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RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE WITH VARYING ARGUMENTS

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Abstract. In this paper, we define the subclasses $\mathcal{V}_\delta(A,B)$ and $\mathcal{K}_\delta(A,B)$ of analytic functions by using $\Omega^\delta f(z)$. For functions belonging to these classes, we obtain coefficient estimates, distortion bounds and many more properties.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1.1}$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{N}$ denote the subclass of $\mathcal{A}$ consisting of functions normalized by $f(0) = 0$ and $f'(0) = 1$ which are univalent in $\mathcal{U}$.

Silverman [8] defined the class $\mathcal{V}(\theta_m)$ as the class of all functions in $\mathcal{N}$ such that $\arg a_m = \theta_m$ for all m. If further there exists a real number β such that $\theta_m + (m-1)\beta \equiv \pi \pmod{2\pi}$, then f is said to be in the class $\mathcal{V}(\theta_m, \beta)$. The union of $\mathcal{V}(\theta_m, \beta)$ taken over all possible sequences $\{\theta_m\}$ and all possible real numbers β is denoted by $\mathcal{V}$.

The class $\mathcal{A}$ is closed under convolution or Hadamard product

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m, \qquad z \in \mathcal{U}, \tag{1.2}$$

where f is given by (1.1) and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$.

Fractional derivative of order δ of an analytic function f is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta}\, dt, \qquad 0 \le \delta \le 1.$$

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f is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)$ is greater than 0. Clearly $f(z) = \lim_{\delta\to 0} D_z^\delta f(z)$ and $f'(z) = \lim_{\delta\to 1} D_z^\delta f(z)$.

For the analytic function f of the form (1.1) we put

$$\Omega^\delta f(z) = \Gamma(2-\delta) z^\delta D_z^\delta f(z) = z + \sum_{m=2}^{\infty} K(m,\delta) a_m z^m,$$

where $K(m.\delta) = \frac{\Gamma(m+1)\Gamma(2-\delta)}{\Gamma(m+1-\delta)}$.

Now we define the class $\mathcal{V}_\delta(A,B)$ consisting of functions $f \in \mathcal{V}$ such that

$$\frac{z(\Omega^\delta f(z))'}{\Omega^\delta f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad -1 \le A < B \le 1. \tag{1.3}$$

Here $\omega(z)$ is analytic, $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathcal{U}$. The following basic result is well known.

Lemma 1.1 [Schwarz's Lemma] *Let ω be analytic with $\omega(0) = 0$, and $|\omega(z)| < 1$ for $z \in \mathcal{U}$. Then $|\omega(z)| < |z|$. The equality holds if and only if $\omega(z) = \lambda z$, where $|\lambda| = 1$.*

Let $\mathcal{K}_\delta(A,B)$ denote the class of functions $f \in \mathcal{V}$ such that $zf' \in \mathcal{V}_\delta(A,B)$.

2. Main Results

Theorem 2.1. *A function $f \in \mathcal{V}$ is in $\mathcal{V}_\delta(A,B)$ if and only if*

$$\sum_{m=2}^{\infty} [(B+1)m - (A+1)] K(m,\delta) |a_m| \le (B-A), \tag{2.1}$$

where $-1 \le A < B \le 1$, $m \ge 2$.

Proof. Suppose $f \in \mathcal{V}_\delta(A,B)$. Then

$$\frac{z(\Omega^\delta f(z))'}{\Omega_\delta f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad -1 \le A < B \le 1.$$

From this we get,

$$\omega(z) = \frac{z(\Omega^\delta f(z))' - \Omega^\delta f(z)}{\Omega^\delta f(z) A - z(\Omega^\delta f(z))' B}.$$

By Schwarz's Lemma, we get

$$\Re\left\{ \frac{\sum_{m=2}^{\infty} (1-m) K(m,\delta) a_m z^{m-1}}{(B-A) + \sum_{m=2}^{\infty} (mB - A) K(m,\delta) a_m z^{m-1}} \right\} < 1. \tag{2.2}$$

Since $f \in \mathcal{V}$, f lies in $\mathcal{V}(\theta_m, \beta)$ for some sequence $\{\theta_m\}$ and a real number β, such that $\theta_m + (m-1)\beta \equiv \pi \pmod{2\pi}$. Setting $z = re^{i\beta}$, we get

$$\Re\left\{\frac{\sum\limits_{m=2}^{\infty}(1-m)K(m,\delta)|a_m|r^{m-1}e^{i(\theta_m+\overline{(m-1)\beta})}}{(B-A)+\sum\limits_{m=2}^{\infty}(mB-A)|a_m|r^{m-1}e^{i(\theta_m+\overline{(m-1)\beta})}}\right\} < 1. \tag{2.3}$$

$$\sum_{m=2}^{\infty}(m-1)K(m,\delta)|a_m|r^{m-1} < (B-A) - \sum_{m=2}^{\infty}(mB-A)K(m,\delta)|a_m|r^{m-1},$$

$$\sum_{m=2}^{\infty}[(B+1)m-(A+1)K(m,\delta)|a_m|r^{m-1} < (B-A). \tag{2.4}$$

Letting $r \to 1$, we get (2.1).

Conversely, suppose $f \in \mathcal{V}$ and satisfies (2.1). In view of (2.4), which is implied by (2.1), since $r^{m-1} < 1$, we have

$$\begin{aligned}\left|\sum_{m=2}^{\infty}(1-m)K(m,\delta)a_m z^{m-1}\right| &\le \sum_{m=2}^{\infty}(m-1)K(m,\delta)|a_m|r^{m-1}\\ &< (B-A) - \sum_{m=2}^{\infty}(mB-A)K(m,\delta)|a_m|r^{m-1}\\ &\le \left|(B-A) - \sum_{m=2}^{\infty}(mB-A)K(m,\delta)a_m z^{m-1}\right|\end{aligned}$$

which gives (2.2) and hence it follows that $f \in \mathcal{V}_\delta(A,B)$. ■

Corollary 2.2. *If $f \in \mathcal{V}$ is in $\mathcal{V}_\delta(A,B)$ then*

$$|a_m| \le \frac{(B-A)}{[(B+1)m-(A+1)]K(m,\delta)}, \qquad \text{for} \quad m \ge 2, \quad -1 \le A < B \le 1.$$

The equality holds for the function f given by

$$f(z) = z + \frac{(B-A)}{[(B+1)m-(A+1)]K(m,\delta)}e^{i\theta_m}z^m, \quad z \in \mathcal{U}.$$

For parametric values $a = n+1$, $c = 1$, we get the following result proved by Padmanabhan and Jayamala [4] as corollaries to the above theorem.

Corollary 2.3. *Let $f \in \mathcal{V}$. Then $f \in \mathcal{V}_n(A,B)$ if and only if*

$$\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!\,(m-1)!}C_m|a_m| < (B-A),$$

where $C_m = (B+1)(n+m)-(A+1)(n+1)$.

The equality holds for the function f given by

$$f(z) = z + \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)}\frac{(B-A)}{D_m}e^{i\theta_m}z^m, \quad z \in \mathcal{U}.$$

THEOREM 2.4. *Let $f \in \mathcal{V}$. Then $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is in $\mathcal{K}(A, B, a, c)$ if and only if*

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} m D_m a_m < B - A,$$

where $D_m = [(B+1)(a+m-1) - (A+1)a]$, $-1 \leq A < B \leq 1$, $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$.

Now we examine the extreme points of the class $\mathcal{V}(A, B, a, c)$.

THEOREM 2.5. *Let $f(z) \in \mathcal{V}(A, B, a, c)$ with $\arg a_m = \theta_m$, where $[\theta_m + (m-1)\beta] \equiv \pi \pmod{2\pi}$. Define $f_1(z) = z$ and*

$$f_m(z) = z + \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m} e^{i\theta_m} z^m, \quad m = 2, 3, \ldots,$$

$-1 \leq A < B \leq 1$, $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $z \in \mathcal{U}$. $f \in \mathcal{V}(A, B, a, c)$ if and only if f can be expressed as $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ where $\mu_m \geq 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

Proof. If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1$, $\mu_m \geq 0$, then

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m \mu_m \cdot \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m}$$
$$= \sum_{m=2}^{\infty} \mu_m (B-A) = (1-\mu_1)(B-A) \leq (B-A).$$

Hence $f \in \mathcal{V}(A, B, a, c)$.

Conversely, let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in \mathcal{V}(A, B, a, c)$, define

$$\mu_m = \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} \frac{|a_m| D_m}{(B-A)}, \quad m = 2, 3, \ldots$$

and define $\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem 2.1, $\sum_{m=2}^{\infty} \mu_m \leq 1$ and so $\mu_1 \geq 0$. Since $\mu_m f_m(z) = \mu_m f + a_m z^m$, $\sum_{m=1}^{\infty} \mu_m f_m(z) = z + \sum_{m=2}^{\infty} a_m z^m = f(z)$. ∎

THEOREM 2.6. *Define $f_1(z) = z$ and*

$$f_m(z) = z + \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m} z^m, \quad m = 2, 3, \ldots,$$

$-1 \leq A < B \leq 1$, $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $z \in \mathcal{U}$. Then $f \in \mathcal{K}(A, B, a, c)$ if and only if f can be expressed as $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ where $\mu_m \geq 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

THEOREM 2.7. *The class $\mathcal{V}(A, B, a, c)$ is closed under convex linear combination.*

Proof. Let $f, g \in \mathcal{V}(A, B, a, c)$ and let

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = z + \sum_{m=2}^{\infty} b_m z^m.$$

For η such that $0 \le \eta \le 1$, it suffices to show that the function defined by $h(z) = (1-\eta)f(z) + \eta g(z)$, $z \in \mathcal{U}$ belongs to $\mathcal{V}(A,B,a,c)$. Now

$$h(z) = z + \sum_{m=2}^{\infty} [(1-\eta)a_m + \eta b_m] z^m.$$

Applying Theorem 2.1 to $f, g \in \mathcal{V}(A,B,a,c)$, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m[(1-\eta)a_m + \eta b_m] \\
&= (1-\eta)\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m a_m + \eta \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m b_m \\
&\le (1-\eta)(B-A) + \eta(B-A) = B-A.
\end{aligned}$$

This implies that $h \in \mathcal{V}(A,B,a,c)$. ∎

Corollary 2.8. *If $f_1(z), f_2(z)$ are in $\mathcal{V}(A,B,a,c)$ then the function defined by $g(z) = \frac{1}{2}[f_1(z) + f_2(z)]$ is also in $\mathcal{V}(A,B,a,c)$.*

Theorem 2.9. *The class $\mathcal{K}(A,B,a,c)$ is closed under convex linear combination.*

Theorem 2.10. *Let for $j = 1,2,\dots,m$, $f_j(z) = z + \sum_{m=2}^{\infty} a_{m,j} z^m \in \mathcal{V}(A,B,a,c)$ and $0 < \lambda_j < 1$ such that $\sum_{j=1}^{m} \lambda_j = 1$. Then the function $F(z)$ defined by $F(z) = \sum_{j=1}^{m} \lambda_j f_j(z)$ is also in $\mathcal{V}(A,B,a,c)$.*

Proof. For each $j \in \{1,2,\dots,m\}$ we obtain

$$\sum_{m=2}^{\infty} D_m \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} |a_m| < B-A.$$

Since $F(z) = \sum_{j=1}^{m} \lambda_j (z - \sum_{m=2}^{\infty} a_{m,j} z^m) = z - \sum_{m=2}^{\infty} (\sum_{j=1}^{m} \lambda_j a_{m,j}) z^m$,

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m [\sum_{j=1}^{m} \lambda_j a_{m,j}] \\
&\qquad = \sum_{j=1}^{m} \lambda_j \left[\sum_{m=2}^{\infty} D_m \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} \right] < \sum_{j=1}^{m} \lambda_j (B-A) < B-A.
\end{aligned}$$

Therefore $F(z) \in \mathcal{V}(A,B,a,c)$. ∎

Theorem 2.11. *Let $f(z) \in \mathcal{V}(A,B,a,c)$ and Komato operator of f is defined by*

$$k(z) = \int_0^1 \frac{(c+1)^\gamma}{\Gamma(\gamma)} t^c \left(\log \frac{1}{t}\right)^{\gamma-1} \frac{f(tz)}{t}\, dt,$$

$c > -1$, $\gamma \ge 0$. Then $k(z) \in \mathcal{V}(A,B,a,c)$.

Proof. We have

$$\int_0^1 t^c \left(\log \frac{1}{t}\right)^{\gamma-1} dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma}$$

$$\int_0^1 t^{m+c-1} \left(\log \frac{1}{t}\right)^{\gamma-1} dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma}, \quad m = 2, 3, \ldots,$$

$$\begin{aligned} k(z) &= \frac{(c+1)^\gamma}{\Gamma(\gamma)} \left[\int_0^1 t^c \left(\log \frac{1}{t}\right)^{\gamma-1} z\, dt + \sum_{m=2}^{\infty} z^m \int_0^1 a_m t^{m+c-1} \left(\log \frac{1}{t}\right)^{\gamma-1} dt\right] \\ &= z + \sum_{m=2}^{\infty} \left(\frac{c+1}{c+m}\right)^\gamma a_m z^m. \end{aligned}$$

Since $f \in \mathcal{V}(A, B, a, c)$ and since $\left(\frac{c+1}{c+m}\right)^\gamma < 1$, we have

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}[(1+A) - m(1+B)] \left(\frac{c+1}{c+m}\right)^\gamma a_m < B - A. \quad \blacksquare$$

In the next theorem we will find distortion bound for $L(a, c)f(z)$.

THEOREM 2.12. *If $f \in \mathcal{V}(A, B, a, c)$, then*

$$|z| - \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2 \le |L(a, c)f(z)| \le |z| + \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2.$$

Proof. Let $f(z) \in \mathcal{V}(A, B, a, c)$. Using Theorem 2.1,

$$\sum_{m=2}^{\infty} a_m \le \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}.$$

Therefore

$$|L(a, c)f(z)| \le |z| + |z|^2 \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} a_m < |z| + \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2$$

and

$$|L(a, c)f(z)| \ge |z| - |z|^2 \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} a_m > |z| - \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2. \quad \blacksquare$$

REMARK 2.13. (i) For parametric values of $a = 1$ and $c = 1$ we get

$$|z| - \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2 \le f(z) \le |z| + \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2.$$

(ii) For parametric values of $a = 2$ and $c = 1$ we get

$$1 - \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z| \le f'(z) \le 1 + \frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|.$$

THEOREM 2.14. *Let $f \in \mathcal{V}(A, B, a, c)$. Then for every $0 \leq \delta < 1$ the function*

$$\mathcal{H}_\delta(z) = (1-\delta)f(z) + \delta \int_0^z \frac{f(t)}{t}\, dt.$$

Proof. We have $\mathcal{H}_\delta(z) = z + \sum_{m=2}^{\infty} \left(1 + \frac{\delta}{m} - \delta\right) a_m z^m$. Since $\left(1 + \frac{\delta}{m} - \delta\right) < 1$, $m \geq 2$, so by Theorem 2.1,

$$\sum_{m=2}^{\infty} \left(1 + \frac{\delta}{m} - \delta\right) D_m a_m \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} < \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m a_m < B - A.$$

Therefore $\mathcal{H}_\delta(z) \in \mathcal{V}(A, B, a, c)$. ■

REFERENCES

[**1**] H.S. Al-Amiri, *On Ruscheweyh derivatives*, Ann. Polon. Math. **38** (1980), 87–84.

[**2**] B.C. Carlson, D.B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), 737–745.

[**3**] G Murugusundara Moorty, K. Vijaya, R.K. Raina, *Harmonic functions with varying arguments defined by Dziok-Srivastava operator*, Arch. Math. (Brno) **45** (2009), 37–46.

[**4**] K.S. Padmanabhan, M. Jayamala, *A class of univalent functions with varying arguments*, Inter. J. Math. Math. Sci **15** (1992), 517–522.

[**5**] P. Sharma, P. Srivastava, *A note on analytic functions with varying arguments*, Intern. J. Math. Archive **1** (2010), 5–9.

[**6**] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc **49** (1975), 109–115.

[**7**] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives Theory and Applications*, Gordon and Breach Science Publishing, Switzerland, 1993.

[**8**] H. Silverman, *Univalent functions with varying raguments*, Houston J. Math. **7** (1981).

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