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THE QUASI-HADAMARD PRODUCTS OF UNIFORMLY CONVEX FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract. The purpose of this paper is to obtain many interesting results about the quasi-Hadamard products of uniformly convex functions defined by Dziok-Srivastava operator belonging to the class $T_{q,s}([\alpha_1]; \alpha, \beta)$.

1. Introduction

Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ \ (a_n \geq 0), \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : \ |z| < 1\}$. Let $C(\gamma)$ and $T^*(\gamma)$ denote the subclasses of T which are, respectively, convex and starlike functions of order γ, $0 \leq \gamma < 1$. For convenience, we write $C(0) = C$ and $T^*(0) = T^*$ (see [9]).

A function $f \in T$ is said to be in $UST(\beta, \gamma)$, the class of β-uniformly starlike functions of order γ, $-1 \leq \gamma < 1$, if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (\beta \geq 0). \tag{1.2}$$

Replacing $f(z)$ in (1.2) by $zf^{'}(z)$ we have the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \beta \left|\frac{zf^{''}(z)}{f'(z)}\right| \quad (\beta \geq 0),$$

required for the function f to be in the subclass $UCT(\beta, \gamma)$ of β-uniformly convex functions of order γ (see [2]).

Let $f_j(z) \in T$ $(j = 1, \ldots, t)$ be given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geqslant 0; j = 1, 2, \ldots, t). \tag{1.3}$$

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Then the quasi-Hadamard product (or convolution) of these functions is defined by

$$(f_1 * f_2 * \cdots * f_t)(z) = z - \sum_{n=2}^{\infty}\Big(\prod_{j=1}^{t} a_{n,j}\Big)z^n. \tag{1.4}$$

For positive real parameters $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_s$, $\beta_i \in C \setminus Z^-$; $Z^- = \{0, -1, -2, \dots\}$; $i = 1, 2, \dots, s$, the Dziok-Srivastava operator (see [3] and [4]) $H_{q,s}(\alpha_1) : T \to T$ is given by

$$H_{q,s}(\alpha_1)f(z) = z\ {}_qF_s(\alpha_1, \dots \alpha_q; \beta_1, \dots \beta_s) * f(z) = z - \sum_{n=2}^{\infty} \Psi_n a_n z^n, \tag{1.5}$$

where

$$\Psi_n = \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}\ (n-1)!}, \tag{1.6}$$

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1, & n = 0 \\ \theta(\theta+1)\cdots(\theta+n-1), & n \in N. \end{cases}$$

and ${}_qF_s(\alpha_1, \dots, \alpha_q;\ \beta_1, \dots, \beta_s;\ z)$ $(q \le s+1;\ s, q \in N_0 = N \cup \{0\},\ N = \{1, 2, \dots\};\ z \in U)$ is the generalized hypergeometric function.

$${}_qF_s(\alpha_1, \dots, \alpha_q;\ \beta_1, \dots, \beta_s;\ z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} z^n.$$

For $-1 \le \gamma < 1$, $\beta \ge 0$, and for all $z \in U$, Aouf and Murugusundaramoorthy [1] defined the subclass $T_{q,s}([\alpha_1]; \gamma, \beta)$ of functions of T which satisfy:

$$\mathrm{Re}\left(\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - \gamma\right) > \beta\left|\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - 1\right|, z \in U. \tag{1.7}$$

They also proved [1] that the necessary and sufficient condition for functions $f(z)$ of the form (1.1) to be in the class $T_{q,s}([\alpha_1]; \gamma, \beta)$ is that:

$$\sum_{n=2}^{\infty}[n(1+\beta) - (\gamma+\beta)]\Psi_n a_n \le 1 - \gamma. \tag{1.8}$$

We note that for suitable choices of q, s, γ and β, we obtain the following subclasses studied by various authors.

(1) For $q = 2$ and $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in (1.7), the class $T_{2,1}([1]; \gamma, \beta)$ reduces to the class $S_pT(\gamma, \beta)$ $(-1 \le \gamma < 1, \beta \ge 0)$ and the class $S_pT(\gamma, 1)$ which for $\beta = 1$ reduces to the class $S_pT(\gamma)$ (see [2]).

(2) For $q = 2$, $s = 1, \alpha_1 = a\,(a > 0)$, $\alpha_2 = 1$ and $\beta_1 = c\ (c > 0)$ in (1.7), the class $T_{2,1}([a]; \gamma, \beta)$ reduces to the class $S_pT(a, c; \gamma, \beta)$ $(-1 \le \gamma < 1,\ \beta \ge 0)$ (see [5]).

(3) For $q = 2$, $s = 1$, $\alpha_1 = \lambda + 1\,(\lambda > -1)$, $\alpha_2 = 1$ and $\beta_1 = 1$ in (1.7), the class $T_{2,1}(\lambda + 1, 1; 1; \gamma, \beta)$ reduces to the class $S_pT(\lambda; \gamma, \beta)$ $(-1 \le \gamma < 1,\ \beta \ge 0)$ (see [8]).

(4) For $q=2,\ s=1,\ \alpha_1=v+1\,(v>-1),\ \alpha_2=1$ and $\beta_1=v+2$ in (1.7), the class $T_{2,1}(v+1,1;v+2;\gamma,\beta)$ reduces to the class $S_pT(v;\gamma,\beta)$ $(-1\le\gamma<1,\ \beta\ge 0)$ (see [1]).

(5) For $q=2,\ s=1,\ \alpha_1=2,\ \alpha_2=1$ and $\beta_1=2-\mu\,(\mu\ne 2,3,\dots)$ in (1.7), the class $T_{2,1}(2,1;2-\mu;\gamma,\beta)$ reduces to the class $S_pT(\mu;\gamma,\beta)$ $(-1\le\gamma<1,\ \beta\ge 0)$ (see [1]).

2. Main results

Unless otherwise mentioned, we shall assume in the remainder of this paper that the parameters $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_s$ are positive real numbers, $-1\le\gamma<1$, $\beta\ge 0$, $z\in U$, Ψ_n is defined by (1.6), $\Psi_n\ge 1$ and $j=1,2,\dots,t$.

THEOREM 1. *Let the functions $f_j(z)$ defined by (1.3) be in the class $T_{q,s}([\alpha_1];\gamma_j,\beta)$. Then we have $(f_1*\cdots*f_t)(z)\in T_{q,s}([\alpha_1];\delta,\beta)$, where*

$$\delta=1-\frac{(1+\beta)\prod_{j=1}^{t}(1-\gamma_j)}{\prod_{j=1}^{t}(2+\beta-\gamma_j)\Psi_2^{t-1}-\prod_{j=1}^{t}(1-\gamma_j)}. \tag{2.1}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z)=z-\frac{(1-\gamma_j)}{(2+\beta-\gamma_j)\Psi_2}z^2. \tag{2.2}$$

Proof. Employing the technique used earlier by Schild and Silverman [7] and Owa [6], we prove Theorem 1 by using mathematical induction on t. For $t=2$, (1.8) gives

$$\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\gamma_j+\beta)]\Psi_n}{(1-\gamma_j)}a_{n,j}\le 1\ \ (j=1,2). \tag{2.3}$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty}\sqrt{\prod_{j=1}^{2}\tfrac{[n(1+\beta)-(\gamma_j+\beta)]\Psi_n}{(1-\gamma_j)}}\sqrt{a_{n,2}a_{n,2}}\le 1. \tag{2.4}$$

To prove the case when $t=2$, we need to find the largest $\delta(-1\le\delta<1)$ such that

$$\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\delta+\beta)]\Psi_n}{(1-\delta)}a_{n,1}a_{n,2}\le 1, \tag{2.5}$$

thus, it suffices to show that

$$\frac{[n(1+\beta)-(\delta+\beta)]\Psi_n}{(1-\delta)}a_{n,1}a_{n,2}\le\frac{\sqrt{\prod_{j=1}^{2}[n(1+\beta)-(\gamma_j+\beta)]\Psi_n}}{\sqrt{\prod_{j=1}^{2}(1-\gamma_j)}}\sqrt{a_{n,1}a_{n,2}}$$

or, equivalently, to

$$\sqrt{a_{n,1}a_{n,2}}\le\frac{(1-\delta)\sqrt{\prod_{j=1}^{2}[n(1+\beta)-(\gamma_j+\beta)]\Psi_n}}{[n(1+\beta)-(\delta+\beta)]\Psi_n\sqrt{\prod_{j=1}^{2}(1-\gamma_j)}}.$$

By noting that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\sqrt{\prod_{j=1}^{2}(1-\gamma_j)}}{\sqrt{\prod_{j=1}^{2}[n(1+\beta)-(\gamma_j+\beta)]\Psi_n}},$$

consequently, we need only to prove that

$$\frac{\prod_{j=1}^{2}(1-\gamma_j)}{\prod_{j=1}^{2}[n(1+\beta)-(\gamma_j+\beta)]\Psi_n} \leq \frac{(1-\delta)}{[n(1+\beta)-(\delta+\beta)]}$$

which is equivalent to

$$\delta \leq 1 - \frac{(n-1)(1+\beta)\prod_{j=1}^{2}(1-\gamma_j)}{\prod_{j=1}^{2}[n(1+\beta)-(\gamma_j+\beta)]\Psi_n - \prod_{j=1}^{2}(1-\gamma_j)}.$$

Since

$$B(n) = 1 - \frac{(n-1)(1+\beta)\prod_{j=1}^{2}(1-\gamma_j)}{\prod_{j=1}^{2}[n(1+\beta)-(\gamma_j+\beta)]\Psi_n - \prod_{j=1}^{2}(1-\gamma_j)},$$

is an increasing function of n $(n \geqslant 2)$, then

$$\delta \leq B(2) = 1 - \frac{(1+\beta)\prod_{j=1}^{2}(1-\gamma_j)}{\prod_{j=1}^{2}(2+\beta-\gamma_j)\Psi_2 - \prod_{j=1}^{2}(1-\gamma_j)}.$$

Therefore, the result is true for $t = 2$.

Suppose that the result is true for any positive integer $t = k$. Then we have

$$(f_1 * \cdots * f_k * f_{k+1})(z) \in T_{q,s}([\alpha_1];\lambda,\beta),$$

where

$$\lambda = 1 - \frac{(1+\beta)(1-\gamma_{k+1})(1-\delta)}{(2+\beta-\gamma_{k+1})(2+\beta-\delta)\Psi_2 - (1-\gamma_{k+1})(1-\delta)},$$

and δ is given by (2.2). After simple calculations, we have

$$\lambda = 1 - \frac{(1+\beta)\prod_{j=1}^{k+1}(1-\gamma_j)}{\prod_{j=1}^{k+1}(2+\beta-\gamma_j)\Psi_2^k - \prod_{j=1}^{k+1}(1-\gamma_j)}. \tag{2.6}$$

This shows that the result is true for $t = k+1$. Therefore, by mathematical induction, the result is true for any positive integer $t\,(t \geq 2)$.

Taking the functions $f_j(z)$ given by (2.2), we have

$$(f_1 * \cdots * f_t)(z) = z - \prod_{j=1}^{t}\frac{(1-\gamma_j)}{(2+\beta-\gamma_j)\Psi_2}z^2 = z - H_2 z^2, \tag{2.7}$$

which shows that

$$\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\delta+\beta)]\Psi_n}{(1-\delta)}H_2 = \frac{(2+\beta-\delta]\Psi_2}{(1-\delta)} \cdot \prod_{j=1}^{t}\frac{(1-\gamma_j)}{(2+\beta-\gamma_j)\Psi_2} = 1;$$

Consequently, the result is sharp for functions $f_j(z)$ given by (2.2). This completes the proof of Theorem 1. ■

Letting $\gamma_j = \gamma$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the functions* $f_j(z)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma,\beta)$*. Then we have* $(f_1 * \cdots * f_t)(z) \in T_{q,s}([\alpha_1];\delta,\beta)$*,*

$$\delta = 1 - \frac{(1+\beta)(1-\gamma)^t}{(2+\beta-\gamma)^t \Psi_2^{t-1} - (1-\gamma)^t}. \tag{2.8}$$

The result is sharp for the functions $f_j(z)$ *given by*

$$f_j(z) = z - \frac{(1-\gamma)}{(2+\beta-\gamma)\Psi_2} z^2. \tag{2.9}$$

Putting $t = 2$ in Corollary 1, we obtain the following corollary.

Corollary 2. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma,\beta)$*. Then we have* $(f_1 * f_2)(z) \in T_{q,s}([\alpha_1];\delta,\beta)$*, where*

$$\delta = 1 - \frac{(1+\beta)(1-\gamma)^2}{(2+\beta-\gamma)^2 \Psi_2 - (1-\gamma)^2}.$$

The result is sharp.

Next, similarly by applying the method of proof of Theorem 1, we easily get the following result.

Theorem 2. *Let the functions* $f_j(z)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma,\zeta_j)$*,* $\zeta_j \geq 0$*. Then we have* $(f_1 * \cdots * f_t)(z) \in T_{q,s}([\alpha_1];\gamma,\eta)$*, where*

$$\eta = \frac{\prod_{j=1}^{t}(2+\zeta_j-\gamma)\Psi_2^{t-1}}{(1-\gamma)^{t-1}} + \gamma - 2. \tag{2.10}$$

The result is sharp for the functions $f_j(z)$ *given by*

$$f_j(z) = z - \frac{(1-\gamma)}{(2+\zeta_j-\gamma)\Psi_2} z^2. \tag{2.11}$$

Let $\zeta_j = \beta$ $(j = 1, 2, \dots, t)$ in Theorem 2, we obtain the following corollary.

Corollary 3. *Let the functions* $f_j(z)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma,\beta)$*. Then we have* $(f_1 * \cdots * f_t)(z) \in T_{q,s}([\alpha_1];\gamma,\eta)$*, where*

$$\eta = \frac{(2+\beta-\gamma)^t \Psi_2^{t-1}}{(1-\gamma)^{t-1}} + \gamma - 2.$$

The result is sharp for the functions $f_j(z)$ *given by* (2.9)*.*

Putting $t = 2$ in Corollary 3, we obtain the following corollary.

COROLLARY 4. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1]; \gamma, \beta)$. *Then we have* $(f_1 * f_2)(z) \in T_{q,s}([\alpha_1]; \gamma, \eta)$, *where*

$$\eta = \frac{(2 + \beta - \gamma)^2 \Psi_2}{(1 - \gamma)} + \gamma - 2.$$

The result is sharp.

THEOREM 3. *Let the functions* $f_j(z)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1]; \gamma_j, \beta)$. *Then the function*

$$F(z) = z - \sum_{n=2}^{\infty} \Big(\sum_{j=1}^{t} a_{n,j}^m \Big) z^n \quad (m > 1) \tag{2.12}$$

belongs to the class $T_{q,s}([\alpha_1]; \delta_t, \beta)$, *where*

$$\delta_t = 1 - \frac{t(1 + \beta)(1 - \gamma)^m}{(2 + \beta - \gamma)^m \Psi_2^{m-1} - t(1 - \gamma)^m} \quad (\gamma = \min_{1 \le j \le t} \{\gamma_j\}), \tag{2.13}$$

and $(2 + \beta - \gamma)^m \Psi_2^{m-1} \ge t(2 + \beta)(1 - \gamma)^m$. *The result is sharp for the functions* $f_j(z)$ $(j = 1, 2, \ldots, t)$ *given by* (2.2).

Proof. By virtue of (1.8), we have

$$\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\gamma_j + \beta)]\Psi_n}{(1 - \gamma_j)} a_{n,j} \le 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \Big(\frac{[n(1 + \beta) - (\gamma_j + \beta)]\Psi_n}{(1 - \gamma_j)} \Big)^m a_{n,j}^m \le \Big(\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\gamma_j + \beta)]\Psi_n}{(1 - \gamma_j)} a_{n,j} \Big)^m \le 1. \tag{2.14}$$

It follows from (2.14) that

$$\sum_{n=2}^{\infty} \left(\frac{1}{t} \sum_{j=1}^{t} \left(\frac{[n(1 + \beta) - (\gamma_j + \beta)]\Psi_n}{(1 - \gamma_j)} \right)^m a_{n,j}^m \right) \le 1.$$

By setting $\gamma = \min_{1 \le j \le t} \{\gamma_j\}$, the last inequality gives

$$\sum_{n=2}^{\infty} \left(\frac{1}{t} \left(\frac{[n(1 + \beta) - (\gamma + \beta)]\Psi_n(\alpha_1)}{(1 - \gamma)} \right)^m \sum_{j=1}^{t} a_{n,j}^m \right) \le 1.$$

Therefore, to prove our result we need to fined the largest δ_t such that

$$\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\delta_t + \beta)]\Psi_n(\alpha_1)}{(1 - \delta_t)} \sum_{j=1}^{t} a_{n,j}^m \le 1,$$

that is, that

$$\frac{[n(1+\beta)-(\delta_t+\beta)]\Psi_n}{(1-\delta_t)} \leq \frac{1}{t}\left(\frac{[n(1+\beta)-(\gamma+\beta)]\Psi_n}{(1-\gamma)}\right)^m$$

which leads to

$$\delta_t \leq 1-\frac{t(n-1)(1+\beta)(1-\gamma)^m}{[n(1+\beta)-(\gamma+\beta)]^m\left(\Psi_n\right)^{m-1}-t(1-\gamma)^m}.$$

Now let

$$R(n)=1-\frac{t(n-1)(1+\beta)(1-\gamma)^m}{[n(1+\beta)-(\gamma+\beta)]^m\Psi_n^{m-1}-t(1-\gamma)^m} \quad (n \geq 2).$$

Since $R(n)$ is an increasing function of n $(n \geqslant 2)$, then we have

$$\delta_t \leq R(2)=1-\frac{t(1+\beta)(1-\gamma)^m}{(2+\beta-\gamma)^m\Psi_2^{m-1}-t(1-\gamma)^m},$$

and by noting that $(2+\beta-\gamma)^m\Psi_2^{m-1} \geq t(2+\beta)(1-\gamma)^m$, we can see that $0 \leq \delta_t < 1$.

The result is sharp for the functions $f_j(z)$ $(j=1,2,\ldots,t)$ given by (2.2). This completes the proof of Theorem 3. ∎

Putting $m=2$ and $\gamma_j=\gamma$ $(j=1,\ldots,t)$ in Theorem 3, we obtain the following corollary.

Corollary 5. *Let the functions* $f_j(z)$ $(j=1,2,\ldots,t)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma,\beta)$. *Then the function*

$$G(z)=z-\sum_{n=2}^{\infty}\left(\sum_{j=1}^{t}a_{n,j}^2\right)z^n, \tag{2.15}$$

belongs to the class $T_{q,s}([\alpha_1];\delta_t,\beta)$, *where*

$$\delta_t=1-\frac{t(1+\beta)(1-\gamma)^2}{(2+\beta-\gamma)^2\Psi_2-t(1-\gamma)^2} \tag{2.16}$$

and $(2+\beta-\gamma)^2\Psi_2 \geq t(2+\beta)(1-\gamma)^2$. *The result is sharp for the functions* $f_j(z)$ $(j=1,2,\ldots,t)$ *given by* (2.9).

Similarly by applying the method of proof of Theorem 3, we easily get the following result.

Theorem 4. *Let the functions* $f_j(z)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma,\zeta_j), \zeta_j \geq 0$. *Then the function* $F(z)$ *defined by* (2.12) *belongs to the class* $T_{q,s}([\alpha_1];\gamma,\eta_t)$, *where*

$$\eta_t=\frac{(2+\beta-\gamma)^m\Psi_2^{m-1}}{t(1-\gamma)^{m-1}}+\gamma-2 \quad (\beta=\min_{1\leq j\leq t}\{\zeta_j\}),$$

and $(2+\beta-\gamma)^m\Psi_2^{m-1} \geq t(2-\gamma)(1-\gamma)^{m-1}$. *The result is sharp for the functions* $f_j(z)$ $(j=1,2,\dots,t)$ *given by* (2.11).

Putting $m = 2$ and $\zeta_j = \beta$ $(j = 1,2,\dots,t)$ in Theorem 4, we obtain the following corollary.

Corollary 6. *Let the functions* $f_j(z)$ $(j=1,2,\dots,t)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma,\beta)$. *Then the function* $G(z)$ *defined by* (2.15) *belongs to the class* $T_{q,s}([\alpha_1];\gamma,\eta_t)$, *where*

$$\eta_t = \frac{(2+\beta-\gamma)^2\Psi_2}{t(1-\gamma)} + \gamma - 2$$

and $(2+\beta-\gamma)^2\Psi_2 \geq t(2-\gamma)(1-\gamma)$. *The result is sharp for the functions* $f_j(z)$ *given by* (2.9).

Theorem 5. *Let the functions* $f_j(z)$ $(j=1,2,\dots,t)$ *defined by* (1.3) *be in the class* $T_{q,s}([\alpha_1];\gamma_j,\beta)$ $(j=1,2,\dots,t)$ *and let the functions* $g_m(z)$ *defined by*

$$g_m(z) = z - \sum_{n=2}^{\infty} b_{n,m}z^n \quad (b_{n,m} \geqslant 0;\ m=1,2,\dots,s), \tag{2.17}$$

be in the class $T_{q,s}([\alpha_1];\gamma_m,\beta)$ $(m=1,2,\dots,s)$, *then*

$$(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in T_{q,s}([\alpha_1];\Omega,\beta),$$

where

$$\Omega = 1 - \frac{(1+\beta)\prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}{\prod_{j=1}^{t}(2+\beta-\gamma_j)\prod_{m=1}^{s}(2+\beta-\gamma_m)\Psi_2^{t+s-1} - \prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}. \tag{2.18}$$

The result is sharp for the functions $f_j(z)$ *given by* (2.2) *and the functions* $g_m(z)$ *given by*

$$g_m(z) = z - \frac{(1-\gamma_m)}{(2+\beta-\gamma_m)\Psi_2}z^2 \quad (m=1,2,\dots,s). \tag{2.19}$$

Proof. From Theorem 1 we note that, if $f(z) \in T_{q,s}([\alpha_1];\delta,\beta)$ and $g(z) \in T_{q,s}([\alpha_1];\mu,\beta)$, then $(f*g)(z) \in T_{q,s}([\alpha_1];\Omega,\beta)$, where

$$\Omega = 1 - \frac{(1+\beta)(1-\delta)(1-\mu)}{(2+\beta-\delta)(2+\beta-\mu)\Psi_2 - (1-\delta)(1-\mu)}. \tag{2.20}$$

Since Theorem 1 leads to $(f_1 * f_2 * \cdots * f_t)(z) \in T_{q,s}([\alpha_1];\delta,\beta)$, where δ is defined by (2.1) and $(g_1 * g_2 * \cdots * g_s)(z) \in T_{q,s}([\alpha_1];\mu,\beta)$ with

$$\mu = 1 - \frac{(1+\beta)\prod_{m=1}^{s}(1-\gamma_m)}{\prod_{m=1}^{s}(2+\beta-\gamma_m)\Psi_2^{s-1} - \prod_{m=1}^{s}(1-\gamma_m)}. \tag{2.21}$$

Then, we have $(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta)$, where Ω is given by (2.18), this completes the proof of Theorem 5. ■

Letting $\gamma_j = \gamma$ $(j = 1, 2, \ldots, t)$ and $\gamma_m = \gamma$ $(m = 1, 2, \ldots, s)$ in Theorem 5, we obtain the following corollary.

Corollary 7. *Let the functions* $f_j(z)$ $(j = 1, 2, \ldots, t)$ *defined by* (1.3) *and let the functions* $g_m(z)$ *defined by* (2.17) *be in the class* $T_{q,s}([\alpha_1]; \gamma, \beta)$. *Then we have* $(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \ldots * g_s)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta)$, *where*

$$\Omega = 1 - \frac{(1+\beta)(1-\gamma)^{t+s}}{(2+\beta-\gamma)^{t+s}\Psi_2^{t+s-1} - (1-\gamma)^{t+s}}.$$

The result is sharp for the functions $f_j(z)$ *given by* (2.9) *and the functions* $g_m(z)$ *given by*

$$g_m(z) = z - \frac{(1-\gamma)}{(2+\beta-\gamma)\Psi_2} z^2 \quad (m = 1, 2, \ldots, s).$$

Letting $t = s = 2$ in Corollary 7, we obtain the following corollary.

Corollary 8. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (1.3) *and let the functions* $g_m(z)$ $(m = 1, 2)$ *defined by* (2.17) *be in the class* $T_{q,s}([\alpha_1]; \gamma, \beta)$. *Then we have* $(f_1 * f_2 * g_1 * g_2)(z) \in T_{q,s}([\alpha_1]; \Omega, \beta)$, *where*

$$\Omega = 1 - \frac{(1+\beta)(1-\gamma)^4}{(2+\beta-\gamma)^4\Psi_2^3 - (1-\gamma)^4}.$$

The result is sharp.

Putting $q = 2$, $s = 1$ and $\alpha_1 = \alpha_2 = \beta_1 = 1$ in Theorem 5, we obtain the following corollary.

Corollary 9. *Let* $f_j(z)$ $(j = 1, 2, \ldots, t)$ *defined by* (1.3) *be in the class* $S_pT(\gamma_j, \beta)$ *and let the functions* $g_m(z)$ $(m = 1, 2, \ldots, s)$ *defined by* (2.17) *be in the class* $S_pT(\gamma_m, \beta)$ $(m = 1, 2, \ldots, s)$, *then*

$$(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in S_pT(\tau, \beta),$$

where

$$\tau = 1 - \frac{(1+\beta)\prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}{\prod_{j=1}^{t}(2+\beta-\gamma_j)\prod_{m=1}^{s}(2+\beta-\gamma_m) - \prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}.$$

The result is sharp for the functions $f_j(z)$ *given by*

$$f_j(z) = z - \frac{(1-\gamma_j)}{(2+\beta-\gamma_j)} z^2 \quad (j = 1, 2, \ldots, t)$$

and the functions $g_m(z)$ *given by*

$$g_m(z) = z - \frac{(1-\gamma_m)}{(2+\beta-\gamma_m)} z^2 \quad (m = 1, 2, \ldots, s).$$

Putting $q = 2$, $s = 1$, $\alpha_1 = a(a > 0)$, $\alpha_2 = 1$ and $\beta_1 = c\ (c > 0)$ in Theorem 5, we obtain the following corollary.

Corollary 10. *Let $f_j(z)$ $(j = 1, 2, \dots, t)$ defined by (1.3) be in the class $S_pT(a, c; \gamma_j, \beta)$ and let the functions $g_m(z)$ $(m = 1, 2, \dots, s)$ defined by (2.17) be in the class $S_pT(a, c; \gamma_m, \beta)$ $(m = 1, 2, \dots, s)$, then*

$$(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in S_pT(a, c; \zeta, \beta),$$

where

$$\zeta = 1 - \frac{c^{t+s-1}(1+\beta)\prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}{a^{t+s-1}\prod_{j=1}^{t}(2+\beta-\gamma_j)\prod_{m=1}^{s}(2+\beta-\gamma_m) - c^{t+s-1}\prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}.$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{(1-\gamma_j)c}{(2+\beta-\gamma_j)a} z^2 \quad (j = 1, 2, \dots, t)$$

and the functions $g_m(z)$ given by

$$g_m(z) = z - \frac{(1-\gamma_m)c}{(2+\beta-\gamma_m)a} z^2 \ (m = 1, 2, \dots, s).$$

Putting $q = 2$, $s = 1$, $\alpha_1 = \lambda + 1(\lambda > -1)$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 5, we obtain the following corollary.

Corollary 11. *Let $f_j(z)$ $(j = 1, 2, \dots, t)$ defined by (1.3) be in the class $S_pT(\lambda; \gamma_j, \beta)$ and let the functions $g_m(z)$ $(m = 1, 2, \dots, s)$ defined by (2.17) be in the class $S_pT(\lambda; \gamma_m, \beta)$ $(m = 1, 2, \dots, s)$, then*

$$(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in S_pT(\lambda; \nu, \beta),$$

where

$$\nu = 1 - \frac{(1+\beta)\prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}{(\lambda+1)^{t+s-1}\prod_{j=1}^{t}(2+\beta-\gamma_j)\prod_{m=1}^{s}(2+\beta-\gamma_m) - \prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}.$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{(1-\gamma_j)}{(2+\beta-\gamma_j)(\lambda+1)} z^2 \quad (j = 1, 2, \dots, t)$$

and the functions $g_m(z)$ given by

$$g_m(z) = z - \frac{(1-\gamma_m)}{(2+\beta-\gamma_m)(\lambda+1)} z^2 \quad (m = 1, 2, \ldots, s).$$

Putting $q = 2$, $s = 1$, $\alpha_1 = v + 1 (v > -1)$, $\alpha_2 = 1$ and $\beta_1 = v + 2$ in Theorem 5, we obtain the following corollary.

Corollary 12. *Let $f_j(z)$ $(j = 1, 2, \ldots, t)$ defined by (1.3) be in the class $S_pT(v; \gamma_j, \beta)$ and let the functions $g_m(z)$ $(m = 1, 2, \ldots, s)$ defined by (2.17) be in the class $S_pT(v; \gamma_m, \beta)$ $(m = 1, 2, \ldots, s)$, then*

$$(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in S_pT(v; \sigma, \beta),$$

where

$$\sigma = 1 - \frac{(v+2)^{t+s-1}(1+\beta)\prod_{j=1}^{t}(1-\gamma_j)\prod_{j=1}^{s}(1-\gamma_m)}{(v+1)^{t+s-1}\prod_{j=1}^{t}(2+\beta-\gamma_j)\prod_{j=1}^{s}(2+\beta-\gamma_m) - (v+2)^{t+s-1}\prod_{j=1}^{t}(1-\gamma_j)\prod_{j=1}^{s}(1-\gamma_m)}.$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{(1-\gamma_j)(v+2)}{(2+\beta-\gamma_j)(v+1)} z^2 \quad (j = 1, 2, \ldots, t)$$

and the functions $g_m(z)$ given by

$$g_m(z) = z - \frac{(1-\gamma_m)(v+2)}{(2+\beta-\gamma_m)(v+1)} z^2 \quad (m = 1, 2, \ldots, s).$$

Putting $q = 2$, $s = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\beta_1 = 2 - \mu$ $(\mu \neq 2, 3, \ldots)$ in Theorem 5, we obtain the following corollary.

Corollary 13. *Let $f_j(z)$ $(j = 1, 2, \ldots, t)$ defined by (1.3) be in the class $S_pT(\mu; \gamma_j, \beta)$ and let the functions $g_m(z)$ $(m = 1, 2, \ldots, s)$ defined by (2.17) be in the class $S_pT(\mu; \gamma_m, \beta)$ $(m = 1, 2, \ldots, s)$, then*

$$(f_1 * f_2 * \cdots * f_t * g_1 * g_2 * \cdots * g_s)(z) \in S_pT(\mu; \kappa, \beta),$$

where

$$\kappa = 1 - \frac{(2-\mu)^{t+s-1}(1+\beta)\prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}{2^{t+s-1}\prod_{j=1}^{t}(2+\beta-\gamma_j)\prod_{m=1}^{s}(2+\beta-\gamma_m) - (2-\mu)^{t+s-1}\prod_{j=1}^{t}(1-\gamma_j)\prod_{m=1}^{s}(1-\gamma_m)}.$$

The result is sharp for the functions $f_j(z)$ *given by*

$$f_j(z) = z - \frac{(2-\mu)(1-\gamma_j)}{2(2+\beta-\gamma_j)} z^2 \quad (j = 1, 2, \ldots, t)$$

and the functions $g_m(z)$ *given by*

$$g_m(z) = z - \frac{(2-\mu)(1-\gamma_m)}{2(2+\beta-\gamma_m)} z^2 \quad (m = 1, 2, \ldots, s).$$

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REFERENCES

[**1**] M.K. Aouf, G. Murugusundaramoorthy, *On a subclass of uniformly convex functions defined by the Dziok-Srivastava operator*, Austral. J. Math. Anal. Appl. **5** (2008), 1–17.

[**2**] R. Bharati, R. Parvatham, A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math. **28** (1997), 17–32.

[**3**] J. Dziok, H.M. Srivastava, *Classes of analytic functions with the generalized hypergeometric function*, Appl. Math. Comput. **103** (1999), 1–13.

[**4**] J. Dziok, H.M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct. **14** (2003), 7–18.

[**5**] G. Murugusundaramoorthy, N. Magesh, *A new subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient*, J. Inequal. Pure Appl. Math. **5** (2004), 1–20.

[**6**] S. Owa, *The quasi-Hadamard products of certain analytic functions*, in: H.M. Srivastava, S. Owa (Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992, 244–251.

[**7**] A. Schild, H. Silverman, *Convolutions of univalent functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklodowska Sect. A **29** (1975), 99–107.

[**8**] S. Shams, S.R. Kulkarni, *On class of univalent functions defined by Ruscheweyh derivatives*, Kyungpook Math. J. **43** (2003), 579–585.

[**9**] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

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