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ON HAUSDORFFNESS AND COMPACTNESS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract. Without directly involving the role of points, we introduce and study the notions of fuzzy λ-Hausdorff spaces and fuzzy μ-compact spaces. A characterization of a map f from a fuzzy λ-Hausdorff space X to a fuzzy μ-compact space Y, where $\lambda = f^{-1}(\mu)$, to be fuzzy λ-continuous is obtained, which puts such a characterization for the continuity of f in ordinary topological setting, for fuzzy topological spaces. These notions and results have been formulated for intuitionistic fuzzy topological spaces also.

1. Introduction

Since the introduction of the notion of fuzzy sets by Zadeh [16], several mathematical structures have been studied in fuzzy setting. Among them are fuzzy topological spaces [4], fuzzy proximity [13], fuzzy groups [15], and fuzzy vector spaces [12]. Atanassov [1] generalized fuzzy sets into the concept termed *intuitionistic fuzzy sets* by realizing the pair (X, λ), where λ is a fuzzy set of X to be a triple (X, λ, λ'). Indeed, an *intuitionistic fuzzy set* (IFS) A of X is an object having the structure $A = \langle X, \mu_A, \nu_A\rangle$, where *fuzzy sets* $\mu_A \in I^X$ and $\nu_A \in I^X$ denote the *degree of membership* and the *degree of non-membership* of each element $x \in X$ to A, respectively, such that for each $x \in X$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

In [1], notions such as the containment, equality, complement, union and intersection in the case of IFSs have been extended in a natural way. The IFS $0_\sim$ is denoted by $\langle X, \underline{0}, \underline{1}\rangle$, and the IFS $1_\sim$ by $\langle X, \underline{1}, \underline{0}\rangle$, where $\underline{\alpha}$ (or, simply α) denotes the constant fuzzy set mapping whole of the set X to $\alpha \in [0, 1]$.

To remain as self-contained as possible, we describe below the notions of an image and a pre-image of an IFS under a map f from a set X to a set Y.

(i) If $A = \langle X, \mu_A, \nu_A\rangle$ is an IFS of X, then the *image of* A is an IFS of Y defined by

$$f(A) = \langle Y, f(\mu_A), f(\nu_A)\rangle\,,$$

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where

$$f(\mu_A)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \inf_{x\in f^{-1}(y)} \nu_A(x), & \text{if } f^{-1}(y) \neq \phi, \\ 1, & \text{otherwise,} \end{cases}$$

for each $y \in Y$.

(ii) If $B = \langle Y, \mu_B, \nu_B\rangle$ is an IFS of Y, then the *inverse image of* B is an IFS of X defined by

$$f^{-1}(B) = \left\langle X, f^{-1}(\mu_B), f^{-1}(\nu_B)\right\rangle.$$

Definition 1.1. [5] An *intuitionistic fuzzy topology* on X is a family $i\tau X$ of IFSs of X satisfying the following axioms:

(i) $0_\sim$, $1_\sim \in i\tau X$,

(ii) $G_1 \cap G_2 \in i\tau X$, for any $G_1, G_2 \in i\tau X$, and

(iii) $\bigcup G_\alpha \in i\tau X$, for any $\{G_\alpha | \alpha \in \Lambda$, where Λ is a nonempty index set$\} \subset i\tau X$.

The pair $(X, i\tau X)$ is called an *intuitionistic fuzzy topological space* (IFTS) and an IFS in $i\tau X$ is known to be an *intuitionistic fuzzy open set* (IFOS) of X. The complement A' of an IFOS A is called an *intuitionistic fuzzy closed set* (IFCS) of X.

Several topological notions for intuitionistic fuzzy topological spaces have been considered and studied. Among many available papers, we refer to [6, 8, 9, 10, 11].

Below, we recall the product of two fuzzy topological spaces as well as that of two intuitionistic fuzzy topological spaces. The product is both commutative and associative upto fuzzy homeomorphism (intuitionistic fuzzy homeomorphism) [3].

1. [2] Let $(X, \tau X)$ and $(Y, \tau Y)$ be fuzzy topological spaces in the sense of Chang [4]. Then the fuzzy product space of X and Y is the cartesian product $X \times Y$ of sets X and Y together with the fuzzy topology $\tau X \times Y$, generated by the family $\{p_X^{-1}(\lambda_\alpha),\ p_Y^{-1}(\mu_\beta) |\ \lambda_\alpha \in \tau X,\ \mu_\beta \in \tau Y$, where p_X and p_Y are projections of $X \times Y$ onto X and Y, respectively$\}$. Because $p_X^{-1}(\lambda_\alpha) = \lambda_\alpha \times 1$, $p_Y^{-1}(\mu_\beta) = 1 \times \mu_\beta$ and $\lambda_\alpha \times 1 \cap 1 \times \mu_\beta = \lambda_\alpha \times \mu_\beta$; the family $\mathcal{B} = \{\lambda_\alpha \times \mu_\beta \mid \lambda_\alpha \in \tau X, \mu_\beta \in \tau Y\}$ forms a base for the fuzzy product topology $\tau X \times Y$ on $X \times Y$.

2. [9] Let $A = \langle X, \mu_A, \nu_A\rangle$ and $B = \langle Y, \mu_B, \nu_B\rangle$ be IFSs of X and Y, respectively. Then the *product of intuitionistic fuzzy sets* A and B denoted by $A \times B$ is defined by

$$A \times B = \langle X \times Y,\, \mu_A \times \mu_B,\, \nu_A \times_c \nu_B\rangle,$$

where

$$(\mu_A \times \mu_B)(x, y) = \min(\mu_A(x), \mu_B(y)),$$

and

$$(\nu_A \times_c \nu_B)(x, y) = \max(\nu_A(x), \nu_B(y)),$$

for $(x, y) \in X \times Y$. Obviously, $0 \leq \mu_A \times \mu_B + \nu_A \times_c \nu_B \leq 1$.

Lemma 1.2. *Let p_X and p_Y be projection maps of $X \times Y$ onto X and Y, respectively. Suppose that A is an IFS of X, and B is an IFS of Y. Then*

(a) $p_X^{-1}(A) = A \times 1_\sim$,

(b) $p_Y^{-1}(B) = 1_\sim \times B$, *and*

(c) $(A \times 1_\sim) \cap (1_\sim \times B) = A \times B$.

Let $(X, i\tau X)$ and $(Y, i\tau Y)$ be intuitionistic fuzzy topological spaces. Then the *product of intuitionistic fuzzy spaces X and Y* is the underlying set $X \times Y$ with the *intuitionistic fuzzy topology $i\tau X \times Y$*, generated by the family $\{p_X^{-1}(A_\alpha),\ p_Y^{-1}(B_\beta) \mid A_\alpha \in i\tau X,\ B_\beta \in i\tau Y$, where p_X and p_Y are projection maps of $X \times Y$ onto X and Y, respectively$\}$. The family $\mathbf{B} = \{A_\alpha \times B_\beta \mid A_\alpha \in i\tau X,\ B_\beta \in i\tau Y\}$ forms a base for the *intuitionistic fuzzy topology $i\tau X \times Y$* on $X \times Y$.

Lemma 1.3. *Let A be an IFCS of an intuitionistic fuzzy topological space X, and B be an IFCS of an intuitionistic fuzzy topological space Y. Then $A \times B$ is an IFCS in the intuitionistic fuzzy product space $X \times Y$.*

We state below the characterizations of a Hausdorff space and a compact space which we use to introduce these notions for fuzzy topological spaces, and also for intuitionistic fuzzy topological spaces.

1. [7] A topological space X is Hausdorff iff the diagonal of X is closed in $X \times X$.

2. [7, 14] A Hausdorff space X is compact iff the projection $p_Z : X \times Z \longrightarrow Z$ is closed for every Hausdorff space Z.

We establish fuzzy analogue of a well known theorem according to which the continuity of a function from a Hausdorff space to a compact Hausdorff space is characterized [7; Ch. XI, Theorem 2.7].

2. Lemmas

In this section, we establish certain lemmas which become tools to obtain results in Sections 3 and 4. Throughout this section X and Y denote sets.

Definition 2.1. [3] Let λ be a fuzzy set of X and $d_X : X \longrightarrow X \times X$ be the diagonal map on X defined by $d_X(x) = (x, x),\ x \in X$. Then the fuzzy set $d_X(\lambda)$ of $X \times X$ is called the *fuzzy λ-diagonal* of X.

Definition 2.2. [3] Let f be a map from X to Y and let λ be a fuzzy set of X. Then the fuzzy set $g(\lambda)$ of $X \times Y$, where $g : X \longrightarrow X \times Y$ is defined by $g(x) = (x, f(x)),\ x \in X$, is called the *fuzzy λ-graph* of f.

Lemma 2.3. *For a map f from X to Y,*

$$p_X\big(p_Y^{-1}(\mu) \cap g(\lambda)\big) = f^{-1}(\mu) \cap \lambda,$$

where λ is a fuzzy set of X and μ is a fuzzy set of Y.

Proof. For $x \in X$, we have

$$\begin{aligned} p_X\big(p_Y^{-1}(\mu) \cap g(\lambda)\big)(x) &= \sup_{y\in Y} \big(p_Y^{-1}(\mu) \cap g(\lambda)\big)(x,y) \\ &= \mu(y) \wedge \lambda(x), \qquad \text{where } \ y = f(x) \\ &= (f^{-1}(\mu) \cap \lambda)(x), \end{aligned}$$

and hence the result. ∎

LEMMA 2.4. *For a map f from X to Y and a fuzzy set μ of Y, the following holds:*

$$g(f^{-1}(\mu)) = (f \times I_Y)^{-1}(d_Y(\mu)),$$

where I_Y is the identity map on Y.

Proof. For $(x, y) \in X \times Y$, we have

$$g(f^{-1}(\mu))(x,y) = \begin{cases} f^{-1}(\mu)(x), & \text{if } y = f(x), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} (f \times I_Y)^{-1}(d_Y(\mu))(x,y) &= d_Y(\mu)(f \times I_Y)(x,y) = d_Y(\mu)(f(x),y) \\ &= \begin{cases} \mu(y), & \text{if } y = f(x), \\ 0, & \text{otherwise} \end{cases} = \begin{cases} f^{-1}(\mu)(x), & \text{if } y = f(x), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

LEMMA 2.5. *Let $A = \langle X, \mu_A, \nu_A\rangle$ be an intuitionistic fuzzy set of X. Then*

(i) $\langle X \times X, d_X(\mu_A), d_X(\nu_A)\rangle$ is an intuitionistic fuzzy set of $X \times X$,

(ii) for a map $f : X \longrightarrow Y$, $\langle X \times Y, g(\mu_A), g(\nu_A)\rangle$ is an intuitionistic fuzzy set of $X \times Y$.

Proof. Straightforward.

DEFINITION 2.6. Let $A = \langle X, \mu_A, \nu_A\rangle$ be an intuitionistic fuzzy set of X. Then the A-*diagonal* of X is defined to be the intuitionistic fuzzy set $d_X(A) = \langle X \times X,\ d_X(\mu_A),\ d_X(\nu_A)\rangle$ of $X \times X$.

DEFINITION 2.7. Let f be a map from a set X to a set Y, and let $A = \langle X, \mu_A, \nu_A\rangle$ be an intuitionistic fuzzy set of X. Then the A-*graph* of f is defined to be the intuitionistic fuzzy set $g(A) = \langle X \times Y, g(\mu_A), g(\nu_A)\rangle$ of $X \times Y$.

LEMMA 2.8. *For a map f from X to Y,*

$$p_X\big(p_Y^{-1}(B) \cap g(A)\big) = f^{-1}(B) \cap A,$$

where $A = \langle X, \mu_A, \nu_A\rangle$ is an intuitionistic fuzzy set of X and $B = \langle Y, \mu_B, \nu_B\rangle$ is an intuitionistic fuzzy set of Y.

Proof. For $x \in X$, we have

$$\begin{aligned} p_X\big(p_Y^{-1}(\mu_B)\cap g(\mu_A)\big)(x) &= \sup_{y\in Y}\ \big(p_Y^{-1}(\mu_B)\cap g(\mu_A)\big)(x,y) \\ &= \mu_B(y)\wedge\mu_A(x), \qquad \text{where } y=f(x) \\ &= \big(f^{-1}(\mu_B)\cap\mu_A\big)(x), \end{aligned}$$

and

$$\begin{aligned} p_X\big(p_Y^{-1}(\nu_B)\cup g(\nu_A)\big)(x) &= \inf_{y\in Y}\ \big(p_Y^{-1}(\nu_B)\cup g(\nu_A)\big)(x,y) \\ &= \nu_B(y)\vee\nu_A(x), \qquad \text{where } y=f(x) \\ &= \big(f^{-1}(\nu_B)\cup\nu_A\big)(x). \end{aligned}$$

LEMMA 2.9. *For a map $f : X \longrightarrow Y$ and an intuitionistic fuzzy set $B = \langle Y, \mu_B, \nu_B\rangle$ of Y, the following holds*:

$$g(f^{-1}(B)) = (f\times I_Y)^{-1}(d_Y(B)).$$

Proof. For $(x,y)\in X\times Y$, we have

$$g(f^{-1}(\mu_B))(x,y) = \begin{cases} f^{-1}(\mu_B)(x), & \text{if } y=f(x),\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\begin{aligned} (f\times I_Y)^{-1}(d_Y(\mu_B))(x,y) &= d_Y(\mu_B)(f\times I_Y)(x,y) = d_Y(\mu_B)(f(x),y) \\ &= \begin{cases} \mu_B(y), & \text{if } y=f(x),\\ 0, & \text{otherwise}\end{cases} = \begin{cases} f^{-1}(\mu_B)(x), & \text{if } y=f(x),\\ 0, & \text{otherwise.}\end{cases} \end{aligned}$$

Similarly, we have $g(f^{-1}(\nu_B)) = (f\times I_Y)^{-1}(d_Y(\nu_B))$.

3. Fuzzy λ-Hausdorff spaces and fuzzy λ-compact spaces

In this section, we introduce the notions of fuzzy λ-Hausdorff spaces and fuzzy μ-compact spaces, and obtain a characterization for a map f from a fuzzy λ-Hausdorff space to a fuzzy μ-compact space, where $\lambda = f^{-1}(\mu)$, to be fuzzy λ-continuous, which is a generalization of a fuzzy continuous mapping.

DEFINITION 3.1. Let λ be a fuzzy set of a fuzzy topological space X. Then X is said to be a *fuzzy λ-Hausdorff space* if the fuzzy λ-diagonal $d_X(\lambda)$ is a fuzzy closed set of $X\times X$.

EXAMPLE 3.2. Let X be a set and $x_0\in X$. Choose an $s\in I$. Define $\lambda : X\to I$ by $\lambda(x) = s$ if $x = x_0$, otherwise 0. Consider the fuzzy topological space $(X,\tau X)$, where $\tau X = \{0,1,\lambda'\}$. That $(X,\tau X)$ is a fuzzy λ-Hausdorff space follows by just observing that $d_X(\lambda) = \lambda\times\lambda$.

It is pertinent to note that fuzzy λ-Hausdorffness is not lost if more fuzzy open sets are added in τX. However, for $s, t \in I$, $s \neq t$ and $x_0, x_1 \in X$, $x_0 \neq x_1$, if we consider $\lambda : X \to I$ defined by $\lambda(x_0) = s$, $\lambda(x_1) = t$, and $\lambda(x) = 0$, otherwise, then the fuzzy topology $\{0, 1, \lambda'\}$ on X does not make X to be a fuzzy λ-Hausdorff space.

THEOREM 3.3. *Let X be a fuzzy λ-Hausdorff space and Y be a fuzzy μ-Hausdorff space. Then $X \times Y$ is a fuzzy $\lambda \times \mu$-Hausdorff space.*

Proof. It follows by noting that

(i) $d_{X\times Y}(\lambda \times \mu) = d_X(\lambda) \times d_Y(\mu)$, and

(ii) the product of two fuzzy closed sets is a fuzzy closed set.

DEFINITION 3.4. Let λ be a fuzzy set of a fuzzy topological space X. Then X is said to be a *fuzzy λ-compact space* if X is fuzzy λ-Hausdorff, and the projection mapping p_Z from $X \times Z$ to Z is a fuzzy closed mapping for each fuzzy ν-Hausdorff space Z.

EXAMPLE 3.5. Recall the fuzzy λ-Hausdorff space $(X, \tau X)$ as described in Example 3.2, with $s = 1$. To see that it is a fuzzy λ-compact space, we consider a fuzzy ν-Hausdorff space $(Z, \tau Z)$, and the projection mapping $p_Z : X \times Z \to Z$. Let G' be a fuzzy closed set of $X \times Z$, where $G = \bigcup_{(\alpha,\beta)\in\mathcal{A}}(\lambda_\alpha \times \mu_\beta)$, $\lambda_\alpha \in \tau X$ and $\mu_\beta \in \tau Z$, and $\mathcal{A}$ is an index set. We assume that $\mathcal{A}$ is nonempty, because otherwise, the result follows trivially. Now, for $z \in Z$, we have

$$\begin{aligned} p_Z(G')(z) &= \sup_{x\in X} G'(x,z) \\ &= \sup_{x\in X}\Big[\bigcap_{(\alpha,\beta)\in\mathcal{A}} \big((\lambda'_\alpha \times 1)(x,z) \cup (1 \times \mu'_\beta)(x,z)\big)\Big] \\ &= \sup_{x\in X}\Big[\bigcap_{(\alpha,\beta)\in\mathcal{A}} \big(\lambda'_\alpha(x) \cup \mu'_\beta(z)\big)\Big] \\ &= \bigcap_{(\alpha,\beta)\in\mathcal{A}} \left[\left(\sup_{x\in X} \ \lambda'_\alpha(x)\right) \cup \mu'_\beta(z)\right]. \end{aligned}$$

Since $\sup_{x\in X} \lambda'_\alpha(x)$ is either 1 or 0, $p_Z(G')$ is either 1 or $\bigcap_{\beta\in\mathcal{A}} \mu'_\beta$, and thus in both the cases $p_Z(G')$ is a fuzzy closed set of Z, as required.

However, if we choose $s \neq 1$ and consider the fuzzy λ-Hausdorff space X as described in Example 3.2, then X is not a fuzzy λ-compact space. Indeed, when $\nu : X \to I$ is defined by $\nu(x) = t > s$ if $x = x_0$, and 0 otherwise, then for the fuzzy ν-Hausdorff space $(X, \tau_1 X)$, where $\tau_1 X = \{0, 1, \nu'\}$, the projection p_2 of $X \times X$ onto the second factor is not fuzzy closed, in view of the fact that $p_2(\lambda \times \nu) = \lambda$, and that λ is not a fuzzy closed set of $(X, \tau_1 X)$.

THEOREM 3.6. *Let X be a fuzzy λ-compact space and Y be a fuzzy μ-compact space. Then $X \times Y$ is a fuzzy $\lambda \times \mu$-compact space.*

Proof. Let Z be a fuzzy ν-Hausdorff space. We show that the projection $p_Z : (X \times Y) \times Z \longrightarrow Z$ is fuzzy closed. By Theorem 3.3, $Y \times Z$ is a fuzzy $\mu \times \nu$-Hausdorff space. Now, the result follows by noting that p_Z is the composition of the projections

$$X \times (Y \times Z) \longrightarrow Y \times Z \quad \text{and} \quad Y \times Z \longrightarrow Z,$$

both of which are fuzzy closed mappings on account of the fact that X is fuzzy λ-compact and Y is fuzzy μ-compact.

THEOREM 3.7. *Let f be a fuzzy continuous map from a fuzzy topological space X to a fuzzy μ-Hausdorff space Y. Then the λ-graph $g(\lambda)$ of f is a fuzzy closed set of $X \times Y$, where $\lambda = f^{-1}(\mu)$.*

Proof. Follows from Lemma 2.4 by noting that $f \times I_Y$ is fuzzy continuous.

Before proceeding further, we introduce the notion of a fuzzy λ-continuous map as a generalization of a fuzzy continuous map.

DEFINITION 3.8. Let f be a map from a fuzzy topological space X to a fuzzy topological space Y and λ be a fuzzy set of X. Then f is called *fuzzy λ-continuous* if for each fuzzy closed set ν of Y, $\lambda \cap f^{-1}(\nu)$ is a fuzzy closed set of X.

Note that a fuzzy 1-continuous map is a fuzzy continuous map.

The following is an example of a fuzzy λ-continuous map which is not fuzzy continuous.

EXAMPLE 3.9. Let $X = Y = I$, where $I \equiv [0, 1]$. Consider the fuzzy sets λ, μ and ν of I defined as follows:

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{2}, \\ x - \frac{1}{2}, & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \qquad \mu(x) = \begin{cases} 1, & \text{if } 0 \le x \le \frac{1}{4}, \\ -4x + 2, & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

and

$$\nu(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{4}, \\ 2x - \frac{1}{2}, & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Let $\tau X = \{0,\, 1,\, \lambda'\}$ and $\tau Y = \{0,\, 1,\, \mu,\, \nu'\}$. Then $(X, \tau X)$ and $(Y, \tau Y)$ are fuzzy topological spaces. Define $f : X \longrightarrow Y$ by $f(x) = \frac{x}{2}$, $x \in X$. Since

$$f^{-1}(\mu')(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{2}, \\ 2x - 1, & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

$\big(f^{-1}(\mu')\big)' \notin \tau X$ and f is not fuzzy continuous. However, noting that $\lambda \le f^{-1}(\mu')$, it can be seen that f is fuzzy λ-continuous.

THEOREM 3.10. *Let f be a fuzzy λ-continuous map from a fuzzy topological space X to a fuzzy topological space Y, where λ is a fuzzy set of X. Then the map $f \times I_Y : X \times Y \longrightarrow Y \times Y$ is a fuzzy $\lambda \times 1$-continuous.*

Proof. For a fuzzy closed set ν of $Y \times Y$, we need to show that $(\lambda \times 1) \cap (f \times I_Y)^{-1}(\nu)$ is a fuzzy closed set of $X \times Y$. Set $\nu = G'$, with $G = \bigcup_{(\alpha,\beta)\in\mathcal{A}}(A_\alpha \times B_\beta)$, where A_α and B_β are fuzzy open sets of Y and $\mathcal{A}$ is an index set. In case $\mathcal{A} = \phi$, the result follows trivially. Therefore, we assume that $\mathcal{A}$ is nonempty. Using Lemma 2.2 in [2], we obtain

$$G' = \Big(\bigcup_{(\alpha,\beta)\in\mathcal{A}} (A_\alpha \times B_\beta)\Big)' = \bigcap_{(\alpha,\beta)\in\mathcal{A}} (A_\alpha \times B_\beta)' = \bigcap_{(\alpha,\beta)\in\mathcal{A}} (A'_\alpha \times 1 \cup 1 \times B'_\beta).$$

To complete the proof, we show that

$$(\lambda \times 1) \cap (f \times I_Y)^{-1}(\nu) = \bigcap_{(\alpha,\beta)\in\mathcal{A}} \big((\lambda \cap f^{-1}(A'_\alpha) \times 1) \cup (\lambda \times B'_\beta)\big), \tag{1}$$

and note that $\lambda \cap f^{-1}(A'_\alpha)$ is a fuzzy closed set of X on account of the fact that f is fuzzy λ-continuous. For (1), we choose $(x,y) \in X \times Y$, and have

$$\begin{aligned}
&\big((\lambda \times 1) \cap (f \times I_Y)^{-1}(\nu)\big)(x,y)\\
&= \min\big((\lambda \times 1)(x,y), (f \times I_Y)^{-1}(G')(x,y)\big)\\
&= \min\Big((\lambda \times 1)(x,y), \bigwedge_{(\alpha,\beta)\in\mathcal{A}} \big((f^{-1}(A'_\alpha) \times 1)(x,y) \vee (f^{-1}(1) \times B'_\beta)(x,y)\big)\Big)\\
&= \bigcap_{(\alpha,\beta)\in\mathcal{A}} \big(\lambda \cap f^{-1}(A'_\alpha) \times 1) \cup (\lambda \times B'_\beta)\big)(x,y).
\end{aligned}$$

THEOREM 3.11. *Let f be a map from a fuzzy λ-Hausdorff space X to a fuzzy μ-compact space Y. If the λ-graph $g(\lambda)$ of f is a fuzzy closed set of $X \times Y$, then f is fuzzy λ-continuous.*

Proof. Follows from Lemma 2.3.

Next, we have

THEOREM 3.12. *Let f be a map from a fuzzy λ-Hausdorff space X to a fuzzy μ-compact space Y, where $\lambda = f^{-1}(\mu)$. Then f is fuzzy λ-continuous if and only if λ-graph $g(\lambda)$ of f is a fuzzy closed set of $X \times Y$.*

Proof. In view of Theorem 3.11, we need to show that if f is fuzzy λ-continuous, then λ-graph $g(\lambda)$ of f is a fuzzy closed set of $X \times Y$. By Lemma 2.4, $g(\lambda) = (f \times I_Y)^{-1}(d_Y(\mu))$. Since $(f \times I_Y)^{-1}(d_Y(\mu)) \le \lambda \times 1$,

$$g(\lambda) = (\lambda \times 1) \cap (f \times I_Y)^{-1}(d_Y(\mu)),$$

and hence, the result follows from Theorem 3.10.

If X and Y in Theorem 3.12 are ordinary topological spaces, then for $\mu = 1$, we have the following well known result characterizing the continuity of a map f from a Hausdorff space X to a compact Hausdorff space Y.

COROLLARY 3.13. [7] *A map f from a Hausdorff space X to a compact Hausdorff space Y is continuous if and only if the graph $G(f)$ of f is a closed set of $X \times Y$.*

4. Intuitionistic fuzzy *A*-Hausdorff spaces and intuitionistic fuzzy *A*-compact spaces

Throughout this section X, Y and Z denote intuitionistic fuzzy topological spaces.

DEFINITION 4.1. Let A be an intuitionistic fuzzy set of X. Then X is said to be

(i) an *intuitionistic fuzzy A-Hausdorff space* if the μ_A-diagonal $d_X(\mu_A)$ of X is an intuitionistic fuzzy closed set of $X \times X$, and that the ν_A-diagonal $d_X(\nu_A)$ is an intuitionistic fuzzy open set of $X \times X$.

(ii) an *intuitionistic fuzzy A-compact space* if X is an intuitionistic fuzzy A-Hausdorff space and for an intuitionistic fuzzy B-Hausdorff space Z, the projection $X \times Z$ to Z is an intuitionistic fuzzy closed mapping.

EXAMPLE 4.2. Let X be a set and $x_0 \in X$. Choose $s, t \in I$ such that $s+t \leq 1$. Consider the intuitionistic fuzzy set $A = \langle X, \mu_A, \nu_A \rangle$, where $\mu_A : X \to I$ is defined by $\mu_A(x) = s$ if $x = x_0$, otherwise 0, and $\nu_A : X \to I$ is defined by $\nu_A(x) = t$ if $x = x_0$, otherwise 1. Let $i\tau X = \{0_\sim, 1_\sim, A'\}$. Then $(X, i\tau X)$ is an intuitionistic fuzzy topological space. That X is an intuitionistic fuzzy A-Hausdorff space follows by noting (i) $d_X(\mu_A) = \mu_A \times \mu_A$, (ii) $d_X(\nu_A) = \nu_A \times_c \nu_A$, and (iii) $d_X(A) = A \times A$.

However, for $s, s_1 \in I$, $s \neq s_1$ and $x_0, x_1 \in X$, $x_0 \neq x_1$, if we consider $\mu_A : X \to I$ defined by $\mu_A(x_0) = s$, $\mu_A(x_1) = s_1$, and $\mu_A(x) = 0$, otherwise, and ν_A as in the above example, then the intuitionistic fuzzy topology $\{0_\sim, 1_\sim, A'\}$ on X does not make X to be an intuitionistic fuzzy A-Hausdorff space.

It may be noted that addition of more intuitionistic fuzzy open sets in an intuitionistic fuzzy A-Hausdorff space keeps intuitionistic fuzzy A-Hausdorffness intact.

EXAMPLE 4.3. Consider the intuitionistic fuzzy A-Hausdorff space $(X, i\tau X)$ as described in Example 4.2, with $s = 1$ and $t = 0$. To see that it is an intuitionistic fuzzy A-compact space, we consider an intuitionistic fuzzy A-Hausdorff space $(Z, i\tau Z)$, and the projection mapping $p_Z : X \times Z \to Z$. Letting G to be the fuzzy open set of $X \times Z$ as described in Example 3.5, consider an intuitionistic fuzzy open set $D = \langle X \times Z, \mu_D, \nu_D \rangle$ of $(X \times Z, i\tau X \times Z)$, where $\mu_D = G$ and $\nu_D = 1 - G$. Following similar steps to those in Example 3.5, we obtain that p_Z is an intuitionistic fuzzy closed mapping.

Choosing $s \neq 1$ and $t \in I$, we obtain an example of an intuitionistic fuzzy A-Hausdorff space which is not an intuitionistic fuzzy A-compact space.

Following results for intuitionistic fuzzy topological spaces are analogous to those for fuzzy topological spaces obtained in Section 3.

THEOREM 4.4. *Let X be an intuitionistic fuzzy A-Hausdorff space and Y be an intuitionistic fuzzy B-Hausdorff space. Then $X \times Y$ is an intuitionistic fuzzy $A \times B$-Hausdorff space.*

Proof. It follows from the following facts:

(i) $d_{X\times Y}(\mu_A \times \mu_B) = d_X(\mu_A) \times d_Y(\mu_B)$,

(ii) $d_{X\times Y}(\nu_A \times_c \nu_B) = d_X(\nu_A) \times_c d_Y(\nu_B)$, and

(iii) The product of two intuitionistic fuzzy closed sets is an intuitionistic fuzzy closed set.

THEOREM 4.5. *Let X be an intuitionistic fuzzy A-compact space and Y be an intuitionistic fuzzy B-compact space. Then $X \times Y$ is an intuitionistic fuzzy $A \times B$-compact space.*

Proof. It is similar to that of Theorem 3.6.

THEOREM 4.6. *Let f be an intuitionistic fuzzy continuous map from an intuitionistic fuzzy space X to an intuitionistic fuzzy B-Hausdorff space Y. Then $f^{-1}(B)$-graph $g(f^{-1}(B))$ of f is an intuitionistic fuzzy closed set of $X \times Y$.*

Proof. It follows from Lemma 2.9 and the fact that $f \times I_Y$ is an intuitionistic fuzzy continuous map.

For a map f from an IFTS X to an IFTS Y and an IFS A of X, defining intuitionistic fuzzy A-continuity of f likewise to that of fuzzy λ-continuity, we have the following:

THEOREM 4.7. *Let f be a map from an intuitionistic fuzzy A-Hausdorff space to an intuitionistic fuzzy B-compact space, where $A = f^{-1}(B)$. Then f is intuitionistic fuzzy A-continuous iff A-graph $g(A)$ of f is intuitionistic fuzzy closed in $X \times Y$.*

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