МАТЕМАТИЧКИ ВЕСНИК
64, 1 (2012), 1–16
March 2012

оригинални научни рад
research paper

UNIQUENESS RESULTS OF MEROMORPHIC FUNCTIONS WHOSE NONLINEAR DIFFERENTIAL POLYNOMIALS HAVE ONE NONZERO PSEUDO VALUE

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Abstract. In this paper we deal with some uniqueness questions of meromorphic functions whose certain nonlinear differential polynomials have a nonzero pseudo value. The results in this paper improve the corresponding ones given by M. L. Fang, X. Y. Zhang and W. C. Lin, L. P. Liu, and so on.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [8], [20] and [24]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o\{T(r,h)\}$, as $r\to\infty$ and $r\notin E$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same $a-$points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same $a-$points ignoring multiplicities (see [24]). We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a)=S(r,f)$ (see [24]). Let l be a positive integer or ∞. Next we denote by $E_{l)}(a;f)$ the set of of those $a-$points of f in the complex plane, where each point is of multiplicity $\le l$ and counted according to its multiplicity. By $\overline{E}_{l)}(a;f)$ we denote the reduced form of $E_{l)}(a;f)$. If $\overline{E}_{l)}(a;f)=\overline{E}_{l)}(a;g)$, we say that a is a $l-$order pseudo common value of f and g (see [15]). Obviously, if

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2010 AMS Subject Classification: 30D30, 30D35.

Keywords and phrases: Meromorphic function; entire function; weighted sharing; uniqueness.

This work was supported by the Natural Science Foundation of Jiang-Xi Province in China (Grant No. 2010GQS0119 and No. 2010GQS014) and the Youth Foundation of Education Department of Jiangxi (No. GJJ10050 and No. GJJ10223) of China.

$E_{\infty)}(a;f) = E_{\infty)}(a;g)(\overline{E}_{\infty)}(a;f) = \overline{E}_{\infty)}(a;g)$, resp.) then f and g share a CM (IM, resp.). We define m^* $m^* := \chi_\mu m$, where $\chi_\mu = 0$, if $\mu = 0$, $\chi_\mu = 1$ if $\mu \neq 0$.

In 1976, C. C. Yang posed the following question.

Question A. What can be said about the relationship between two entire functions f and g, if f, g share 0 CM and $f^{(n)}, g^{(n)}$ share 1 CM, where n is a nonnegative integer, and $2\delta(0, f) > 1$?

In 1990, H. X. Yi dealt with Question A (see [21], [22] [23]). In 1997, I. Lahiri posed the following question.

Question B. (see [12]) What can be said if two non-linear differential polynomials generated by two meromorphic functions share 1 CM?

Afterwards some research works concerning Question B have been done by many mathematicians such as ([2–6,8–10,13,14,16-19,24,25]). A recent increment to uniqueness theory has been to considering weighted sharing instead of sharing IM/CM, this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2000, and since then investigated by I. Lahiri, his students and some of Chinese colleagues. In this direction, many research works concerning Question B have been done by many mathematicians, such as ([1,9–10,13–14]). The notion of weighted sharing is defined as follows.

Definition 1.1. [9,10] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

We also need the following five definitions.

Definition 1.2. (see [11, Definition 1]) Let p be a positive integer and $a \in \mathbb{C} \cup \infty$. Then by $N(r, a; f| \leq p)$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}(r, a; f| \leq p)$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N(r, a; f| \geq p)$ we denote the counting function of those $a-$points of f (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}(r, a; f| \geq p)$ we denote the corresponding reduced counting function (ignoring multiplicities), where $N(r, a; f| \leq p), \overline{N}(r, a; f| \leq p), N(r, a; f| \geq p)$ and $\overline{N}(r, a; f| \geq p)$ mean $N(r, f| \leq p), \overline{N}(r, f| \leq p), N(r, f| \geq p)$ and $\overline{N}(r, f| \geq p)$ respectively, if $a = \infty$.

Definition 1.3. Let a be an any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2) + \cdots + \overline{N}(r, a; f| \geq k),$$

and

$$\delta_k(a; f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k(r, a; f)}{T(r, f)}.$$

Definition 1.4. [2] Let k and r be two positive integers such that $1 \leq r < k-1$ and for $a \in \mathbb{C} \cup \{\infty\}$, $\overline{E}_{k)}(a;f) = \overline{E}_{k)}(a;g), E_{r)}(a;f) = E_{r)}(a;g)$. Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)(\overline{N}_L(r,a;g))$ the reduced counting function of those a-points of f and g for which $p > q \geq r+1(q > p \geq r+1)$, by $\overline{N}_E^{(r+1}(r,a;f)$ the reduced counting function of those a-points of f and g for which $p = q \geq r+1$, by $\overline{N}_{f\geq k+1}(r,a;f|g \neq a)(\overline{N}_{g\geq k+1}(r,a;g|f \neq a))$ the reduced counting functions of those a-points of f and g for which $p \geq k+1$ and $q = 0(q \geq k+1$ and $p = 0)$.

Definition 1.5. [2] If $r = 0$ in definition 1.2 then we use the same notations as in definition 1.2 except by $\overline{N}_E^{1)}(r,a;f)$ we mean the common simple a-points of f and g and by $\overline{N}_E^{(2}(r,a;f)$ we mean the reduced counting functions of those a-points of f and g for which $p = q \geq 2$.

Definition 1.6. [2] Let $a, b \in \mathbb{C} \cup \{\infty\}$, We denote by $N(r,a;f|g = b)$ the counting function of those $a-$points of f, counted according to multiplicity, which are b-points of g; by $N(r,a;f|g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

We recall the following result proved by Zhang and Lin in 2008, which extended two uniqueness theorems of Fang in [4].

Theorem A. [24] *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n > 2k + m^* + 4$, and λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then*

(i) when $\lambda\mu \neq 0$, $f \equiv g$,

(ii) when $\lambda\mu = 0$, either $f \equiv tg$, where t is a constant satisfying $t^{n+m} = 1$, or $f = c_1 e^{cz}, g = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k \lambda^2 (c_1c_2)^{n+m*}[(n+m^*)c]^{2k} = 1 \quad or \quad (-1)^k \mu^2 (c_1c_2)^{n+m*}[(n+m^*)c]^{2k} = 1.$$

Using the idea of weighted sharing, Liu proved the following result, which generalized and improved Theorem A.

Theorem B. [16] *Let f and g be two nonconstant meromorphic functions, and let n, m and k be three positive integers, and λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $E_l(1, [f^n(\mu f^m + \lambda)]^{(k)}) = E_l(1, [g^n(\mu g^m + \lambda)]^{(k)})$, and one of the following conditions holds,*

(1) $l \geq 2$ and $n > 3m^* + 3k + 8$;

(2) $l = 1$ and $n > 4m^* + 5k + 10$;

(3) $l = 0$ and $n > 6m^* + 9k + 14$.

Then: (i) when $\lambda\mu \neq 0$, if $m \geq 2$ and $\delta(\infty, f) > \frac{3}{n+m}$, then $f \equiv g$; if $m = 1$ and $\Theta(\infty, f) > \frac{3}{n+1}$, then $f \equiv g$;

(ii) when $\lambda\mu = 0$, *if* $f \neq \infty$ *and* $g \neq \infty$, *then either* $f \equiv tg$, *where* t *is a constant satisfying* $t^{n+m^*} = 1$, *or* $f = c_1e^{cz}, g = c_2e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying*

$$(-1)^k\lambda^2(c_1c_2)^{n+m*}[(n+m^*)c]^{2k} = 1 \quad or \quad (-1)^k\mu^2(c_1c_2)^{n+m*}[(n+m^*)c]^{2k} = 1.$$

Regarding Theorem B, it is natural to ask the following question.

QUESTION 1.1. What can be said about the relationship between two meromorphic functions f and g, if the condition $E_l(1, [f^n(\mu f^m+\lambda)]^{(k)}) = E_l(1, [g^n(\mu g^m+\lambda)]^{(k)})$ in Theorem B is replaced with the condition $E_{l)}(1, [f^n(\mu f^m + \lambda)]^{(k)}) = E_{l)}(1, [g^n(\mu g^m + \lambda)]^{(k)})$?

We will prove the following two theorems, which improves Theorems A and B, and deals with Question 1.1.

THEOREM 1.1. *Let* f *and* g *be two nonconstant meormorphic functions, and let* n, m *and* k *be three positive integers with* $n > \frac{13}{3}k + \frac{13}{3}m^* + \frac{28}{3}$, *and* λ, μ *be constants such that* $|\lambda| + |\mu| \neq 0$. *If* $\overline{E}_{l)}(1, [f^n(\mu f^m + \lambda)]^{(k)}) = \overline{E}_{l)}(1, [g^n(\mu g^m + \lambda)]^{(k)})$ *and* $E_{1)}(1, [f^n(\mu f^m + \lambda)]^{(k)}) = E_{1)}(1, [g^n(\mu g^m + \lambda)]^{(k)})$, *where* $l \geq 3$ *is an integer, then the conclusions of Theorem B still hold.*

THEOREM 1.2. *Let* f *and* g *be two nonconstant meromorphic functions, and let* n, m *and* k *be three positive integers with* $n > 3k+3m^*+6$, *and* λ, μ *be constants such that* $|\lambda| + |\mu| \neq 0$. *If* $\overline{E}_{l)}(1, [f^n(\mu f^m + \lambda)]^{(k)}) = \overline{E}_{l)}(1, [g^n(\mu g^m + \lambda)]^{(k)})$ *and* $E_{2)}(1, [f^n(\mu f^m + \lambda)]^{(k)}) = E_{2)}(1, [g^n(\mu g^m + \lambda)]^{(k)})$, *where* $l \geq 4$ *is an integer, then the conclusions of Theorem B still hold.*

2. Some lemmas

LEMMA 2.1. [8] *Let* $f(z)$ *be a non-constant meromorphic function,* k *a positive integer, and let* c *be a non-zero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + N(r, 0; f) + N(r, c; f^{(k)}) - N(r, 0; f^{(k+1)}) + S(r, f) \\ &= \overline{N}(r, f) + N_{k+1}(r, 0; f) + \overline{N}(r, c; f^{(k)}) - N_0(r, 0; f^{(k+1)}) + S(r, f), \end{aligned} \tag{1}$$

where $N_0(r, 0; f^{(k+1)})$ *is the counting function which only counts those points such that* $f^{(k+1)} = 0$ *but* $f(f^{(k)} - c) \neq 0$.

LEMMA 2.2. [20] *Let* f *be a nonconstant meromorphic function and* $P(f) = a_0 + a_1f + a_2f^2 + \cdots + a_nf^n$, *where* $a_0, a_1, a_2, \ldots, a_n$ *are constants and* $a_n \neq 0$. *Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

LEMMA 2.3. [15, Proof of Lemma 2.3] *Let* f *be a nonconstant meromorphic function, and let* $k \geq 1$ *and* $p \geq 1$ *be two positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.4. [10] *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geq k)+S(r,f).$$

Lemma 2.5. [2] *Let F,G be two nonconstant meromorphic functions such that $E_{1)}(1;F)=E_{1)}(1;G)$ and $H\not\equiv 0$. Then*

$$N_E^{1)}(r,1;F)\leq N(r,\infty;H)+S(r,F)+S(r,G),$$

where $H=(\frac{F''}{F'}-\frac{2F'}{F-1})-(\frac{G''}{G'}-\frac{2G'}{G-1})$.

Lemma 2.6. [1] *Let $\overline{E}_{l)}(1;F)=\overline{E}_{l)}(1;G), E_{1)}(1;F)=E_{1)}(1;G)$ and $H\not\equiv 0$, where $l\geq 3$. Then*

$$\begin{aligned}N(r,\infty;H)\leq{}&\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}(r,\infty;F|\geq 2)\\&+\overline{N}(r,\infty;G|\geq 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\\&+\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G'),\end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 2.7. [2] *Let $\overline{E}_{l)}(1;F)=\overline{E}_{l)}(1;G), E_{1)}(1;F)=E_{1)}(1;G)$ and $H\not\equiv 0$, where $l\geq 3$. Then*

$$\begin{aligned}&2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+l\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)\\&\qquad-\overline{N}_{F>2}(r,1;G)\leq N(r,1;G)-\overline{N}(r,1;G).\end{aligned}$$

Lemma 2.8. *Let $\overline{E}_{l)}(1;F)=\overline{E}_{l)}(1;G), E_{1)}(1;F)=E_{1)}(1;G)$, where $l\geq 3$. Then*

$$\begin{aligned}&\overline{N}_{F>2}(r,1;G)+2\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\\&\qquad\leq\frac{2}{3}\overline{N}(r,0;F)+\frac{2}{3}\overline{N}(r,\infty;F)-\frac{2}{3}N_0(r,0;F')+S(r,F).\end{aligned}$$

Proof. We note that any 1-point of F with multiplicity ≥ 3 is counted at most twice. Hence by using Lemma 2.4 we see that

$$\begin{aligned}&\overline{N}_{F>2}(r,1;G)+2\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\\&\qquad\leq\overline{N}(r,1;F|\geq 3;G|=2)+2\overline{N}(r,1;F|G\neq 1)\\&\qquad\leq\frac{2}{3}N(r,0;F'|F=1)\\&\qquad\leq\frac{2}{3}N(r,0;F'|F\neq 0)-\frac{2}{3}N_0(r,0;F')\\&\qquad\leq\frac{2}{3}\overline{N}(r,0;F)+\frac{2}{3}\overline{N}(r,\infty;F)-\frac{2}{3}N_0(r,0;F')+S(r,F),\end{aligned}$$

where by $\overline{N}(r,1;F|\geq 3;G|=2)$ we mean the reduced counting function of 1 points of F with multiplicity not less than 3 which are the 1-points of G with multiplicity 2. Thus, we complete the proof of the lemma. ∎

LEMMA 2.9. *Let* $\overline{E}_{l)}(1;(F^*)^{(k)})=\overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{1)}(1;(F^*)^{(k)})=E_{1)}(1;(G^*)^{(k)})$ *and* $H^*\not\equiv 0$, *where* $l\geq 3$. *Then*

$$\begin{aligned}T(r,F^*)\leq{}&(\frac{8}{3}+\frac{2}{3}k)\overline{N}(r,\infty;F^*)+\frac{5}{3}\overline{N}(r,0;F^*)+\frac{2}{3}N_k(r,0;F^*)\\&+N_{k+1}(r,0;F^*)+(k+2)\overline{N}(r,\infty;G^*)+\overline{N}(r,0;G^*)\\&+N_{k+1}(r,0;G^*)+S(r,F^*)+S(r,G^*)\end{aligned}$$

where

$$H^*\equiv\left[\frac{(F^*)^{(k+2)}}{(F^*)^{(k+1)}}-\frac{2(F^*)^{(k+1)}}{(F^*)^{(k)}-1}\right]-\left[\frac{(G^*)^{(k+2)}}{(G^*)^{(k+1)}}-\frac{2(G^*)^{(k+1)}}{(G^*)^{(k)}-1}\right].$$

Proof. Let $F=(F^*)^{(k)}$ and $G=(G^*)^{(k)}$, then the condition of this lemma is $\overline{E}_{l)}(1;F)=\overline{E}_{l)}(1;G), E_{1)}(1;F)=E_{1)}(1;G)$ and $H^*=H\not\equiv 0$. From the definition of H^*, and Lemma 2.5, we have

$$\begin{aligned}N_E^{1)}(r,1;F)&\leq\overline{N}(r,0;H^*)\leq T(r,H^*)+O(1)\\&\leq N(r,\infty;H^*)+S(r,F^*)+S(r,G^*).\end{aligned}\tag{2}$$

On the other hand, by the assumptions, we can see that possible poles of H^* occur at the zeros of F' and G', and the common 1-points of F and G whose multiplicities are different, and the poles of F^* and G^*, and those 1-points of $F(G)$ which are not the 1-points of $G(F)$, and the zeros of $F'(G')$ which are not the zeros of $F^*(F-1)(G^*(G-1))$. So from Lemma 2.6 and (2), we have

$$\begin{aligned}N(r,\infty;H^*)\leq{}&\overline{N}(r,0;F^*)+\overline{N}(r,0;G^*)+\overline{N}(r,\infty;F^*)+\overline{N}(r,\infty;G^*)\\&+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\\&+\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G'),\end{aligned}\tag{3}$$

From Lemmas 2.7 and (2),(3) , we get

$$\begin{aligned}&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\&\quad\leq N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\&\quad\quad+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+\overline{N}(r,1;G)\\&\quad\leq\overline{N}(r,0;F^*)+\overline{N}(r,\infty;F^*)+\overline{N}(r,0;G^*)\\&\quad\quad+\overline{N}(r,\infty;G^*)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\&\quad\quad+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)\\&\quad\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\end{aligned}$$

$$
\begin{aligned}
&+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+T(r,G)-m(r,1;G)\\
&+O(1)-2\overline{N}_L(r,1;F)-2\overline{N}_L(r,1;G)-\overline{N}_E^{(2}(r,1;F)\\
&-l\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}_{F>2}(r,1;G)+\overline{N}_0(r,0;F')\\
&+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\leq{}&\overline{N}(r,0;F^*)+\overline{N}(r,\infty;F^*)+\overline{N}(r,0;G^*)+\overline{N}(r,\infty;G^*)\\
&+T(r,G)-m(r,1;G)+2\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\\
&+\overline{N}_{F>2}(r,1;G)-(l-1)\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)\\
&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned}
$$

From Lemma 2.8, we can get

$$
\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\leq\overline{N}(r,0;F^*)+\overline{N}(r,\infty;F^*)+\overline{N}(r,0;G^*)\\
&\quad+\overline{N}(r,\infty;G^*)+T(r,G)-m(r,1;G)\\
&\quad+\frac{2}{3}\overline{N}(r,0;F)+\frac{2}{3}\overline{N}(r,\infty;F)\\
&\quad-(l-1)\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}_0(r,0;F')\\
&\quad+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \qquad (4)
\end{aligned}
$$

Using Lemma 2.1 for F^* and G^*, we get

$$
\begin{aligned}
T(r,F^*)\leq{}&\overline{N}(r,\infty;F^*)+N_{k+1}(r,0;F^*)+\overline{N}(r,1;F)\\
&-N_0(r,0;F')+S(r,F^*), \qquad (5)
\end{aligned}
$$

and

$$
\begin{aligned}
T(r,G^*)\leq{}&\overline{N}(r,\infty;G^*)+N_{k+1}(r,0;G^*)+\overline{N}(r,1;G)\\
&-N_0(r,0;G')+S(r,G^*). \qquad (6)
\end{aligned}
$$

Adding (5) and (6), we get

$$
\begin{aligned}
&T(r,F^*)+T(r,G^*)\\
&\leq\overline{N}(r,\infty;F^*)+N_{k+1}(r,0;F^*)+\overline{N}(r,\infty;G^*)\\
&\quad+N_{k+1}(r,0;G^*)+\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad-N_0(r,0;F')-N_0(r,0;G')+S(r,F^*)+S(r,G^*). \qquad (7)
\end{aligned}
$$

Since

$$T(r,G)=T(r,(G^*)^{(k)})\leq T(r,G^*)+k\overline{N}(r,\infty;G^*)+S(r,G^*), \qquad (8)$$

from (2), (7), (8) and $S(r,F)=S(r,F^*), S(r,G)=S(r,G^*)$, we get

$$T(r,F^*)\leq\overline{N}(r,\infty;F^*)+N_{k+1}(r,0;F^*)+\overline{N}(r,\infty;G^*)+N_{k+1}(r,0;G^*)$$

$$
\begin{aligned}
&+\overline{N}(r,0;F^*)+\overline{N}(r,\infty;F^*)+\overline{N}(r,0;G^*)+\overline{N}(r,\infty;G^*)\\
&+k\overline{N}(r,\infty;G^*)-m(r,1;G)+\frac{2}{3}\overline{N}(r,0;F)+\frac{2}{3}\overline{N}(r,\infty;F)\\
&+S(r,F^*)+S(r,G^*). \qquad (9)
\end{aligned}
$$

Since $F=(F^*)^{(k)}$ and $G=(G^*)^{(k)}$, from Lemma 2.3, (9) becomes

$$
\begin{aligned}
T(r,F^*) \leq\ & \frac{8}{3}\overline{N}(r,\infty;F^*)+\overline{N}(r,0;F^*)+N_{k+1}(r,0;F^*)\\
&+\frac{2}{3}\overline{N}(r,0;(F^*)^{(k)})+(k+2)\overline{N}(r,\infty;G^*)+\overline{N}(r,0;G^*)\\
&+N_{k+1}(r,0;G^*)+S(r,F^*)+S(r,G^*)\\
\leq\ & (\frac{8}{3}+\frac{2}{3}k)\overline{N}(r,\infty;F^*)+\frac{5}{3}\overline{N}(r,0;F^*)+\frac{2}{3}N_k(r,0;F^*)\\
&+N_{k+1}(r,0;F^*)+(k+2)\overline{N}(r,\infty;G^*)+\overline{N}(r,0;G^*)\\
&+N_{k+1}(r,0;G^*)+S(r,F^*)+S(r,G^*). \quad \blacksquare \qquad (10)
\end{aligned}
$$

Lemma 2.10. *Let* $\overline{E}_{l)}(1;(F^*)^{(k)})=\overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{1)}(1;(F^*)^{(k)})=E_{1)}(1;(G^*)^{(k)})$ *where* $l\geq 3$. *If*

$$
\begin{aligned}
\Delta_{1l} =\ & (\frac{8}{3}+\frac{2}{3}k)\Theta(\infty;F^*)+\frac{5}{3}\Theta(0,F^*)+\frac{2}{3}\delta_k(0,F^*)+\delta_{k+1}(0;F^*)\\
&+(k+2)\Theta(\infty;G^*)+\Theta(0,G^*)+\delta_{k+1}(0;G^*)\\
>\ & \frac{5}{3}k+9,
\end{aligned}
$$

then $(F^*)^{(k)}(G^*)^{(k)}\equiv 1$ *or* $F^*\equiv G^*$.

Proof. From Lemma 2.9, we first suppose that $H\not\equiv 0$, without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,G^*)\leq T(r,F^*)$ for $r\in I$. From Lemma 2.9 we get

$$
\begin{aligned}
T(r,F^*)\leq\ & \{\frac{5}{3}k+10-(\frac{8}{3}+\frac{2}{3}k)\Theta(\infty;F^*)-\frac{5}{3}\Theta(0,F^*)-\frac{2}{3}\delta_k(0,F^*)\\
&-\delta_{k+1}(0;F^*)-(k+2)\Theta(\infty;G^*)-\Theta(0,G^*)-\delta_{k+1}(0;G^*)\\
&+\varepsilon\}T(r,F^*)+S(r,F^*), \qquad (11)
\end{aligned}
$$

for $r\in I$ and $0<\varepsilon<\Delta_{1l}-\frac{5}{3}k-9$, that is $\{\Delta_{1l}-\frac{5}{3}k-9-\varepsilon\}T(r,F^*)\leq S(r,F^*)$, i.e. $\Delta_{1l}-\frac{5}{3}k-9\leq 0$, i.e. $\Delta_{1l}\leq\frac{5}{3}k+9$, which is a contradiction to the condition of Lemma 2.10.

Therefore, we have $H\equiv 0$, then

$$
\frac{(F^*)^{(k+2)}}{(F^*)^{(k+1)}}-\frac{2(F^*)^{(k+1)}}{(F^*)^{(k)}-1}\equiv\frac{(G^*)^{(k+2)}}{(G^*)^{(k+1)}}-\frac{2(G^*)^{(k+1)}}{(G^*)^{(k)}-1}. \qquad (12)
$$

From this equation we get

$$
(G^*)^{(k)}=\frac{(b+1)(F^*)^{(k)}+(a-b-1)}{b(F^*)^{(k)}+(a-b)}, \qquad (13)
$$

where $a(\neq 0),b$ are two constants.

We will prove $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$ or $F^* \equiv G^*$ with the employment of the same argument used in [3]. Now, we consider three cases as follows.

Case 1. $b \neq 0, -1$, If $a-b-1 \neq 0$, then by (13) we know

$$\overline{N}\Big(r, \frac{a-b-1}{b+1}; (F^*)^{(k)}\Big) = \overline{N}(r, 0; (G^*)^{(k)}).$$

By Lemma 2.1 we have

$$\begin{aligned}T(r,F^*) &\le \overline{N}(r,F^*) + N_{k+1}(r,0;F^*) + \overline{N}(r,c;(F^*)^{(k)})\\ &\quad - N_0(r,0;(F^*)^{(k+1)}) + S(r,F^*)\\ &\le \overline{N}(r,F^*) + N_{k+1}(r,0;F^*) + \overline{N}\Big(r, \frac{a-b-1}{b+1}; (F^*)^{(k)}\Big) + S(r,F^*)\\ &\le \overline{N}(r,F^*) + N_{k+1}(r,0;F^*) + k\overline{N}(r,G^*) + \overline{N}(r,0;G^*) + S(r,F^*).\end{aligned}$$

Hence, from the assumptions of this lemma, we deduce that $T(r,F^*) \le S(r,F^*)$, $r \in I$ a contradiction.

If $a-b-1=0$, then by (13) we know $(G^*)^{(k)} = ((b+1)(F^*)^{(k)})/(b(F^*)^{(k)}+1)$. Obviously,

$$\overline{N}\Big(r, \frac{1}{b}; (F^*)^{(k)}\Big) = \overline{N}(r, (G^*)^{(k)}).$$

By Lemma 2.1 we have

$$\begin{aligned}T(r,F^*) &\le \overline{N}(r,F^*) + N_{k+1}(r,0;F^*) + \overline{N}(r,c;(F^*)^{(k)})\\ &\quad - N_0(r,0;(F^*)^{(k+1)}) + S(r,F^*)\\ &\le \overline{N}(r,F^*) + N_{k+1}(r,0;F^*) + \overline{N}\Big(r, \frac{1}{b}; (F^*)^{(k)}\Big) + S(r,F^*)\\ &\le \overline{N}(r,F^*) + N_{k+1}(r,0;F^*) + \overline{N}(r,G^*) + S(r,F^*) + S(r,G^*).\end{aligned}$$

Hence, from the assumptions of this lemma, we deduce that $T(r,F^*) \le S(r,F^*)$, $r \in I$ a contradiction.

Case 2. $b=-1$. Then (13) becomes $(G^*)^{(k)} = a/(a+1-(F^*)^{(k)})$.

If $a+1 \neq 0$, then $\overline{N}(r, a+1; (F^*)^{(k)}) = \overline{N}(r, (G^*)^{(k)})$. Similarly, we can deduce a contradiction as in Case 1.

If $a+1=0$, then $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$.

Case 3. $b=0$. Then (13) becomes $(G^*)^{(k)} = ((F^*)^{(k)} + a - 1)/a$.

If $a-1 \neq 0$, then $\overline{N}\Big(r, 1-a; (F^*)^{(k)}\Big) = \overline{N}\Big(r, 0; (G^*)^{(k)}\Big)$. Similarly, we can again deduce a contradiction as in Case 1.

If $a-1=0$, then $(F^*)^{(k)} \equiv (G^*)^{(k)}$. From this equation, we obtain

$$F^* = G^* + p(z),$$

where $p(z)$ is a polynomial, then $T(r,F^*) = T(r,G^*) + S(r,F^*)$. If $p(z) \not\equiv 0$, then by Lemma 2.2, we have

$$\begin{aligned}T(r,F^*) &\le \overline{N}(r,F^*) + \overline{N}(r,0;F^*) + \overline{N}(r,p;F^*) + S(r,F^*)\\ &\le \overline{N}(r,F^*) + \overline{N}(r,0;F^*) + \overline{N}(r,0;G^*) + S(r,F^*).\end{aligned}$$

Hence, from the assumptions of this lemma, we deduce that $T(r,F^*) \leq S(r,F^*)$, $r \in I$, a contradiction. Thus, we deduce that $p(z) \equiv 0$, that is $F^* \equiv G^*$.

Therefore, we complete the proof of Lemma 2.10. ■

LEMMA 2.11. *Let* $\overline{E}_{l)}(1;(F^*)^{(k)}) = \overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{2)}(1;(F^*)^{(k)}) = E_{2)}(1;(G^*)^{(k)})$ *and* $H^* \not\equiv 0$, *where* $l \geq 4$. *Then*

$$\begin{aligned}T(r,F^*)+T(r,G^*) \leq{}& (k+4)\overline{N}(r,\infty;F^*)+(k+4)\overline{N}(r,\infty;G^*)\\&+2N_{k+1}(r,0;F^*)+2N_{k+1}(r,0;G^*)+2\overline{N}(r,0;F^*)\\&+2\overline{N}(r,0;G^*)+S(r,F^*)+S(r,G^*).\end{aligned}$$

where H^* *is defined as Lemma 2.9.*

Proof. Let $F = (F^*)^{(k)}$ and $G = (G^*)^{(k)}$, then $\overline{E}_{l)}(1;F) = \overline{E}_{l)}(1;G)$, $E_{2)}(1;F) = E_{2)}(1;G)$. Since $H^* \not\equiv 0$, using the same argument of as in Lemma 2.11 and by Lemma 2.1, we can get

$$\begin{aligned}&T(r,F^*)+T(r,G^*)\\&\leq \overline{N}(r,\infty;F^*)+N_{k+1}(r,0;F^*)+\overline{N}(r,\infty;G^*)+N_{k+1}(r,0;G^*)\\&\quad+\overline{N}(r,1;(F^*)^{(k)})+\overline{N}(r,1;(G^*)^{(k)})-N_0(r,0;(F^*)^{(k+1)})\\&\quad-N_0(r,0;(G^*)^{(k+1)})+S(r,F^*)+S(r,G^*)\\&\leq \overline{N}(r,\infty;F^*)+N_{k+1}(r,0;F^*)+\overline{N}(r,\infty;G^*)+N_{k+1}(r,0;G^*)\\&\quad+N(r,1;(F^*)^{(k)}|=1)+\overline{N}(r,1;(F^*)^{(k)}|\geq 2)+\overline{N}(r,1;(G^*)^{(k)})\\&\quad-N_0(r,0;(F^*)^{(k+1)})-N_0(r,0;(G^*)^{(k+1)})+S(r,F^*)+S(r,G^*)\\&\leq \overline{N}(r,\infty;F^*)+N_{k+1}(r,0;F^*)+\overline{N}(r,\infty;G^*)+N_{k+1}(r,0;G^*)\\&\quad+\overline{N}(r,0;F^*)+\overline{N}(r,0;G^*)+\overline{N}(r,\infty;F^*)+\overline{N}(r,\infty;G^*)\\&\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)+\overline{N}(r,1;G)\\&\quad+\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}(r,1;F|\geq 2)+S(r,F^*)+S(r,G^*).\end{aligned}$$

Since

$$\overline{N}(r,1;F|=l;G|=l-1)+\cdots+\overline{N}(r,1;F|=l;G|=3) \leq \overline{N}(r,1;F|=l),$$

and

$$\overline{N}(r,1;G|=l;F|=l-1)+\cdots+\overline{N}(r,1;G|=l;F|=3) \leq \overline{N}(r,1;G|=l),$$

we see that

$$\begin{aligned}&\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{F\geq l+1}(r,1;F|G\neq 1)\\&\qquad+\overline{N}_{G\geq l+1}(r,1;G|F\neq 1)+\overline{N}(r,1;F|\geq 2)+\overline{N}(r,1;(G)\\&\quad\leq \overline{N}(r,1;F|=l;G|=l-1)+\cdots+\overline{N}(r,1;F|=l;G|=3)\end{aligned}$$

$$
\begin{aligned}
&+\overline{N}(r,1;F|\geq l+2)+\overline{N}(r,1;G|=l;F|=l-1)+\cdots\\
&+\overline{N}(r,1;G|=l;F|=3)+\overline{N}(r,1;G|\geq l+2)\\
&+\overline{N}(r,1;G|\geq l+2)+\overline{N}(r,1;F|\geq l+1)\\
&+\overline{N}(r,1;G|\geq l+1)+\overline{N}(r,1;F|=2)+\cdots\\
&+\overline{N}(r,1;F|=l)+\overline{N}(r,1;F|\geq l+1)+\overline{N}(r,1;G|=1)\\
&+\cdots+\overline{N}(r,1;G|=l)+\overline{N}(r,1;G|\geq l+1)\\
\leq{}&\frac{1}{2}N(r,1;F|=1)+\overline{N}(r,1;F|=2)+\cdots+2\overline{N}(r,1;F|=l)\\
&+2\overline{N}(r,1;F|\geq l+1)+\overline{N}(r,1;F|\geq l+2)+\frac{1}{2}N(r,1;G|=1)\\
&+\overline{N}(r,1;G|=2)+\cdots+2\overline{N}(r,1;G|=l)+2\overline{N}(r,1;G|\geq l+1)\\
&+\overline{N}(r,1;G|\geq l+2)\\
\leq{}&\frac{1}{2}[N(r,1;F)+N(r,1;G)]\\
\leq{}&\frac{1}{2}[T(r,F)+T(r,G)].
\end{aligned}
$$

Since

$$T(r,F)=T(r,(F^*)^{(k)})\leq T(r,F^*)+k\overline{N}(r,\infty;F^*)+S(r,F^*),$$

and

$$T(r,G)=T(r,(G^*)^{(k)})\leq T(r,G^*)+k\overline{N}(r,\infty;G^*)+S(r,G^*),$$

we can get

$$
\begin{aligned}
T(r,F^*)+T(r,G^*)\leq{}&(k+4)\overline{N}(r,\infty;F^*)+(k+4)\overline{N}(r,\infty;G^*)\\
&+2N_{k+1}(r,0;F^*)+2N_{k+1}(r,0;G^*)+2\overline{N}(r,0;F^*)\\
&+2\overline{N}(r,0;G^*)+S(r,F^*)+S(r,G^*).
\end{aligned}
$$

Thus, we complete the proof of the lemma. ∎

LEMMA 2.12. *Let* $\overline{E}_{l)}(1;(F^*)^{(k)})=\overline{E}_{l)}(1;(G^*)^{(k)})$, $E_{2)}(1;(F^*)^{(k)})=E_{2)}(1;(G^*)^{(k)})$ *and where* $l\geq 4$. *If*

$$
\begin{aligned}
\Delta_{2l}={}&(\frac{1}{2}k+2)[\Theta(\infty;F^*)+\Theta(\infty;G^*)]+\Theta(0;F^*)\\
&+\Theta(0;G^*)+\delta_{k+1}(0;F^*)+\delta_{k+1}(0;G^*)>k+5,
\end{aligned}
$$

then $(F^*)^{(k)}(G^*)^{(k)}\equiv 1$ *or* $F^*\equiv G^*$.

Proof. We omit the proof since the proof can be carried out in the line of proof of Lemma 2.11 by using the Lemma 2.12. ∎

3. Proofs of Theorems

Let $F^* = f^n(\mu f^m + \lambda), G^* = g^n(\mu g^m + \lambda)$, by Lemma 2.2, we can get

$$\Theta(0; F^*) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; F^*)}{T(r, F^*)} \geq 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; f^n) + \overline{N}(r, 0; \mu f^m + \lambda)}{(n + m^*)T(r, f)},$$

i.e.

$$\Theta(0; F^*) \geq 1 - \frac{m^* + 1}{n + m^*}. \tag{14}$$

Similarly, we have

$$\Theta(0; G^*) \geq 1 - \frac{m^* + 1}{n + m^*}. \tag{15}$$

And since

$$\begin{aligned}\Theta(\infty; F^*) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; F^*)}{T(r, F^*)} = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; f^n)}{(n + m^*)T(r, f)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; f)}{(n + m^*)T(r, f)} \geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{(n + m^*)T(r, f)},\end{aligned}$$

we have

$$\Theta(\infty; F^*) \geq 1 - \frac{1}{n + m^*}. \tag{16}$$

Similarly, we have

$$\Theta(\infty; G^*) \geq 1 - \frac{1}{n + m^*}. \tag{17}$$

Next, by the definition of $N_k(r, a; f)$ we have

$$\delta_{k+1}(0; F^*) = 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; F^*)}{T(r, F^*)} \geq 1 - \limsup_{r\to\infty} \frac{(k + 1)\overline{N}(r, 0; F^*)}{T(r, F^*)}.$$

Therefore

$$\delta_{k+1}(0; F^*) \geq 1 - \limsup_{r\to\infty} \frac{(k + 1)\overline{N}(r, 0; f) + N_{k+1}(r, 0; \mu f^m + \lambda))}{(n + m^*)T(r, f)},$$

i.e.

$$\delta_{k+1}(0; F^*) \geq 1 - \frac{m^* + k + 1}{n + m^*}. \tag{18}$$

Similarly, we get

$$\delta_k(0; F^*) \geq \frac{n - k}{n + m^*} \quad \delta_k(0; G^*) \geq \frac{n - k}{n + m^*} \quad \delta_{k+1}(0; G^*) \geq \frac{n - k - 1}{n + m^*}. \tag{19}$$

Proof of Theorem 1.1.

From the condition of Theorem 1.1, we have $\overline{E}_{l)}(1; F^{(k)}) = \overline{E}_{l)}(1; G^{(k)})$, $E_{1)}(1; F^{(k)}) = E_{1)}(1; G^{(k)})$, where $l \geq 3$.

From (14)–(19) and Lemma 2.10, we have

$$\Delta_{1l} \geq (\frac{5}{3}k + \frac{14}{3})\frac{n + m^* - 1}{n + m^*} + \frac{5}{3}\frac{n - 1}{n + m^*} + \frac{2}{3}\frac{n - k}{n + m^*} + 2\frac{n - k - 1}{n + m^*}. \tag{20}$$

Since $n > \frac{13}{3}m^* + \frac{13}{3}k + \frac{28}{3}$, we can get $\Delta_{1l} > \frac{5}{3}k + 9$. From Lemma 2.10, we have $F^* \equiv G^*$ or $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$.

We will prove the conclusions of Theorem 1.1 with the employment of the same argument used in Theorem 1 in [16].

We will consider two cases as follows.

Case 1. $F^* \equiv G^*$. That is,

$$f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda). \tag{21}$$

If $\lambda\mu = 0$, then from $|\lambda| + |\mu| \neq 0$, we can get $f^{n+m} \equiv g^{n+m}$ or $f^n \equiv g^n$. Then we can get $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m^*} = 1$.

If $\lambda\mu \neq 0$, then we set $h = \frac{f}{g}$. If $h \not\equiv 1$, then substituting $f = hg$ into (21) we have

$$g^m = -\frac{\lambda}{\mu} \cdot \frac{1 - h^n}{1 - h^{n+m}} = -\frac{\lambda}{\mu} \cdot \frac{1 + h + \cdots + h^{n-1}}{1 + h + \cdots + h^{n+m-1}}. \tag{22}$$

If $m = 1$, (22) is $g = -\frac{\lambda}{\mu} \cdot \frac{1+h+\cdots+h^{n-1}}{1+h+\cdots+h^n}$, from $f = hg$, we have $f = -\frac{\lambda}{\mu} \cdot \frac{(1+h+\cdots+h^{n-1})h}{1+h+\cdots+h^n}$, where h is a nonconstant meromorphic function. It follows that $T(r,f) = T(r,gh) = (n+1)T(r,h) + S(r,f)$. On the other hand, by the second fundamental theorem, we can get

$$\overline{N}(r,f) = \sum_{i=1}^{n} \overline{N}\Big(r, a_i; h\Big) \geq (n-2)T(r,h) + S(r,f), \tag{23}$$

where $a_i(\neq 1)(i = 1, 2, \ldots, n)$ are distinct roots of the algebraic equation $h^{n+1} = 1$.

From (23), we have

$$\Theta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \leq 1 - \limsup_{r\to\infty} \frac{(n-2)T(r,h) + S(r,f)}{(n+1)T(r,h) + S(r,f)} \leq \frac{3}{n+1}.$$

Thus, we get a contradiction with the assumption $\Theta(\infty, f) > \frac{3}{n+1}$. Therefore, $h \equiv 1$, that is, $f(z) \equiv g(z)$.

If $m \geq 2$, from (22), we have $f^m = -\frac{\lambda}{\mu} \cdot \frac{(1+h+\cdots+h^{n-1})h^m}{1+h+\cdots+h^{n+m-1}}$. It follows that $T(r,f) = \Big(1 + \frac{n}{m}\Big)T(r,h) + S(r,f)$ and every poles of f of order p must be a zero of $h^{n+m} - 1$ of order mp. Therefore, $N(r,f) = \frac{1}{m}\sum_{i=1}^{n+m} N\Big(r, a_i; h\Big)$, where $a_i(\neq 1)(i = 1, 2, \ldots, (n+m-1))$ are distinct root of the algebraic equation $h^{n+m} = 1$. Thus, we have

$$\begin{aligned} N(r,f) &= \frac{1}{m}\sum_{i=1}^{n+m-1} N\Big(r, a_i; h\Big) \geq \frac{1}{m}\sum_{i=1}^{n+m-1} N\Big(r, a_i; h\Big) \\ &\geq \frac{n+m-3}{m}T(r,h) + S(r,f). \end{aligned} \tag{24}$$

From (24), we have

$$\delta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)} \leq \frac{3}{n+m}.$$

Thus, we can get a contradiction with the assumption $\delta(\infty, f) > \frac{3}{n+m}$.

Case 2. $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$. That is,

$$[f^n(\mu f^m + \lambda)]^{(k)}[g^n(\mu g^m + \lambda)]^{(k)} \equiv 1. \tag{25}$$

Next, we consider two subcases.

Subcase 2.1. $\lambda\mu = 0$. By $|\lambda| + |\mu| \neq 0$, we have $\lambda = 0, \mu \neq 0$ or $\lambda \neq 0, \mu = 0$. If $\lambda = 0, \mu \neq 0$, from (26), we have $[\mu f^{n+m}]^{(k)}[\mu g^{n+m}]^{(k)} \equiv 1$.

Thus, if z_0 is a zero of $[\mu f^{n+m}]^{(k)}$, then z_0 is a pole of $[\mu g^{n+m}]^{(k)}$. This contradicts that $g \neq \infty$. Hence $f(z) \neq 0, g(z) \neq 0$. Thus, we have $[\mu f^{n+m}]^{(k)}n \neq 0$ and $[\mu g^{n+m}]^{(k)} \neq 0$. From [7], we have $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, here c_1, c_2 and c are three constants satisfying $(-1)^k \mu^2 (c_1c_2)^{n+m}[(n+m)c]^{2k} = 1$ when $k \geq 2$. When $k = 1$, we can also get that $-\mu^2(c_1c_2)^{n+m}[(n+m)c]^2 = 1$ with the employment of the same argument used in Theorem 1 in [16].

When $\lambda \neq 0, \mu = 0$, by using the same argument as above, we can also get the results which is $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k \mu^2 (c_1c_2)^{n+m}[(n+m)c]^{2k} = 1$.

Subcase 2.2. $\lambda\mu \neq 0$. We can rewrite (25) as

$$[f^n(f - a_1)\cdots(f - a_m)]^{(k)}[g^n(g - a_1)\cdots(g - a_m)]^{(k)} \equiv 1, \tag{26}$$

where $a_1, a_2, \ldots, a_m$ are roots of $\mu\omega^m + \lambda = 0$.

Let z_0 be zero of f of order p. From (27) we know that z_0 is a pole of g. Let z_0 be a pole of g of order q. From (26), we have $np - k = (n+m)q + k$, i.e. $n(p - q) = mq + 2k$, which implies that $p \geq q + 1$ and $mq + 2k \geq n$. From $n > \frac{13}{3}k + \frac{13}{3}m + \frac{28}{3}$, we can get $p \geq 6$.

Let z_i^1 be a zero of $f - a_i (i = 1, \ldots, m)$ of order p_i^1, then z_i^1 is a zero of $f^n(\mu f^m + \lambda)$ of order $p_i^1 - k$. Hence, from (26), we get z_i^1 is a pole of g of order q_i^1 and $p_i^1 - k = (n+m)q_i^1 + k$, i.e. $p_i^1 = (n+m)q_i^1 + 2k$. Thus, we have $p_i^1 \geq n+m+2k$.

Let z_2 be a zero of f' of order p_2 that not a zero of $f(f - a_1)\cdots(f - a_m)$, as above, we have $p_2 \geq n + m + 2k - 1$. So we have similar results for the zeros of $g^n(\mu g^m + \lambda)$.

From (26), we have

$$\begin{aligned}\overline{N}(r, f) &\leq \overline{N}(r, 0; g) + \sum_{i=1}^{m} \overline{N}(r, a_i; f) + \overline{N}(r, 0; g') \\ &\leq \frac{1}{6}N(r, 0; g) + \frac{1}{n+m+2k}\sum_{i=1}^{m} N(r, a_i; g) + \frac{1}{n+m+2k-1}N(r, 0; g').\end{aligned}$$

That is,

$$\overline{N}(r, f) \leq \Big(\frac{1}{6} + \frac{m}{n+m+2k} + \frac{1}{n+m+2k-1}\Big)T(r, g) + S(r, g). \tag{27}$$

From (27) and the second fundamental theorem we have

$$\begin{aligned} mT(r,f) &\le \overline{N}(r,f) + \sum_{i=1}^{m} \overline{N}(r,a_i;g) + \overline{N}(r,0;f) + S(r,f) \\ &\le \Big(\frac{1}{6} + \frac{m}{n+m+2k} + \frac{1}{n+m+2k-1}\Big)T(r,g) \\ &\quad + \Big(\frac{1}{6} + \frac{m}{n+m+2k}\Big)T(r,f) + S(r,f) + S(r,g). \end{aligned} \tag{28}$$

Similarly, we have

$$\begin{aligned} mT(r,g) &\le \Big(\frac{1}{6} + \frac{m}{n+m+2k} + \frac{1}{n+m+2k-1}\Big)T(r,f) \\ &\quad + \Big(\frac{1}{6} + \frac{m}{n+m+2k}\Big)T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{29}$$

From (28) and (29), we have

$$\begin{aligned} m(T(r,f)+T(r,g)) \le &\Big(\frac{1}{3} + \frac{2m}{n+m+2k} + \frac{1}{n+m+2k-1}\Big)[T(r,f)+T(r,g)] \\ &+ S(r,f) + S(r,g). \end{aligned}$$

From this and $n > \frac{13}{3}k + \frac{13}{3}m + \frac{28}{3}$, we have

$$T(r,f) + T(r,g) \le \Big(\frac{1}{3} + \frac{1}{11} + \frac{1}{21}\Big)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),$$

i.e. $0.52[T(r,f)+T(r,g)] \le S(r,f) + S(r,g)$. Then, we get a contradiction.

Thus, we complete the proof of Theorem 1.1. ∎

Proof of Theorem 1.2.

From the condition of Theorem 1.2, we have $\overline{E}_{l)}(1;F^{(k)}) = \overline{E}_{l)}(1;G^{(k)})$, $E_{2)}(1;F^{(k)}) = E_{2)}(1;G^{(k)})$, where $l \ge 4$. From (14)–(19) and Lemma 2.12, we have

$$\Delta_{1l} \ge (k+4)\frac{n+m^*-1}{n+m^*} + 2\frac{n-1}{n+m^*} + 2\frac{n-k-1}{n+m^*}. \tag{30}$$

Since $n > 3m^* + 3k + 6$, we can get $\Delta_{1l} > k+5$. From Lemma 2.12, we have $F^* \equiv G^*$ or $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$.

Proceeding as in the proof of Theorem 1.1, we can get the conclusion of Theorem 1.2. Thus, we complete the proof of Theorem 1.2. ∎

Acknowledgements. We thank the referee(s) for reading the manuscript very carefully and making a number of valuable and kind comments which improved the presentation.

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(received 09.07.2010; in revised form 16.04.2011; available online 01.07.2011)

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