МАТЕМАТИЧКИ ВЕСНИК
64, 2 (2012), 147–158
June 2012

оригинални научни рад
research paper

JACOBI TYPE AND GEGENBAUER TYPE GENERALIZATION OF CERTAIN POLYNOMIALS

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Abstract. This paper deals with the Jacobi type and Gegenbauer type generalizations of certain polynomials and their generating functions. Relationships among those generalized polynomials have also been indicated.

1. Introduction

In 1947, Sister Celine (Fasenmyer [2]) obtained some basic formal properties of the hypergeometric polynomials. Sister Celine's polynomials are defined by the following generating relation

$$(1-t)^{-1}{}_rF_s\left[\begin{matrix}a_1,\dots,a_r;\\ b_1,\dots,b_s;\end{matrix}\ -\frac{4xt}{(1-t)^2}\right]=\sum_{n=0}^{\infty}f_n\left[\begin{matrix}a_1,\dots,a_p;\\ b_1,\dots,b_s;\end{matrix}\ x\right]t^n \tag{1.1}$$

which yields

$$\begin{aligned}f(a_i;b_i;x)&\equiv f_n(a_1,a_2,\cdots,a_r;b_1,b_2,\cdots,b_s;x)\\ &\equiv {}_{r+2}F_{s+2}\left[\begin{matrix}-n,\ n+1,\ a_1,\dots,a_r;\\ 1,\ \frac{1}{2},\ b_1,\dots,b_s;\end{matrix}\ x\right]\end{aligned}$$

(n is non negative integer) in an attempt to unify and to extend the study of certain sets of polynomials which have attracted considerable attention.

Her polynomials include as special cases Legendre's polynomials $P_n(1-2x)$, some special Jacobi polynomials, Rice's polynomials $H_n(\xi,p,\nu)$, Bateman's $Z_n(x)$, $F_n(z)$ and Pasternak's $F_n^m(z)$ which is a generalization of Bateman's polynomials $F_n(z)$.

2010 AMS Subject Classification: 33C45.

Keywords and phrases: Jacobi type and Gegenbauer type generalization of Sister Celine's polynomials, Bateman's polynomials, Pasternack's polynomials, Hahn polynomial, Rice polynomials, generating functions.

1.1. Bateman's polynomials

Bateman defined the following polynomials (see [8], [7; eq. (1), p. 289])

$$F_n(z) = {}_3F_2\left[\begin{matrix} -n,\ n+1,\ \frac{1}{2}(1+z); \\ 1,\ \ 1; \end{matrix}\ 1\right]$$

Bateman [7], [8], obtained the following generating functions

$$\sum_{n=0}^{\infty} F_n(z)t^n = \frac{1}{1-t}\ {}_2F_1\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2}+\frac{1}{2}z; \\ 1; \end{matrix}\ \frac{-4t}{(1-t)^2}\right]$$

$$\sum_{n=0}^{\infty}[F_n(z-2)-F_n(z)]t^n = \frac{2t}{(1-t)^3}\ {}_2F_1\left[\begin{matrix} \frac{3}{2},\ \frac{1}{2}+\frac{1}{2}z; \\ 2; \end{matrix}\ \frac{-4t}{(1-t)^2}\right]$$

1.2. Pasternack's polynomials

The generalization of Bateman's polynomial due to Pasternack is given below

$$F_n^m(z) = {}_3F_2\left[\begin{matrix} -n,\ n+1,\ \frac{1}{2}(z+m+1); \\ 1,\ \ m+1; \end{matrix}\ 1\right]$$

which is a generalization of Bateman's polnomials $F_n(z)$. Generating function of generalization of Pasternack's polynomials is given below

$$\sum_{n=0}^{\infty} F_n^m(z)t^n = \frac{1}{(1-t)}\ {}_2F_1\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2}(z+m+1); \\ m+1; \end{matrix}\ -\frac{4t}{(1-t)^2}\right]$$

In 1936, Bateman [7] was interested in constructing inverse Laplace transforms. For this purpose he introduced the following polynomial

$$Z_n(x) = {}_2F_2\left[\begin{matrix} -n,\ n+1; \\ 1,\ 1; \end{matrix}\ x\right]$$

By theorem 48 (see [7, p. 137]), Rainville writes the generating function as follows

$$\sum_{n=0}^{\infty} Z_n(x)t^n = \frac{1}{1-t}{}_2F_1\left[\begin{matrix} \frac{1}{2},\ n+1; \\ 1; \end{matrix}\ \frac{-4xt}{(1-t)^2}\right]$$

In 1939, Pasternack [7] obtained the following generating function of Bateman's polynomials

$$\sum_{n=0}^{\infty} F_m(-2n-1)\frac{(-t)^n}{n!} = e^{-t}Z_m(t)$$

$$\sum_{n=0}^{\infty} F_n(-2m-1)t^n = (1-t)^{-m-1}(1+t)^m P_m\Big(\frac{1+t^2}{1-t^2}\Big)$$

1.3. Rice's polynomials

S. O. Rice made a considerable study of the polynomials defined by

$$H_n(\xi,p,\nu) = {}_3F_2\left[\begin{matrix} -n,\ \ n+1,\ \xi; \\ 1,\ \ p; \end{matrix}\ \nu\right]$$

The generalized Rice's Polynomial due to Khandekar [4], is given below

$$\frac{n!}{(1+\alpha)_n} H_n^{(\alpha,\beta)}(\xi,p,\nu) = {}_3F_2\left[\begin{matrix} -n, & n+\alpha+\beta+1, \ \xi; \\ & 1+\alpha, \ \ p; \end{matrix} \ \nu\right] \tag{1.2}$$

Generating function of the generalized Rice polynomial due to Khandekar [4], is given below

$$\sum_{n=0}^{\infty} \frac{(2\alpha+1)_n}{(1+\alpha)_n} H_n^{(\alpha,\beta)}(\xi,p,\nu)t^n = (1-t)^{-\alpha-\beta-1}{}_3F_2\left[\begin{matrix} \Delta(2;\alpha+\beta+1), & \xi; \\ 1+\alpha, & p; \end{matrix} \ -\frac{4\nu t}{(1-t)^2}\right]$$

1.4. Hahn polynomials

Hahn polynomial is defined as

$$Q_n(x;\alpha,\beta,\ N) = {}_3F_2\left[\begin{matrix} -n, \ n+\alpha+\beta, \ -x; \\ 1+\alpha, \ -N; \end{matrix} \ 1\right], \quad \alpha,\ \beta > -1, \ \ n,x = 0,1,\ldots,N.$$

The following generating functions are satisfied by the Hahn polynomial

$$\sum_{n=0}^{\infty} \frac{(-N)_n}{(\beta+1)_n n!} Q_n(x;\alpha,\beta,\ N)t^n = {}_1F_1\left[-x;\alpha+1; \ \ -t\right] {}_1F_1\left[\begin{matrix} x-N; \\ \beta+1; \end{matrix} \ t\right],$$

$x = 0,1,\ldots,N$, and

$$\sum_{n=0}^{\infty} \frac{(-N)_n}{(\beta+1)_n n!} Q_n(x;\alpha,\beta,\ N)t^n = {}_2F_0\left[\begin{matrix} -x, \ -x+\beta+N+1; \\ \_\_\_\_; \end{matrix} \ -t\right] \times$$

$$\times \, {}_2F_0\left[\begin{matrix} x-N, \ -x+\alpha+1; \\ \_\_\_\_; \end{matrix} \ t\right], \quad x = 0,1,\ldots,N.$$

Motivated by the Jacobi type generalization of the Rice's polynomials obtained by Khandekar [4], we aim here to obtain Jacobi type generalization of the polynomials mentioned in the first section of this paper.

2. Jacobi type generalization of certain polynomials and their generating functions

Before obtaining the Gegenbauer type generalization of polynomials, we shall first discuss the Jacobi type generalization of polynomials. The Gegenbauer type generalizations shall be discussed in the next section of this paper.

The Jacobi type generalization of Sister Celine's polynomial is given below.

2.1. Jacobi type generalization of Sister Celine's polynomial

Substituting $\lambda = c, p = r, q = s, r = 1, s = 1$ and $\mu = 2\alpha+1$ in the Eq. (3.16) of the reference [3] the following generalized Sister Celine's polynomials are obtained by means of the following generating relation

$$(1-t)^{-c}{}_{2+r}F_{2+s}\left[\begin{matrix} \frac{c}{2}, \ \frac{1+c}{2}, \ a_1,\ldots,a_r; \\ 1+\alpha, \ \frac{1}{2}+\alpha, \ b_1,\ldots,b_s; \end{matrix} \ -\frac{4xt}{(1-t)^2}\right]$$

$$= \sum_{n=0}^{\infty} f_n\left[\begin{matrix} \frac{c}{2}, \ \frac{1+c}{2}, \ a_1,\ldots,a_p; \\ 1+\alpha, \ \frac{1}{2}+\alpha, \ b_1,\ldots,b_s; \end{matrix} \ x\right] t^n$$

which produces the following relation

$$f_n\left[\begin{matrix}\frac{c}{2},\ \frac{1+c}{2},\ a_1,\dots,a_p;\\ 1+\alpha,\ \frac{1}{2}+\alpha,\ b_1,\dots,b_s;\end{matrix}\ x\right]=\frac{(c)_n}{n!}\,{}_{2+r}F_{2+s}\left[\begin{matrix}-n,\ n+c,\ a_1,\dots,a_r;\\ 1+\alpha,\ \frac{1}{2}+\alpha,\ b_1,\dots,b_s;\end{matrix}\ x\right]$$

Case (i). For $\alpha=0$ it reduces to

$$(1-t)^{-c}\,{}_{2+r}F_{2+s}\left[\begin{matrix}\frac{c}{2},\ \frac{1+c}{2},\ a_1,\dots,a_r;\\ 1,\ \frac{1}{2},\ b_1,\dots,b_s;\end{matrix}\ -\frac{4xt}{(1-t)^2}\right]$$
$$=\sum_{n=0}^{\infty}f_n\left[\begin{matrix}\frac{c}{2},\ \frac{1+c}{2},\ a_1,\dots,a_r;\\ 1,\ \frac{1}{2},\ b_1,\dots,b_s;\end{matrix}\ x\right]t^n$$

Case (ii). For $\alpha=0$ and $c=1$, it reduces to original Sister Celine polynomial (see [7; eq. (1), pp. 290] of Rainville) and eq. (1.1).

Case (iii). For $c=1+\alpha+\beta$, it gives Jacobi type generalization of Sister Celine's polynomial

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n}{n!}\,{}_{2+r}F_{2+s}\left[\begin{matrix}-n,\ n+1+\alpha+\beta,\ a_1,\dots,a_r;\\ 1+\alpha,\ \frac{1}{2}+\alpha,\ b_1,\dots,b_s;\end{matrix}\ x\right]t^n$$
$$=\frac{1}{(1-t)^{1+\alpha+\beta}}\,{}_{2+r}F_{2+s}\left[\begin{matrix}\frac{1+\alpha+\beta}{2},\ \frac{2+\alpha+\beta}{2},\ a_1,\dots,a_r;\\ 1+\alpha,\ \frac{1}{2}+\alpha,\ b_1,\dots,b_s;\end{matrix}\ -\frac{4xt}{(1-t)^2}\right]$$

For $\alpha=\beta$, we get the following ultraspherical type generalization of Sister Celine's polynomials

$$\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{n!}\,{}_{2+r}F_{2+s}\left[\begin{matrix}-n,\ n+2\alpha+1,\ a_1,\dots,a_r;\\ 1+\alpha,\ \frac{1}{2}+\alpha,\ b_1,\dots,b_s;\end{matrix}\ x\right]t^n$$
$$=\frac{1}{(1-t)^{1+2\alpha}}\,{}_rF_s\left[\begin{matrix}a_1,\dots,a_r;\\ b_1,\dots,b_s;\end{matrix}\ -\frac{4xt}{(1-t)^2}\right]$$

Case (iv). For $c=1+\alpha+\beta$ and $r=s=0$ we have

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n}{n!}\,{}_2F_2\left[\begin{matrix}-n,\ n+1+\alpha+\beta;\\ 1+\alpha,\ \frac{1}{2}+\alpha;\end{matrix}\ x\right]t^n$$
$$=\frac{1}{(1-t)^{1+\alpha+\beta}}\,{}_2F_2\left[\begin{matrix}\frac{1+\alpha+\beta}{2},\ \frac{2+\alpha+\beta}{2};\\ 1+\alpha,\ \frac{1}{2}+\alpha;\end{matrix}\ -\frac{4xt}{(1-t)^2}\right]$$

Case (v). When$\alpha=\beta=r=s=0$, we have the following form of Sister Celine's polynomial

$$\sum_{n=0}^{\infty}{}_2F_2\left[\begin{matrix}-n,\ n+1;\\ 1,\ \frac{1}{2};\end{matrix}\ x\right]t^n=\frac{1}{1-t}\exp\left[-\frac{4xt}{(1-t)^2}\right].$$

2.2. Jacobi type generalization of Bateman's polynomials

In 1999, M. A. Khan and A. K. Shukla [3] defined the Jacobi type generalization of Bateman's polynomials as follows

$$F_n^{(\alpha,\beta)}(p,z)={}_3F_2\left[\begin{matrix}-n,\ n+\alpha+\beta+1,\ \frac{1}{2}(1+z);\\ 1+\alpha,\ p;\end{matrix}\ 1\right].\tag{2.1}$$

We define the generating function for the Jacobi type generalization of Bateman's polynomials defined by Khan and Shukla.

$$\sum_{n=0}^{\infty}\frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}}F_n^{(\alpha,\beta)}(p,z)t^n$$
$$=\sum_{n=0}^{\infty}\frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k(\frac{1}{2}(1+z))_kt^n}{(1+\alpha)_k(p)_kk!}$$
$$=\sum_{n=0}^{\infty}\frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}}\sum_{k=0}^{n}\frac{(-1)^k(1)_n}{(n-k)!}\frac{(n+\alpha+\beta+1)_k(\frac{1}{2}(1+z))_kt^n}{(1+\alpha)_k(p)_kk!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(1)_{n+\alpha+\beta+k}}{(n-k)!}\frac{(\frac{1}{2}(1+z))_kt^n}{(1)_{\alpha+\beta}(1+\alpha)_k(p)_kk!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1)_{n+\alpha+\beta+2k}}{n!}\frac{(\frac{1}{2}(1+z))_kt^{n+k}}{(1)_{\alpha+\beta}(1+\alpha)_k(p)_kk!}$$
$$=\sum_{k=0}^{\infty}\frac{(1)_{\alpha+\beta}(1+\alpha+\beta)_{2k}(\frac{1}{2}(1+z))_k(-t)^k}{(1)_{\alpha+\beta}(1+\alpha)_k(p)_kk!}\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta+2k)_nt^n}{n!}$$
$$=\frac{1}{(1-t)^{1+\alpha+\beta}}\sum_{k=0}^{\infty}\frac{2^{2k}(\frac{1+\alpha+\beta}{2})_k(\frac{2+\alpha+\beta}{2})_k(\frac{1}{2}(1+z))_k(-t)^k}{(1+\alpha)_k(p)_kk!}\frac{1}{(1-t)^{2k}}$$
$$=\frac{1}{(1-t)^{1+\alpha+\beta}}\ {}_3F_2\left[\begin{matrix}\frac{1+\alpha+\beta}{2}, & \frac{2+\alpha+\beta}{2}, & \frac{1}{2}(1+z); & -\frac{4t}{(1-t)^2}\\ & 1+\alpha, p; & & \end{matrix}\right].$$

Another kind of generating function can also be defined for the Jacobi type generalization of the Bateman's polynomials, given below

$$\sum_{n=0}^{\infty}\frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}}[F_n^{(\alpha,\beta)}(p,z-2)-F_n^{(\alpha,\beta)}(p,z)]t^n$$
$$=\sum_{n=0}^{\infty}\frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(1+\alpha)_k(p)_kk!}\left\{\left(\tfrac{1}{2}(-1+z)\right)_k-\left(\tfrac{1}{2}(1+z)\right)_k\right\}t^n$$
$$=\sum_{n=1}^{\infty}\frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}}\sum_{k=1}^{n}\frac{(-1)^k(1)_n}{(n-k)!}\frac{(n+\alpha+\beta+1)_k}{(1+\alpha)_k(p)_kk!}\left\{(-k)\left(\tfrac{1}{2}(1+z)\right)_{k-1}\right\}t^n$$
$$=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k+1}(1)_{n+\alpha+\beta+k}}{(n-k)!}\frac{(\frac{1}{2}(1+z))_{k-1}t^n}{(1)_{\alpha+\beta}(1+\alpha)_k(p)_k(k-1)!}$$
$$=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(-1)^{k+2}(1)_{n+\alpha+\beta+k+1}}{(n-k-1)!}\frac{(\frac{1}{2}(1+z))_kt^{n-1}}{(1)_{\alpha+\beta}(1+\alpha)_{k+1}(p)_{k+1}k!}$$
$$=\sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1)_{n+\alpha+\beta+2k}}{(n-2)!}\frac{(\frac{1}{2}(1+z))_kt^{n+k-2}}{(1)_{\alpha+\beta}(1+\alpha)_{k+1}(p)_{k+1}k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1)_{n+\alpha+\beta+2k+2}}{n!}\frac{(\frac{1}{2}(1+z))_kt^{n+k}}{(1)_{\alpha+\beta}(1+\alpha)_{k+1}(p)_{k+1}k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1)_{\alpha+\beta}(1+\alpha+\beta)_{n+2k+2}(\frac{1}{2}(1+z))_k(-t)^kt^n}{(1)_{\alpha+\beta}(1+\alpha)_{k+1}(p)_{k+1}n!\,k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_2(3+\alpha+\beta)_{n+2k}(\frac{1}{2}(1+z))_k(-t)^k t^n}{(1+\alpha)_{k+1}(p)_{k+1} n!\, k!}$$

$$= (1+\alpha+\beta)_2 \sum_{k=0}^{\infty} \frac{(3+\alpha+\beta)_{2k}(\frac{1}{2}(1+z))_k(-t)^k}{(1+\alpha)_{k+1}(p)_{k+1} n! k!} \sum_{n=0}^{\infty} \frac{(3+\alpha+\beta+2k)t^n}{n!}$$

$$= \frac{(1+\alpha+\beta)_2}{(1-t)^{3+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{2^{2k}(\frac{3+\alpha+\beta}{2})_k(\frac{4+\alpha+\beta}{2})_k(\frac{1}{2}(1+z))_k(-t)^k}{(1+\alpha)(2+\alpha)_k(p)(p+1)_k k!} \frac{1}{(1-t)^{2k}}$$

$$= \frac{(1+\alpha+\beta)_2}{(1-t)^{3+\alpha+\beta}(1+\alpha)(p)} \,{}_3F_2\left[\begin{matrix}\frac{3+\alpha+\beta}{2}, & \frac{4+\alpha+\beta}{2}, & \frac{1}{2}(1+z); \\ & 2+\alpha, p+1; & \end{matrix}\; -\frac{4t}{(1-t)^2}\right].$$

For $\alpha=\beta$, we get ultraspherical type generalization of Bateman's polynomials

$$F_n^{(\alpha,\alpha)}(p,z) = {}_3F_2\left[\begin{matrix}-n, & n+2\alpha+1, & \frac{1}{2}(1+z); \\ & 1+\alpha, \; p; & \end{matrix}\; 1\right]$$

Generating function for these polynomials is given below

$$\sum_{n=0}^{\infty} \frac{(1+n)_{2\alpha}}{(1)_{2\alpha}}\{F_n^{(\alpha,\alpha)}(p,z-2) - F_n^{(\alpha,\alpha)}(p,z)\}t^n$$

$$= \frac{(1+2\alpha)_2}{(1-t)^{3+2\alpha}(1+\alpha)p}\,{}_2F_1\left[\begin{matrix}\frac{3+2\alpha}{2}, & \frac{1}{2}(1+z); \\ & p+1; \end{matrix}\; -\frac{4t}{(1-t)^2}\right]$$

2.3. Jacobi type generalization of Pasternack's polynomial

By substituting $z=z+m$, $p=1+m$ in eq. (2.1), Khan and Shukla [3] obtained Jacobi type generalization of Pasternack's generalized Bateman's polynomial $F_n^m(z)$, given below

$$F_{n,m}^{(\alpha,\beta)}(z) = {}_3F_2\left[\begin{matrix}-n, & n+\alpha+\beta+1, & \frac{1}{2}(1+z+m); \\ & 1+\alpha, \; 1+m; & \end{matrix}\; 1\right]. \tag{2.2}$$

We derive here the generating function of these polynomials

$$\sum_{n=0}^{\infty} \frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}} F_{n,m}^{(\alpha,\beta)}(z)t^n$$

$$= \frac{1}{(1-t)^{1+\alpha+\beta}}\,{}_3F_2\left[\begin{matrix}\frac{1+\alpha+\beta}{2}, & \frac{2+\alpha+\beta}{2}, & \frac{1}{2}(1+z+m); \\ & 1+\alpha, 1+m; & \end{matrix}\; -\frac{4t}{(1-t)^2}\right].$$

Substitution of $\alpha=\beta$ gives us its ultraspherical type generalization

$$F_{n,m}^{(\alpha,\alpha)}(z) = {}_3F_2\left[\begin{matrix}-n, & n+2\alpha+1, & \frac{1}{2}(1+z+m); \\ & 1+\alpha, \; 1+m; & \end{matrix}\; 1\right]. \tag{2.3}$$

The following is the generating function of the ultraspherical type generalized of the polynomial (2.3)

$$\sum_{n=0}^{\infty} \frac{(1+n)_{2\alpha}}{(1)_{2\alpha}} F_{n,m}^{(\alpha,\alpha)}(z)t^n = \frac{1}{(1-t)^{1+2\alpha}}\,{}_2F_1\left[\begin{matrix}\frac{1+2\alpha}{2}, & \frac{1}{2}(1+z+m); \\ & 1+m; \end{matrix}\; -\frac{4t}{(1-t)^2}\right].$$

2.4. Jacobi type generalization of Bateman's polynomial $Z_n(x)$

Jacobi type generalization of Bateman's polynomials $Z_n(x)$ was considered to be new in the last decade. For $\alpha = \beta = \nu - \frac{1}{2}$, it reduces to Gegenbauer generalization of the Bateman's polynomials and also for $\alpha = \beta = 0$, it reduces to Bateman's polynomial. Khan and Shukla [3] adopted the symbol $Z_n^{(\alpha,\beta)}(b,x)$ to denote the Jacobi type generalization of Bateman's polynomial $Z_n(x)$.

$$Z_n^{(\alpha,\beta)}(b,x) = {}_2F_2\left[\begin{matrix} -n,\ n+\alpha+\beta+1; \\ 1+\alpha,\ b+1; \end{matrix}\ x\right]. \tag{2.4}$$

We determine the generating function for the polynomial $Z_n^{(\alpha,\beta)}(b,x)$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}} Z_n^{(\alpha,\beta)}(b,x)t^n &= \sum_{n=0}^{\infty} \frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k x^k t^n}{(1+\alpha)_k(b+1)_k k!} \\
&= \sum_{n=0}^{\infty} \frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(-1)^k(1)_n}{(n-k)!} \frac{(n+\alpha+\beta+1)_k x^k t^n}{(1+\alpha)_k(b+1)_k k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k(1)_{n+\alpha+\beta+k}}{(n-k)!} \frac{x^k t^n}{(1)_{\alpha+\beta}(1+\alpha)_k(b+1)_k k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k(1)_{n+\alpha+\beta+2k}}{n!} \frac{x^k t^{n+k}}{(1)_{\alpha+\beta}(1+\alpha)_k(b+1)_k k!} \\
&= \sum_{k=0}^{\infty} \frac{(1)_{\alpha+\beta}(1+\alpha+\beta)_{2k}(-xt)^k}{(1)_{\alpha+\beta}(1+\alpha)_k(b+1)_k k!} \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta+2k)_n t^n}{n!} \\
&= \frac{1}{(1-t)^{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{2^{2k}(\frac{1+\alpha+\beta}{2})_k(\frac{2+\alpha+\beta}{2})_k(-xt)^k}{(1+\alpha)_k(b+1)_k k!} \frac{1}{(1-t)^{2k}} \\
&= \frac{1}{(1-t)^{1+\alpha+\beta}}\ {}_2F_2\left[\begin{matrix} \frac{1+\alpha+\beta}{2},\ \frac{2+\alpha+\beta}{2}; \\ 1+\alpha, b+1; \end{matrix}\ -\frac{4xt}{(1-t)^2}\right].
\end{aligned}$$

The generating function given below establishes a relation between $F_{m,n}^{(\alpha,\beta)}$ and $Z_n^{(\alpha,\beta)}(m,t)$; we have

$$F_{m,n}^{(\alpha,\beta)}(z) = {}_3F_2\left[\begin{matrix} -m,\ m+\alpha+\beta+1,\ \frac{1}{2}(1+z+n); \\ 1+\alpha,\ 1+m; \end{matrix}\ 1\right], \tag{2.5}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{m,n}^{(\alpha,\beta)}(-2n-1)\frac{(-t)^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-m)_k(m+\alpha+\beta+1)_k(-n)_k(-t)^n}{(1+\alpha)_k(1+m)_k k! n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-m)_k(m+\alpha+\beta+1)_k}{(1+\alpha)_k(1+m)_k k! n!} \frac{(-1)^k(1)_n(-t)^n}{(n-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-m)_k(m+\alpha+\beta+1)_k}{(1+\alpha)_k(1+m)_k k!} \frac{(-1)^k(-t)^{n+k}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \sum_{k=0}^{\infty} \frac{(-m)_k(m+\alpha+\beta+1)_k t^k}{(1+\alpha)_k(1+m)_k k!}
\end{aligned}$$

$$= e^{-t}\,{}_2F_2\left[\begin{matrix}-m, & m+\alpha+\beta+1; \\ & 1+\alpha, 1+m;\end{matrix}\ t\right] = e^{-t} Z_m^{(\alpha,\beta)}(m,t).$$

For $\alpha = \beta$,we get the ultraspherical type generalization of it, as given below:

$$F_{m,n}^{(\alpha,\alpha)}(z) = {}_3F_2\left[\begin{matrix}-m,\ m+2\alpha+1,\ \frac{1}{2}(1+z+n); \\ 1+\alpha,\ 1+m;\end{matrix}\ 1\right],$$

$$\sum_{n=0}^{\infty} F_{m,n}^{(\alpha,\alpha)}(-2n-1)\frac{(-t)^n}{n!} = e^{-t}{}_2F_2\left[\begin{matrix}-m, & m+2\alpha+1; \\ & 1+\alpha, 1+m;\end{matrix}\ t\right] = e^{-t} Z_m^{(\alpha,\alpha)}(m,t).$$

In particular, for $\alpha = \beta$, we obtain the ultraspherical generalization of Bateman's polynomial due to Khan and Shukla [3]

$$Z_n^{(\alpha,\alpha)}(b,x) = {}_2F_2\left[\begin{matrix}-n,\ n+2\alpha+1; \\ 1+\alpha,\ b+1;\end{matrix}\ x\right]$$

Generating function for the ultraspherical generalized Bateman's polynomials is given below

$$\sum_{n=0}^{\infty} \frac{(1+n)_{2\alpha}}{(1)_{2\alpha}} Z_n^{(\alpha,\alpha)}(b,x)t^n = \frac{1}{(1-t)^{1+2\alpha}}\,{}_1F_1\left[\begin{matrix}\frac{1+2\alpha}{2}; \\ b+1;\end{matrix}\ -\frac{4xt}{(1-t)^2}\right]$$

Substitution of $\alpha = \beta$ reduces Khandekar's polynomial to ultraspherical type generalization of Rice's polynomials, therefore, we have

$$\frac{n!}{(1+\alpha)_n} H_n^{(\alpha,\alpha)}(\xi,p,\nu) = {}_3F_2\left[\begin{matrix}-n, & n+2\alpha+1,\ \xi; \\ & 1+\alpha,\ \ p;\end{matrix}\ \nu\right]$$

Generating functions for the ultraspherical type generalized Rice's polynomial is given below

$$\sum_{n=0}^{\infty} \frac{(2\alpha+1)_n}{(1+\alpha)_n} H_n^{(\alpha,\alpha)}(\xi,p,\nu)t^n$$

$$= \sum_{n=0}^{\infty} \frac{(2\alpha+1)_n}{(1+\alpha)_n} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n+2\alpha+1)_k(\xi)_k\nu^k t^n}{(1+\alpha)_k(p)_k k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k(2\alpha+1)_n(n+2\alpha+1)_k(\xi)_k\nu^k t^n}{(n-k)!(1+\alpha)_k(p)_k k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k(2\alpha+1)_{n+k}(\xi)_k\nu^k t^n}{(n-k)!(1+\alpha)_k(p)_k k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2\alpha+1)_{n+2k}(\xi)_k(-\nu t)^k t^n}{n!(1+\alpha)_k(p)_k k!}$$

$$= \sum_{k=0}^{\infty} \frac{(2\alpha+1)_{2k}(\xi)_k(-\nu t)^k}{(1+\alpha)_k(p)_k k!} \sum_{n=0}^{\infty} \frac{(2\alpha+1+2k)_n t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{2^{2k}(\frac{1}{2}+\alpha)_k(1+\alpha)_k(\xi)_k(-\nu t)^k}{(1+\alpha)_k(p)_k k!} \frac{1}{(1-t)^{2\alpha+1+2k}}$$

$$= \frac{1}{(1-t)^{2\alpha+1}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}+\alpha)_k(\xi)_k(-4\nu t)^k}{(p)_k k!(1-t)^{2k}}$$

$$= \frac{1}{(1-t)^{2\alpha+1}}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}+\alpha, & \xi; \\ & p;\end{matrix}\ -\frac{4\nu t}{(1-t)^2}\right]$$

Ultraspherical type generalization of the Hahn polynomial is given below

$$Q_n(x;\alpha,\alpha,\ N) = {}_3F_2\left[\begin{matrix} -n,\ n+2\alpha+1,\ -x; \\ 1+\alpha,\ -N; \end{matrix}\ 1\right] \tag{2.6}$$

Polynomials given by (2.6), satisfy the following generating functions

$$\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{n!}Q_n(x;\alpha,\alpha,\ N)t^n = \frac{1}{(1-t)^{2\alpha+1}}\ {}_2F_1\left[\begin{matrix} \frac{1}{2}+\alpha,\ -x; \\ -N; \end{matrix}\ -\frac{4t}{(1-t)^2}\right].$$

3. Gegenbauer type generalization of certain polynomials and their generating functions

In this section we determine Gegenbauer type generalization of the polynomials mentioned in the introduction section of this paper.

3.1. Gegenbauer type generalization of the Sister Celine's polynomial

Gegenbauer type generalization of the Sister Celine's polynomials is given below

$$f_n^{(\nu)}(a_1,\ \cdots,\ a_r;\ b_1,\ \cdots,\ b_s;\ x) = \frac{(2\nu)_n}{n!}\ {}_{2+r}F_{2+s}\left[\begin{matrix} -n,\ \ n+2\nu,\ \ a_1,\ \cdots,\ a_r; \\ \frac{1}{2}+\nu,\ \ \nu,\ \ b_1,\ \cdots,\ b_s; \end{matrix}\ x\right]$$

Generating function of the Gegenbauer type generalization of the Sister Celine's polynomial is given below

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(2\nu)_n}{n!}\ {}_{2+r}F_{2+s}\left[\begin{matrix} -n,\ \ n+2\nu,\ \ a_1,\ \cdots,\ a_r; \\ \frac{1}{2}+\nu,\ \ \nu,\ \ b_1,\ \cdots,\ b_s; \end{matrix}\ x\right]t^n \\
&= \sum_{n=0}^{\infty}\frac{(2\nu)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+2\nu)_k(a_1)_k\cdots(a_r)_k}{(\frac{1}{2}+\nu)_k(\nu)_k(b_1)_k\cdots(b_s)_k}\frac{x^k t^n}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(2\nu)_{n+k}(a_1)_k\cdots(a_r)_k}{(n-k)!(\frac{1}{2}+\nu)_k(\nu)_k(b_1)_k\cdots(b_s)_k}\frac{x^k t^n}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(2\nu)_{n+2k}(a_1)_k\cdots(a_r)_k}{k!(\frac{1}{2}+\nu)_k(\nu)_k(b_1)_k\cdots(b_s)_k}\frac{x^k t^{n+k}}{n!} \\
&= \sum_{k=0}^{\infty}\frac{(2\nu)_{2k}(a_1)_k\cdots(a_r)_k(-xt)^k}{k!(\frac{1}{2}+\nu)_k(\nu)_k(b_1)_k\cdots(b_s)_k}\sum_{n=0}^{\infty}\frac{(2\nu+2k)_n t^n}{n!} \\
&= \sum_{k=0}^{\infty}\frac{2^{2k}(a_1)_k\cdots(a_r)_k(-xt)^k}{k!(b_1)_k\cdots(b_s)_k}\sum_{n=0}^{\infty}\frac{(2\nu+2k)_n t^n}{n!} \\
&= \frac{1}{(1-t)^{2\nu}}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_s)_k}\frac{(-4xt)^k}{(1-t)^{2k}} \\
&= \frac{1}{(1-t)^{2\nu}}\ {}_rF_s\left[\begin{matrix} a_1,\ \cdots,\ a_r; \\ b_1,\ \cdots,\ b_s; \end{matrix}\ -\frac{4xt}{(1-t)^2}\right]
\end{aligned}$$

3.2. Gegenbauer type generalztion of the Bateman's polynomial

By substituting $\alpha = \beta = \nu - \frac{1}{2}$, the Gegenbauer type generalization of the Bateman's polynomial $F_n(z)$ is obtained as given below

$$F_n^{(\nu)}(p,z) = {}_3F_2\left[\begin{matrix} -n,\ n+2\nu,\ \frac{1}{2}(1+z); \\ \nu+\frac{1}{2},\ p; \end{matrix}\ 1\right]. \tag{3.1}$$

One of the generating functions of polynomials (3.1), is determined below

$$\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} F_n^{(\nu)}(p,z)t^n = \frac{1}{(1-t)^{2\nu}}\ {}_2F_1\left[\begin{matrix} \nu,\ \frac{1}{2}(1+z); \\ p; \end{matrix}\ -\frac{4t}{(1-t)^2}\right].$$

Another generating function of polynomial (3.1) is obtained below

$$\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \{F_n^{(\nu)}(p,z-2) - F_n^{(\nu)}(p,z)\}t^n$$

$$= \frac{(2\nu)}{(1-t)^{1+2\nu}(\nu+\frac{1}{2})(p)}\ {}_3F_2\left[\begin{matrix} \nu+\frac{1}{2},\ \nu+1,\ \frac{1}{2}(1+z); \\ \nu+\frac{3}{2},\ p+1; \end{matrix}\ -\frac{4t}{(1-t)^2}\right].$$

By setting $\alpha = \beta = \nu - \frac{1}{2}$ in the Eq. (2.4), another Gegenbauer type generalization of Bateman's polynomials $Z_n^{(\nu)}(b,x)$ is given below

$$Z_n^{(\nu)}(b,x) = {}_2F_2\left[\begin{matrix} -n,\ n+2\nu; \\ \nu+\frac{1}{2},\ 1+b; \end{matrix}\ x\right] \tag{3.2}$$

Gegenbauer type generalized Bateman's polynomial (3.2) satisfies the following generating function

$$\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} Z_n^{(\nu)}(b,x)t^n = \frac{1}{(1-t)^{2\nu}}\ {}_1F_1\left[\begin{matrix} \nu; \\ 1+b; \end{matrix}\ -\frac{4xt}{(1-t)^2}\right].$$

3.3. Gegenbauer type generalization of Pasternack's polynomial

By setting $\alpha = \beta = \nu - \frac{1}{2}$ and $1 + m = p$ in Eq. (2.2), we define here the Gegenbauer type generalization of Pasternack's generalization of the Bateman's polynomial $F_n^m(p,z)$ as given below

$$F_{n,m}^{(\nu)}(p,z) = {}_3F_2\left[\begin{matrix} -n,\ n+2\nu,\ \frac{1}{2}(z+m+1); \\ \nu+\frac{1}{2},\ p; \end{matrix}\ 1\right]. \tag{3.3}$$

For the above polynomial (3.3), we easily obtain the following generating function relation

$$\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} F_{n,m}^{(\nu)}(p,z)t^n = \frac{1}{(1-t)^{2\nu}}\ {}_2F_1\left[\begin{matrix} \nu,\ \frac{1}{2}(z+m+1); \\ p; \end{matrix}\ -\frac{4t}{(1-t)^2}\right]$$

Now substituting $\alpha = \beta = \nu - \frac{1}{2}$ in Eq. (2.5) the following polynomial is obtained

$$F_{m,n}^{(\nu)}(z) = {}_3F_2\left[\begin{matrix} -m,\ m+2\nu,\ \frac{1}{2}(1+z+n); \\ \nu+\frac{1}{2},\ 1+m; \end{matrix}\ 1\right]$$

whose generating function is determined as given below

$$\sum_{n=0}^{\infty} F_{m,n}^{(\nu)}(-2n-1)\frac{(-t)^n}{n!} = e^{-t}{}_2F_2\left[\begin{matrix} -m,\ m+2\nu; \\ \nu+\frac{1}{2}, 1+m; \end{matrix}\ t\right] = e^{-t}Z_m^{(\nu)}(m,t)$$

3.4. Gegenbauer type Generalization of Rice's polynomials

With the steps taken by Khandekar [4] in the definition of Jacobi type generalization of Rice's polynomials, by putting $\alpha = \beta = \nu - \frac{1}{2}$ in Eq. (1.2) we define Gegenbauer generaliation of Rice's polynomials

$$\frac{n!}{\left(\nu+\frac{1}{2}\right)_n} H_n^{(\nu)}(\xi, p, \nu) = {}_3F_2 \left[\begin{matrix} -n, \; n+2\nu, \; \xi; \\ \nu + \frac{1}{2}, p; \end{matrix} \; \nu \right]$$

We define the generating functions of the Gegenbauer type generalization of the Rice's polynomial as follows

$$\sum_{n=0}^{\infty} \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} H_n^{(\nu)}(\xi, p, \nu) t^n = \frac{1}{(1-t)^{2\nu}} \; {}_2F_1 \left[\begin{matrix} \nu, \; \xi; \\ p; \end{matrix} \; -\frac{4\nu t}{(1-t)^2} \right]$$

3.5. Gegenbauer type generalization of Hahn polynomials

We define the Gegenbauer type generalization of Hahn's polynomial as given below

$$Q_n^{(\nu)}(x;\, N) = {}_3F_2 \left[\begin{matrix} -n, \; n+2\nu, \; -x; \\ \frac{1}{2}+\nu, \; -N; \end{matrix} \; 1 \right].$$

The following generating function is satisfied by the Hahn's polynomial

$$\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} Q_n^{(\nu)}(x;\, N) t^n = \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (n+2\nu)_k (-x)_k t^n}{(\frac{1}{2}+\nu)_k (-N)_k k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (2\nu)_{n+k} (-x)_k t^n}{(n-k)! (\frac{1}{2}+\nu)_k (-N)_k k!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (2\nu)_{n+2k} (-x)_k t^{n+k}}{n! (\frac{1}{2}+\nu)_k (-N)_k k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2\nu)_{2k} (-x)_k t^k}{(\frac{1}{2}+\nu)_k (-N)_k k!} \sum_{n=0}^{\infty} \frac{(2\nu+2k)_n t^n}{n!}$$

$$= \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{2^{2k} (\nu)_k (\frac{1}{2}+\nu)_k (-x)_k (-t)^k}{(\frac{1}{2}+\nu)_k (-N)_k k! (1-t)^{2k}}$$

$$= \frac{1}{(1-t)^{2\nu}} \; {}_2F_1 \left[\begin{matrix} \nu, \; -x; \\ -N; \end{matrix} \; -\frac{4t}{(1-t)^2} \right].$$

Acknowledgements. The second author wishes to express his heartfelt thanks to the Human Resource Development Group Council of Scientific & Industrial Research of India for awarding Senior Research Fellowship (NET)(F.No. 10-2(5)/ 2005(i)- E.U.II).

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(received 09.11.2010; in revised form 20.02.2011; available online 20.04.2011)

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