equivalence relation on X whose equivalence classes C_x will be called homogeneous components determined by $x \in X$. Homogeneous components are preserved under homeomorphisms and are indeed homogeneous subspaces of X. Homogeneous components have played a vital role in homogeneity research, see [4, 7, 9]. The concept of homogeneous spaces is a well known concept in topology; there are some papers in the literature where the definition of a homogeneous space is modified in the manner that the role of homeomorphisms is given to pre-homeomorphisms ([1] and [2]) or semihomeomorphisms [3]. The first purpose of the present paper, is to use slight homeomorphisms to introduce slightly homogeneous spaces and slightly homogeneous components. The important thing is that slight homogeneity generalizes both connectedness and homogeneity as two very important concepts in topology.

Brouwer [5] in his development of dimension theory shows that if A and B are two countable dense subsets of the n-dimensional Euclidean space $\mathbb{R}^n$, then there is an autohomeomorphism on $\mathbb{R}^n$ that takes A to B. Fort [11] proves that the

Hilbert cube has this property and R.B. Bennett [4] uses the term countable dense homogeneous (CDH) to denote a separable space provided it has this property. Afterwards, various results concerning CDH were obtained in [7, 8, 9, 10, 12, 16] and others. The second purpose of the present paper is to introduce slight countable dense homogeneity as a generalization of countable dense homogeneity.

Throughout this paper, for any set X, $|X|$ will denote the cardinality of X. For a subset A of a space (X, τ) we write $\overline{A}$, $Int(A)$, $Ext(A)$, and $Bd(A)$ to denote the closure of A, the interior of A, the exterior of A, and the boundary of A, respectively. Moreover, for a subset A of $\mathbb{R}^n$ we write (A, τ_u) to denote the subspace topology on A relative to the usual topology. For a non-empty set X, τ_{disc} and τ_{ind} will denote, respectively, the discrete and the indiscrete topology on X. For spaces (X, τ) and (Y, σ), we denote the product topology on $X \times Y$ by τ_{prod}.

In 1997, A.R. Singal and R.C. Jain [13] introduced the concept of slight continuity as follows.

DEFINITION 1.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be slightly continuous if for any point $x \in X$ and any clopen neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.

PROPOSITION 1.2. [13] *For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent.*

(i) f is slightly continuous.

(ii) The inverse image of every clopen subset of (Y, σ) is an open subset of (X, τ).

(iii) The inverse image of every clopen subset of (Y, σ) is a clopen subset of (X, τ).

DEFINITION 1.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be a slight homeomorphism if it is a bijection and both f, f^{-1} are slightly continuous. The spaces (X, τ) and (Y, σ) are called slightly homeomorphic if and only if there exists a slight homeomorphism $f : (X, \tau) \to (Y, \sigma)$.

From now on we denote the group of all slight homeomorphisms from a space (X, τ) onto itself by $SH(X, \tau)$.

DEFINITION 1.4. A property of a space (X, τ) is called a slightly topological property if and only if every space (Y, σ) slightly homeomorphic to (X, τ) also has the same property.

DEFINITION 1.5. [14] A space (X, τ) is extremally disconnected if for each $O \in \tau, \overline{O} \in \tau$.

DEFINITION 1.6. [15] A space (X, τ) is said to be C-compact if every cover of any closed set in (X, τ) by open sets of (X, τ) has a finite subfamily whose union is dense in the closed set.

PROPOSITION 1.7. [13] *The slightly continuous image of a connected space is connected.*

PROPOSITION 1.8. *If $f : (X, \tau) \to (Y, \sigma)$ is a bijective function such that (X, τ) and (Y, σ) are both connected, then f is a slight homeomorphism.*

PROPOSITION 1.9. [13] *Every continuous function is slightly continuous.*

COROLLARY 1.10. *Every homeomorphism is a slight homeomorphism.*

PROPOSITION 1.11. [13] *Let $f : (X, \tau) \to (Y, \sigma)$ be a slightly continuous function. Then*

(i) If (Y, σ) is an extremally disconnected C-compact Hausdorff space, then f is continuous.

(ii) If (X, τ) is locally connected and (Y, σ) is an extremally disconnected Hausdorff space, then f is continuous.

(iii) If (Y, σ) is a zero-dimensional space, then f is continuous.

COROLLARY 1.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a slight homeomorphism. Then*

(i) If (X, τ) and (Y, σ) are both C-compact, extremally disconnected Hausdorff spaces, then f is a homeomorphism.

(ii) If (X, τ) and (Y, σ) are both locally connected, extremally disconnected Hausdorff spaces, then f is a homeomorphism.

(iii) If (X, τ) and (Y, σ) are zero-dimensional spaces, then f is a homeomorphism.

PROPOSITION 1.13. *Let (X, τ) be a space and A be a clopen subset of X. If $f_1 \in SH(A, \tau_A)$ and $f_2 \in SH(X - A, \tau_{X-A})$ and $f : (X, \tau) \to (X, \tau)$ is defined by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in X - A \end{cases}$$

then $f \in SH(X, \tau)$.

Proof. To see that f is slightly continuous, let U be a clopen subset of X. Then $f^{-1}(U) = f^{-1}(U \cap A) \cup f^{-1}(U \cap (X - A))$ and hence $f^{-1}(U)$ is clopen. Therefore, f is slightly continuous. Similarly, we can show that f^{-1} is slightly continuous. Since f is a bijection, the proof is completed. ∎

PROPOSITION 1.14. *If $f : (X, \tau) \to (Y, \sigma)$ is a slight homeomorphism and A is a clopen subset of X, then the restriction function on A, $f|_A : (A, (\tau)_A) \to (f(A), (\sigma)_{f(A)})$ is a slight homeomorphism.*

Proof. To show that $f|_A$ is slightly continuous, let U be a clopen subset of $f(A)$. Then U is clopen in Y and so $f^{-1}(U)$ is clopen in X, and hence in A. Thus, $(f|_A)$ is slightly continuous. In the same way we can show that $(f|_A)^{-1}$ is slightly

continuous. Since $f|_A : A \rightarrow f(A)$ is a bijection, it follows that $f|_A$ is a slight homeomorphism. ∎

PROPOSITION 1.15. [4] *Every connected CDH space is homogeneous.*

PROPOSITION 1.16. *Let (X, τ) be a space with $|\tau| < \infty$. Then (X, τ) contains a connected clopen set.*

Proof. If (X, τ) is connected, then we are done. If (X, τ) is disconnected, then there exists a non-empty clopen set $U_1 \subset X$. If U_1 is connected then we are done, if not, then there exists a non-empty clopen set $U_2 \subset X$ such that $U_2 \subset U_1$. Since $|\tau| < \infty$, if we go on this way we will reach a non-empty clopen set, say U_n, such that the only non-empty clopen subset of X that is contained in U_n is U_n itself. ∎

2. Slightly homogeneous spaces

DEFINITION 2.1. A space (X, τ) is said to be slightly homogeneous if for any two points $x, y \in X$, there exists $f \in SH(X, \tau)$ such that $f(x) = y$.

As a simple example of a space (X, τ) that is not slightly homogeneous take $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b, c\}\}$.

THEOREM 2.2. *Every connected space is slightly homogeneous.*

Proof. Let (X, τ) be a connected space and let $x, y \in X$. Define $f : (X, \tau) \rightarrow (X, \tau)$ by $f(x) = y$, $f(y) = x$ and $f(t) = t$ for all $t \in X - \{x, y\}$. Then by Proposition 1.8, it follows that $f \in SH(X, \tau)$. Thus, (X, τ) is slightly homogeneous. ∎

THEOREM 2.3. *Every homogeneous space is slightly homogeneous.*

Proof. Corollary 1.10. ∎

REMARK 2.4. It is known (see [17]) that the Sorgenfrey line $(\mathbb{R}, \tau_s)$ is homogeneous and so by Theorem 2.3, it is slightly homogeneous. Since $(\mathbb{R}, \tau_s)$ is disconnected, we conclude that the converse of Theorem 2.2 is not true in general.

Any non-homogeneous connected space is an example of a slightly homogeneous space which is not homogeneous, examples include the space $[0, 1)$, the unit interval or the letter T (all with the usual topology). Also, as an example of a non-homogeneous connected space, take the space $(\mathbb{N}, \tau)$ where $\tau = \{\varnothing\} \cup \{\{n, n + 1, \ldots\} : n \in \mathbb{N}\}$, Consider the set of natural numbers $\mathbb{N}$ with the topology $\tau = \{\varnothing\} \cup \{\{n, n + 1, \ldots\} : n \in \mathbb{N}\}$, if $(\mathbb{N}, \tau)$ is homogeneous, then there is an autohomeomorphism f on $(\mathbb{N}, \tau)$ such that $f(1) = 2$, and then $f(\mathbb{N} - \{1\}) = \mathbb{N} - \{2\}$. This is impossible because $\mathbb{N} - \{1\}$ is open while $\mathbb{N} - \{2\}$ is not open.

THEOREM 2.5. *Every slightly homogeneous extremally disconnected C-compact Hausdorff space is homogeneous.*

Proof. Corollary 1.12 (i). ∎

In general, the condition that the space is Hausdorff in the last result cannot be removed as we shall see in the following example.

EXAMPLE 2.6. Consider the space $(\mathbb{N}, \tau)$ where $\tau = \{\varnothing\} \cup \{\{n, n + 1, \dots\} : n \in \mathbb{N}\}$. Since the closure of every open set in $(\mathbb{N}, \tau)$ is extremally disconnected. To see that this space is C-compact, let F be a non-empty closed subset of $(\mathbb{N}, \tau)$ and let $\mathcal{U}$ be an open cover of F consisting of open subsets of $\mathbb{N}$. Since F is a non-empty closed subset of $\mathbb{N}$, there exists $m \in \mathbb{N}$ such that $F = \{1, 2, \dots, m\}$ or $F = \mathbb{N}$. Take $U_0 \in \mathcal{U}$ such that $1 \in U_0$. Then $\{U_0\}$ is a finite subfamily of $\mathcal{U}$ whose union is dense in F. Therefore, $(\mathbb{N}, \tau)$ is C-compact. Also, $(\mathbb{N}, \tau)$ is a slightly homogeneous space that is not homogeneous.

THEOREM 2.7. *Every slightly homogeneous locally connected extremally disconnected Hausdorff space is homogeneous.*

Proof. Corollary 1.12 (ii). ∎

The space $(\mathbb{N}, \tau)$ in Example 2.6 is a locally connected, extremally disconnected slightly homogeneous space that is not homogeneous, this proves that the condition 'Hausdorff' in Theorem 2.7 cannot be dropped. Also, the space $([0, 1), \tau_u)$ is a locally connected, Hausdorff, and slightly homogeneous space that is not homogeneous which proves that the condition 'extremally disconnected' in Theorem 2.1.7 cannot be dropped.

THEOREM 2.8. *Let (X, τ) be a zero-dimensional slightly homogeneous space. Then (X, τ) is homogeneous.*

Proof. Corollary 1.12 (iii). ∎

THEOREM 2.9. *Being "slightly homogeneous" is a slightly topological property.*

Proof. Let (X, τ) be a slightly homogeneous space and let (Y, σ) be any space slightly homeomorphic to (X, τ). Let $y_1, y_2 \in Y$. Let $f : (X, \tau) \to (Y, \sigma)$ be a slight homeomorphism and let $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since (X, τ) is slightly homogeneous, there exists a slight homeomorphism $g : (X, \tau) \to (X, \tau)$ such that $g(x_1) = x_2$. Define $h : (Y, \sigma) \to (Y, \sigma)$ by $h(y) = (f \circ g \circ f^{-1})(y)$. Then h is a slight homeomorphism and $h(y_1) = y_2$. Thus, (Y, σ) is slightly homogeneous. ∎

COROLLARY 2.10. *Being "slightly homogenous" is a topological property.*

It is known that the product of two homogeneous spaces is homogeneous. For slightly homogeneous spaces we raise the following question.

QUESTION 2.11. Is it true that the product of two slightly homogeneous spaces is a slightly homogeneous space?

Since the product of two homogeneous spaces is homogeneous, by Theorem 2.3, it follows that the answer of Question 2.11 is yes when the two spaces are homogeneous. Also, by the well known fact that the product of two connected

spaces is connected and Theorem 2.2, the answer of Question 2.11 is yes when the two spaces are connected.

The following result answers Question 2.11, partially.

THEOREM 2.12. *The product of two extremally disconnected slightly homogeneous spaces is slightly homogeneous.*

Proof. Let (X, τ) and (Y, σ) be two extremally disconnected slightly homogeneous spaces, and let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Since (X, τ) and (Y, σ) are slightly homogeneous, there exist $f \in SH(X, \tau)$ and $g \in SH(Y, \sigma)$ such that $f(x_1) = x_2$, $g(y_1) = y_2$. Define $h : (X \times Y, \tau_{prod}) \rightarrow (X \times Y, \tau_{prod})$ by $h(x, y) = (f(x), g(y))$. To see that h is slightly continuous, let $(x, y) \in X \times Y$ and let W be a clopen subset of $(X \times Y, \tau_{prod})$ such that $h(x, y) \in W$. Since W is open in $X \times Y$, there are open sets U_1 and U_2 in X and Y, respectively such that $h(x, y) = (f(x), g(y)) \in U_1 \times U_2 \subseteq W$. So $f(x) \in \overline{U_1}$ and $g(y) \in \overline{U_2}$. Since (X, τ) and (Y, σ) are extremally disconnected spaces, $\overline{U_1}$ and $\overline{U_2}$ are clopen subsets of X and Y, respectively. Moreover, since f and g are slightly continuous, there exist $V_1 \in \tau$ and $V_2 \in \sigma$ such that $x \in V_1$, $y \in V_2$ and $f(V_1) \subseteq \overline{U_1}$, $g(V_2) \subseteq \overline{U_2}$. Therefore, $(x, y) \in V_1 \times V_2$, $V_1 \times V_2$ is open in $X \times Y$ and $h(V_1 \times V_2) \subseteq \overline{U_1} \times \overline{U_2} = \overline{U_1 \times U_2} \subseteq \overline{W} = W$. Hence, h is slightly continuous. Similarly, we can show that $h^{-1} : (X \times Y, \tau_{prod}) \rightarrow (X \times Y, \tau_{prod}), h^{-1}(x, y) = (f^{-1}(x), g^{-1}(y))$ is slightly continuous. Since h is clearly a bijection and $h(x_1, y_1) = (x_2, y_2)$, the proof is ended. ∎

DEFINITION 2.13. [6] Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a collection of spaces such that $X_\alpha \cap X_\beta = \varnothing$ for all $\alpha \neq \beta$. Let $X = \bigcup_{\alpha \in \Lambda} X_\alpha$ be topologized by $\{U \subseteq X : U \cap X_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda\}$. Then (X, τ) is called the disjoint sum of the spaces $(X_\alpha, \tau_\alpha), \alpha \in \Lambda$.

THEOREM 2.14. *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a collection of slightly homogeneous spaces with $X_\alpha \cap X_\beta = \varnothing$ and (X_α, τ_α) is slightly homeomorphic to (X_β, τ_β) for all $\alpha, \beta \in \Lambda$. Then the disjoint sum of the spaces $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ is slightly homogenous.*

Proof. Let $x, y \in \bigcup_{\alpha \in \Lambda} X_\alpha$. Then we have two cases.

Case 1. $x, y \in X_\beta$ for some $\beta \in \Lambda$. Since (X_β, τ_β) is slightly homogeneous, there exists $f_\beta \in SH(X_\beta, \tau_\beta)$ such that $f_\beta(x) = y$. Define h from $\bigcup_{\alpha \in \Lambda} X_\alpha$ onto $\bigcup_{\alpha \in \Lambda} X_\alpha$ by

$$h(t) = \begin{cases} f_\beta(t) & \text{if } t \in X_\beta \\ t & \text{if } t \in X - X_\beta \end{cases}$$

Then $h(x) = y$. Also, by Proposition 1.13, h is a slight homeomorphism.

Case 2. $x \in X_\gamma$ and $y \in X_\beta$ for $\gamma, \beta \in \Lambda$, with $\gamma \neq \beta$. Since (X_γ, τ_γ) is slightly homeomorphic to (X_β, τ_β), there exists a slight homeomorphism $g : (X_\gamma, \tau_\gamma) \rightarrow (X_\beta, \tau_\beta)$. Since (X_β, τ_β) is slightly homogeneous, there exists $h \in SH(X_\beta, \tau_\beta)$ such

that $h(g(x)) = y$. Define f on X by

$$f(t) = \begin{cases} (h \circ g)(t) & \text{if } t \in X_\gamma \\ (h \circ g)^{-1}(t) & \text{if } t \in X_\beta \\ t & \text{if } t \in X - (X_\gamma \cup X_\beta) \end{cases}$$

Then f is a bijection and $f(x) = y$. Also, by Proposition 1.13, f is a slight homeomorphism. ∎

In Theorem 2.14, the condition on the spaces to be slightly homeomorphic cannot be dropped as we will see in the following example.

EXAMPLE 2.15. The spaces $((0,1), \tau_u)$ and $(\{3\}, \tau_u)$ are connected and so by Theorem 2.2 they are slightly homogeneous. The disjoint sum of $((0,1), \tau_u)$ and $(\{3\}, \tau_u)$ is not slightly homogeneous because there is no slight homeomorphism $f : ((0,1) \cup \{3\}), \tau_u) \to ((0,1) \cup \{3\}), \tau_u)$ such that $f(3) = \frac{1}{2}$.

THEOREM 2.16. *Let (X, τ) be a space which contains a non-empty clopen connected subset. Then (X, τ) is slightly homogeneous iff (X, τ) is the disjoint sum of connected spaces all of which are slightly homeomorphic to one another.*

Proof. Let (X, τ) be a slightly homogeneous. Let C denote a clopen connected subset of X. Now if $f, g \in SH(X, \tau)$, then by Proposition 1.7, either $f(C) = g(C)$ or $f(C) \cap g(C) = \varnothing$. Since $SH(X, \tau)$ acts transitively on X, X will be the disjoint union of $f(C)$. Moreover, since $f(C)$ for $f \in SH(X, \tau)$ is an open cover of X, it follows that a set is open in X if and only if its intersection with each of the $f(C)$ is open.

The converse follows from Theorems 2.2 and 2.14. ∎

COROLLARY 2.17. *Let (X, τ) be a space with $|\tau| < \infty$. Then (X, τ) is slightly homogeneous if and only if (X, τ) is the disjoint sum of mutually slightly homeomorphic connected spaces.*

Proof. By Proposition 1.16, we obtain a non-empty clopen connected subset. Apply Theorem 2.16 to get the result. ∎

COROLLARY 2.18. *Let (X, τ) be a space with $|X| < \infty$. Then (X, τ) is slightly homogeneous if and only if (X, τ) is the disjoint sum of mutually slightly homeomorphic connected spaces.*

DEFINITION 2.19. Let (X, τ) be a space. We define the equivalence relation $\widetilde{s}$ on X as follows. For $x_1, x_2 \in X$, $x_1 \widetilde{s} x_2$ if there exists $f \in SH(X, \tau)$ such that $f(x_1) = x_2$. A subset of a space (X, τ), which has the form $SC_x = \{y \in X : x \widetilde{s} y\}$ is called the slightly homogeneous component of X at x.

THEOREM 2.20. *Let (X, τ) be a space and let SC_x be a slightly homogeneous component. Then*

1) (X, τ) is slightly homogeneous if and only if it has exactly one slightly homogeneous component.

2) $C_x \subseteq SC_x$ for all $x \in X$.

3) If $f \in SH(X, \tau)$, then $f(SC_x) = SC_x$.

4) If SC_x is clopen, then the subspace (SC_x, τ_{SC_x}) is slightly homogeneous.

5) If SC_x contains a non-empty clopen set U, then it is a union of clopen sets.

6) If B is a slightly homogeneous clopen subspace of X such that $B \cap SC_x \neq \varnothing$, then $B \subseteq SC_x$.

Proof. 1) Obvious.

2) Corollary 1.10.

3) Follows directly from the definition.

4) Let $x_1, x_2 \in SC_x$. Then there exist $f_1, f_2 \in SH(X, \tau)$ such that $f_1(x_1) = x$ and $f_2(x) = x_2$. Define $f : (X, \tau) \to (X, \tau)$ by $f = f_2 \circ f_1$. Then $f \in SH(X, \tau)$ and by Proposition 1.14 and part (3), it follows that $f|_{SC_x} \in SH(SC_x, \tau_{SC_x})$ with $(f|_{SC_x})(x_1) = x_2$.

5) Let $y \in SC_x$. Then there exists $f \in SH(X, \tau)$ such that $f(x) = y$, we are going to find a clopen set that contains y and is contained in SC_x. If $x \in U$, then we are done. If $x \notin U$, then for a point $z \in U$ there exists $g \in SH(X, \tau)$ such that $g(z) = x$. Define $h : (X, \tau) \to (X, \tau)$ by $h = f \circ g$. Then $h \in SH(X, \tau)$ with $h(z) = y$. Therefore, by part (3) $y \in h(U) \subseteq h(SC_x) = SC_x$. Since U is clopen, $h(U)$ is clopen. This completes the proof.

6) Let $b \in B$. Take $z \in B \cap SC_x$. Since B is slightly homogeneous, there exists $h \in SH(B, \tau_B)$ such that $h(b) = z$. Define $f : (X, \tau) \to (X, \tau)$ by

$$f(t) = \begin{cases} h(t) & \text{if } t \in B \\ t & \text{if } t \in X - B \end{cases}$$

By Proposition 1.13, it follows that $f \in SH(X, \tau)$. Since $f(b) = z$, $f(b) \in SC_x$ and so $b \in f^{-1}(SC_x) = SC_x$. ∎

The following example shows that the inclusion in Theorem 2.20 (2) is not equality in general.

EXAMPLE 2.21. Consider the space $([0, 1], \tau_u)$. Since this space is connected, by Theorem 2.2 it is slightly homogeneous, and by Theorem 2.20 (1), it follows that $SC_x = X$ for all $x \in X$. On the other hand, it is not difficult to see that $C_0 = C_1 = \{0, 1\}$ and $C_x = (0, 1)$ for all $x \in (0, 1)$.

3. Slightly countable dense homogeneous spaces

DEFINITION 3.1. A space (X, τ) is said to be slightly countable dense homogeneous (SCDH) if

(i) (X, τ) is separable.

(ii) If A and B are countable dense subsets of (X, τ), then there is $h \in SH(X, \tau)$ such that $h(A) = B$.

THEOREM 3.2. *If (X, τ) is SCDH such that X is infinite, then every dense subset of X is infinite.*

Proof. It is sufficient to see that every proper dense subset of X is infinite. Suppose to the contrary that there exists a finite proper dense subset $A \subseteq X$. Take $x_0 \in X - A$ and let $B = A \cup \{x_0\}$. Then A and B are two countable dense sets in the SCDH space (X, τ), and so there exists $h \in SH(X, \tau)$ such that $h(A) = B$, a contradiction. ∎

THEOREM 3.3. *Let (X, τ) be a connected separable space such that X is un-countable and all dense sets in X are infinite. Then (X, τ) is SCDH.*

Proof. Let A and B be two countable dense sets in X. Then A and B are denumerable. Take a bijection $h : A \to B$. Also since X is uncountable and A and B are countable then the sets $X - A$ and $X - B$ are equipotent; hence, there exists a bijection $g : X - A \to X - B$. Define $f : X \to X$ by $f = h \cup g$. Then f is a bijection; hence, by Proposition 1.8, it follows that $f \in SH(X, \tau)$ with $f(A) = B$. ∎

THEOREM 3.4. *Every CDH space is SCDH.*

Proof. Let (X, τ) be a CDH space. Then (X, τ) is separable. Let A and B be two countable dense sets in X. Then there exists a homeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(A) = B$. By Corollary 1.10, it follows that $f \in SH(X, \tau)$. Hence (X, τ) is SCDH. ∎

The converse of Theorem 3.4 is not true in general, in fact, the space $([0, 1), \tau_u)$ is a connected space that is not homogeneous, and hence by Proposition 1.15, it is not CDH. On the other hand, since it is an infinite T_1 space, then by Theorem 3.3, it is SCDH.

THEOREM 3.5. *Every zero-dimensional SCDH space is (X, τ) is CDH.*

Proof. Let (X, τ) be a zero-dimensional SCDH space. Then (X, τ) is separable. Let A and B be two countable dense sets. Then there exists $f \in SH(X, \tau)$ such that $f(A) = B$. By Corollary 1.12 (iii), f is a homeomorphism and hence (X, τ) is CDH. ∎

THEOREM 3.6. *Let (X, τ) be a space for which X is countable. Then the following are equivalent.*

(i) (X, τ) is CDH.

(ii) (X, τ) is SCDH.

(iii) $\tau = \tau_{disc}$.

Proof. (i) $\Rightarrow$ (ii) Theorem 3.4.

(ii) $\Rightarrow$ (iii) Let A be a countable dense subset of X. Then there exists $f \in SH(X, \tau)$ such that $f(A) = X = f(X)$. Thus, $A = X$. Therefore, the only dense subset of (X, τ) is X and so $\tau = \tau_{disc}$.

(iii) $\Rightarrow$ (i) Obvious. ∎

Applying the last theorem we get that the space $(\mathbb{Q}, \tau_u)$ is a zero-dimensional space that is not SCDH.

THEOREM 3.7. *Being "SCDH" is a topological property.*

Proof. Let (X, τ) be a SCDH space and let (Y, σ) be any space homeomorphic to (X, τ). Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism. Since being separable is a topological property, (Y, σ) is separable. Let S_1 and S_2 be two countable dense sets in Y, then $f^{-1}(S_1)$ and $f^{-1}(S_2)$ are countable dense sets in X. Since (X, τ) is SCDH, there exists $g \in SH(X, \tau)$ such that $g(f^{-1}(S_1)) = f^{-1}(S_2)$. Define $h : (Y, \sigma) \to (Y, \sigma)$ by $h(y) = (f \circ g \circ f^{-1})(y)$. Then g is a slight homeomorphism with $g(S_1) = S_2$. Therefore, (Y, σ) is SCDH. ∎

Being "SCDH" is not a slightly topological property as we will see in the following example.

EXAMPLE 3.8. Let $f : ([0, 1), \tau_{ind}) \to ([0, 1), \tau_u)$ be the identity function. Since $([0, 1), \tau_{ind})$ and $([0, 1), \tau_u)$ are connected, by Proposition 1.8, f is a slight homeomorphism. However, $([0, 1), \tau_u)$ is SCDH while $([0, 1), \tau_{ind})$ is not.

The following proposition will be used in the proof of the next lemma.

PROPOSITION 3.9. *Let (X, τ) be a space, $M \subseteq X, H = Ext(M)$, and G a dense subset of X. Then the set $K = (H \cap G) \cup (M \cap Ext(H))$ is a dense subset of X.*

Proof. Since H is open and G is dense, then $\overline{H \cap G} = \overline{H}$. Therefore, as $\overline{K} = (\overline{H \cap G}) \cup (\overline{M \cap Ext(H)})$, it is sufficient to see that $Ext(H) \subseteq \overline{M \cap Ext(H)}$. Let $x \in Ext(H)$ and $U \in \tau$ with $x \in U$. Since $x \in Ext(H) = Int(\overline{M}) \subseteq \overline{M}$ and $Ext(H) \cap U \in \tau$, then $U \cap (M \cap Ext(H)) = (Ext(H) \cap U) \cap M \neq \varnothing$. Therefore, $x \in \overline{M \cap Ext(H)}$. ∎

The following two lemmas will be used in the next main result.

LEMMA 3.10. *Let (X, τ) be an extremally disconnected SCDH space, G a countable dense subset of X, $x \in X$, $M = G \cap (X - SC_x)$, and $H = Ext(M)$. Then $\overline{H} \subseteq SC_x$.*

Proof. Suppose to the contrary that there exists $y \in \overline{H} \cap (X - SC_x)$. Put $K = (H \cap G) \cup (M \cap Ext(H))$. Since $K \subseteq G$, then K is countable. Also, by Proposition 3.9, K is dense. Therefore, K and $K \cup \{y\}$ are two countable dense sets in X and so there exists $f \in SH(X, \tau)$ such that $f(K \cup \{y\}) = K$. Since $y \in X - SC_x$, $f(y) \in X - SC_x$; thus, $f(y) \in K \cap (X - SC_x) \subseteq G \cap (X - SC_x) = M$; hence, $f(y) \notin Ext(M) = H$. Now since $f(y) \in K$ and $f(y) \notin H \cap G$ implies that $f(y) \in M \cap Ext(H)$, therefore, $f(y) \in Ext(H)$. Since (X, τ) is extremally disconnected and H is open, then $Ext(H)$ is clopen. Since f is slightly continuous, then there exists $O \in \tau$ such that $y \in O$ and $f(O) \subseteq Ext(H)$. Since $y \in O \cap \overline{H}$, $O \cap \overline{H} \in \tau$, and K is dense, then there exists $z \in O \cap \overline{H} \cap K$. If $z \in X - SC_x$,

then $z \in M$; this would imply that $z \notin H$, so as $z \in K$, this would mean that $z \in Ext(H)$, and this contradicts the fact that $z \in \overline{H}$. Thus, we must have $z \in SC_x$ which implies that $f(z) \in SC_x$. Since $z \in O$, then $f(z) \in f(O) \subseteq Ext(H)$. Therefore, $f(z) \in Ext(H) \cap SC_x \cap K$, which is a contradiction because $Ext(H) \cap SC_x \cap K = \varnothing$. ∎

LEMMA 3.11. *Let (X,τ) be SCDH, G a countable dense subset of X, and $x \in X$. Then the set $M = G \cap (X - SC_x)$ is not dense in X.*

Proof. Suppose to the contrary that M is dense. Then M and $M \cup \{x\}$ are two countable dense sets in X and so there exists $f \in SH(X,\tau)$ such that $f(M \cup \{x\}) = M$. Thus, $f(x) \in X - SC_x$. On the other hand, as $x \in SC_x$, then $f(x) \in f(SC_x) = SC_x$, a contradiction. ∎

THEOREM 3.12. *Let (X,τ) be an extremally disconnected SCDH space. Then all slightly homogenous components are clopen.*

Proof. It is sufficient to show that all slightly homogenous components are open. Let $x \in X$. Choose a countable dense set G of X and put $M = G \cap (X - SC_x)$ and $H = Ext(M)$. Since (X,τ) is extremally disconnected, then $\overline{H}$ is clopen. By Lemma 3.11, $\overline{H}$ is non-empty, also by Lemma 3.10, $\overline{H} \subseteq SC_x$. Therefore by Theorem 2.20 (5), it follows that SC_x is open. ∎

The following is an example of an extremally disconnected SCDH space (X,τ) for which there is some $x \in X$ such that $C_x \neq SC_x \neq X$.

EXAMPLE 3.13. Let $X = [0, \infty)$ and the topology τ on X is determined by the following conditions:

a) $\{0\}$ is clopen.

b) Sets of the form $[x, \infty)$ are open, where $x \in X$.

This space is SCDH and extremally disconnected but it is not CDH. Also, $C_1 = \{1\}$ but $SC_1 = (0, \infty)$, and hence $C_1 \neq SC_1 \neq X$.

Example 3.13 shows that there is a difference between extremally disconnected SCDH spaces and CDH spaces.

THEOREM 3.14. *Let (X,τ) be an extremally disconnected SCDH space. Then all slightly homogeneous components are SCDH.*

Proof. Let $x \in X$. According to Theorem 3.12, SC_x is open and hence separable. Let A and B be any two countable dense subsets of SC_x and let S be a countable dense subset of X. Let $A_1 = A \cup (S - SC_x)$ and $B_1 = B \cup (S - SC_x)$. If A_1 is not dense in X, then there exists a non-empty set $U \in \tau$ such that $U \cap (S - SC_x) = \varnothing$ and $U \cap A = \varnothing$, since S is dense in X and $U \cap (X - SC_x)$ is open in X, it follows that $U \cap (X - SC_x) = \varnothing$ and hence U is a non-empty open subset of SC_x, consequently $U \cap A \neq \varnothing$, contradicting the fact that $U \cap A = \varnothing$. Therefore, A_1 is a dense subset of X. Similarly, we can see that B_1 is a dense subset of X. Since A_1 and B_1 are countable, there exists $f \in SH(X,\tau)$ such

that $f(A_1) = B_1$. Now applying Theorem 2.20 (3) to conclude that $f(A) = B$. Define $g : (SC_x, \tau_{SC_x}) \to (SC_x, \tau_{SC_x})$ to be the restriction of f on SC_x. Then by Theorem 3.12 and Proposition 1.14, it follows that g is a slight homeomorphism. Since $g(A) = f(A) = B$, the proof finished. ∎

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