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DOUBLY CONNECTED DOMINATION SUBDIVISION NUMBERS OF GRAPHS

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Abstract. A set S of vertices of a connected graph G is a *doubly connected dominating set* (DCDS) if every vertex not in S is adjacent to some vertex in S and the subgraphs induced by S and $V - S$ are connected. The *doubly connected domination number* $\gamma_{cc}(G)$ is the minimum size of such a set. The *doubly connected domination subdivision number* $\mathrm{sd}_{\gamma_{cc}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the doubly connected domination number. In this paper first we establish upper bounds on the doubly connected domination subdivision number in terms of the order n of G or of its edge connectivity number $\kappa'(G)$. We also prove that $\gamma_{cc}(G)+\mathrm{sd}_{\gamma_{cc}}(G) \le n$ with equality if and only if either $G = K_2$ or for each pair of adjacent non-cut vertices $u, v \in V(G)$, $G[V(G) - \{u, v\}]$ is disconnected.

1. Introduction

In the whole paper, G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). We denote by n its order $|V|$ and by m its size $|E|$. For every vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The S-private neighbors of a vertex v of S are the vertices of $N[v] \setminus N[S \setminus \{v\}]$. The vertex v is its own private neighbor if it is isolated in S. The other private neighbors are external, i.e., belong to $V \setminus S$. The minimum and maximum degrees of G are respectively denoted by δ and Δ. The *edge connectivity number* $\kappa'(G)$ of G is the minimum number of edges whose removal results in a disconnected graph. For every graph, $\kappa'(G) \le \delta$. A *matching* is a set of independent edges and the *matching number* $\alpha'(G)$ is the size of a maximum matching.

A subset S of vertices of G is a *dominating set* if $N[S] = V$, is a *connected dominating set* if the induced subgraph $G[S]$ is connected and is a *doubly connected dominating set* if the induced subgraphs $G[S]$ and $G[V(G) - S]$ are connected.

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The *(connected, doubly connected) domination number* $\gamma(G)$ ($\gamma_c(G)$, $\gamma_{cc}(G)$) is the minimum cardinality of a (connected, doubly connected) dominating set of G, and a (connected, doubly connected) dominating set of minimum cardinality is called a γ-set (γ_c-set, γ_{cc}-set). Since any doubly connected dominating set is also a connected dominating set, $\gamma_c(G) \le \gamma_{cc}(G)$ for any connected graph G with $\Delta < n-1$. The doubly connected domination number was introduced by Cyman et al. in [2] and has been studied by several authors (see for instance [1]).

The *(connected, doubly connected) domination subdivision number* $\mathrm{sd}_{\gamma}(G)$ ($\mathrm{sd}_{\gamma_c}(G)$, $\mathrm{sd}_{\gamma_{cc}}(G)$) of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the (connected, doubly connected) domination number. (An edge $uv \in E(G)$ is *subdivided* if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and vx. The vertex x is called a *subdivision vertex*).

The (connected) domination subdivision number have been studied by several authors (see, for example, [3, 4]). The purpose of this paper is to initialize the study of the doubly connected domination subdivision number $\mathrm{sd}_{\gamma_{cc}}(G)$. Although it may not be immediately obvious that it is defined for all connected graphs of order $n \ge 2$, we will show this shortly.

We make use of the following results in this paper.

Theorem A. *If G is a simple planar triangle-free graph of order $n \ge 3$ and size m, then $m \le 2n-4$.*

Theorem B. (Mantel [6]) *The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$.*

Theorem C. (Cyman [2]) *For every connected graph G on $n \ge 2$ vertices*

$$\gamma_{cc}(G) \le n-1,$$

with equality if and only if either $G = K_2$ or for each pair of adjacent non-cut vertices $u, v \in V(G)$, $G[V(G) - \{u,v\}]$ is disconnected.

Theorem D. (Cyman [2]) *Let G be connected graph with $n \ge 2$ vertices and let G' be obtained from G by subdividing one edge of G. Then $\gamma_{cc}(G) \le \gamma_{cc}(G')$.*

We finish this section with presenting an upper bound on $\gamma_{cc}(G)$ and some observations giving some sufficient conditions for a graph to have small $\mathrm{sd}_{\gamma_{cc}}(G)$.

Theorem 1. *For any connected graph G of order $n \ge 2$, $\gamma_{cc}(G) \le n - \delta(G)$.*

Proof. Let $P : x_1x_2\ldots x_r$ be the longest path in G. Obviously $r \ge \delta(G)+1$. Assume that $G' = G - \{x_1, \ldots, x_\delta\}$. We claim that G' is connected. Suppose to the contrary that G' is disconnected. Let C be the component of G' such that $x_{\delta(G)+1} \notin C$ and let $y_0y_1\ldots y_m$ be the longest path of C. Suppose $S_1 = \{x_1, \ldots, x_\delta\}$, $S_2 = \{y_0, \ldots, y_m\}$ and let $\ell = d(S_1, S_2)$. Then we may assume x_ky_j if $\ell = 1$ and $x_kz_1\ldots z_{\ell-1}y_j$ when $\ell \ge 2$ is the shortest

(S_1, S_2)-path where $1 \le k \le \delta$ and $0 \le j \le m$. Let i be the largest positive integer such that $y_o y_i \in E(G)$. If $j \ge i$, then $y_0 \dots y_j x_k x_{k+1} \dots x_r$ if $\ell = 1$ and $y_0 \dots y_j z_{\ell-1} \dots z_1 x_k x_{k+1} \dots x_r$ when $\ell \ge 2$ is a path of G longer than P which is a contradiction. If $j < i$, then $y_{j+1} \dots y_i y_0 \dots y_j x_k x_{k+1} \dots x_r$ if $\ell = 1$ and $y_{j+1} \dots y_i y_0 \dots y_j z\ell - 1 \dots z_1 x_k x_{k+1} \dots x_r$ when $\ell \ge 2$ is a path of G longer than P which is a contradiction again. Thus G' is connected. Since $\delta(G) > \delta(G[x_1, \dots, x_\delta])$, we deduce that each x_i $(1 \le i \le \delta)$ has at least one neighbor in $V(G) - \{x_1, \dots, x_\delta\}$, and hence $V(G) - \{x_1, \dots, x_\delta\}$ is a dominating set of G. Since also $G[x_1, \dots, x_\delta]$ is connected, $V(G) - \{x_1, \dots, x_\delta\}$ is a doubly connected dominating set of G. Thus $\gamma_{cc}(G) \le n - \delta(G)$ and the proof is complete. ∎

The upper bound is attained for instance for cycles.

Observation 2. *If a connected graph G of order $n \ge 2$ satisfies one of the following properties, then $sd_{\gamma_{cc}}(G) = 1$:*

(i) $\gamma_{cc}(G) = 1$*;*

(ii) $\gamma_{cc}(G) = 2$ *and G contains a $\gamma_{cc}(G)$-set $\{a, b\}$ such that $N(a) \cap N(b) = \emptyset$.*

Proof. (i) Obviously $\mathrm{sd}_{\gamma_{cc}}(K_2) = 1$. Thus we may assume that $n \ge 3$. Then clearly the graph G' obtained from G by subdividing any edge of G has no vertex of degree $n(G') - 1$. Therefore $\gamma_{cc}(G') > 1 = \gamma_{cc}(G)$ and hence $\mathrm{sd}_{\gamma_{cc}}(G) = 1$.

(ii) Every doubly connected dominating set of the graph G' obtained by subdividing the edge ab by one vertex x contains at least one of a, b, say a, and either two vertices in $N(a) \cup N(b)$, or x and b. Hence $\gamma_{cc}(G') \ge 3 > \gamma_{cc}(G)$. Note that this case includes the complete bipartite graph $K_{p,q}$ with $p, q \ge 2$, and the graph obtained from K_4 by subdividing one edge once. ∎

Observation 3. *For any connected graph G with a cut-vertex, $sd_{\gamma_{cc}}(G) \le 2$.*

Proof. Let x be a cut vertex of G and let $G_1, \cdots, G_k$ be the $\{x\}$-components of $G - x$. Let $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$ and let G' be obtained from G by subdividing the edges of e_1, e_2 with subdivision vertices z_1, z_2. Assume D is a $\gamma_{cc}(G')$-set. Since the subgraph $G'[D]$ is connected, we have $x \in D$. On the other hand, since the subgraph $G'[V(G') - D]$ is connected, D contains all vertices of $V(G_i)$ except for one i, say $i = 1$. Thus $z_2 \in D$, and obviously $D - \{z_1, z_2\}$ is a DCDS of G smaller than $\gamma_{cc}(G')$. This completes the proof. ∎

Observation 4. (i) *If $\gamma_{cc}(G) = \gamma_c(G)$, then $sd_{\gamma_{cc}}(G) \le \mathrm{sd}_{\gamma_c}(G)$.*

(ii) *For every connected graph G of order $n \ge 3$, if $\gamma_{cc}(G) = 2$ then $1 \le$ $\mathrm{sd}_{\gamma_{cc}}(G) \le 2$.*

Proof. (i) After having subdivided $\mathrm{sd}_{\gamma_c}(G)$ edges of G, the resulting graph G' satisfies $\gamma_c(G') > \gamma_c(G) = \gamma_{cc}(G)$. Therefore $\gamma_{cc}(G') \ge \gamma_c(G') > \gamma_{cc}(G)$ and $\mathrm{sd}_{\gamma_{cc}}(G) \le \mathrm{sd}_{\gamma_c}(G)$.

(ii) Let $\{u, v\}$ be a $\gamma_{cc}(G)$-set. Then obviously either $\min\{\deg(u), \deg(v)\} = 1$ or each of u and v have external $\{u, v\}$-private neighbor, say u', v', respectively. If $\deg(u) = \min\{\deg(u), \deg(v)\} = 1$, then v is cut vertex and the result follows by Observation 3. Let $\min\{\deg(u), \deg(v)\} \ge 2$ and let G' be obtained from G by subdividing the edges uu', vv'. It is easy to see that $\gamma_{cc}(G') > \gamma_{cc}(G)$ and the proof is complete. ∎

2. Bounds on the doubly connected domination subdivision number

In this section we present some upper bounds on $\mathrm{sd}_{\gamma_{cc}}(G)$ in terms of the edge connectivity number, the minimum degree, the order or the doubly connected dominating number of G.

THEOREM 5. *For any connected graph G of order $n \geq 2$, $\mathrm{sd}_{\gamma_{cc}}(G) \leq \kappa'(G)$.*

Proof. Let $[S, \overline{S}]$ be an edge cut of G of size κ' and G_1, G_2 are connected components of $G - [S, \overline{S}]$. Let G' be obtained from G by subdividing the edges of $[S, \overline{S}]$ and let S' be the set of all subdivision vertices. Let D be a γ_{cc}-set of G' and $D_i = D \cap V(G_i)$ for $i = 1, 2$. If $D \cap S' = \emptyset$ then $D = D_1 \cup D_2$ and $D_i \neq \emptyset$ for each i since D_i must dominate G_i. But then D is not connected, a contradiction. Therefore $D \cap S' \neq \emptyset$ and $D \setminus S'$ is a doubly connected dominating set of G smaller than $\gamma_{cc}(G')$. This implies that $\mathrm{sd}_{\gamma_{cc}}(G) \leq \kappa'(G)$. ∎

Note that the previous bound is obviously attained if G has a cut-edge.

THEOREM 6. *For any connected graph G of order $n \geq 2$,*

$$\mathrm{sd}_{\gamma_{cc}}(G) \leq \max\{1, \delta(G) - 1\}.$$

Proof. If $\delta(G) = 1$, then let u be an end-vertex of G and let $uv \in V(G)$. Then uv is a cut edge of G and the statement is true by Theorem 5.

Now suppose $\delta(G) \geq 2$. Let $v \in V(G)$ be a vertex of minimum degree δ and let $N(v) = \{v_1, \dots, v_\delta\}$. Let G' be obtained from G by subdividing the edges vv_i ($2 \leq i \leq \delta$), with $\delta - 1$ new vertices $z_2, \dots, z_\delta$, respectively. Assume $S = \{z_2, \dots, z_\delta\}$ and let D be a γ_{cc}-set of G'. If $D \cap S \neq \emptyset$, then clearly $D - S$ is a doubly connected dominating set of G smaller than $\gamma_{cc}(G')$. Let $D \cap S = \emptyset$.

First let $v \in D$. Since the subgraph $G[D]$ is connected and since $D \cap S = \emptyset$, we have $v_1 \in D$. It is easy to see that $D - \{v\}$ is a doubly connected dominating set of G smaller than $\gamma_{cc}(G')$. Now let $v \notin D$. Then to dominate v and the subdivision vertices, we must have $N_G(v) \subseteq D$. Since $G[V(G') - D]$ is connected, we deduce that $D = V(G') - \{v, z_2, \dots, z_\delta\}$. Then obviously D is a doubly connected dominating set of G. Assume to the contrary that $|D| = \gamma_{cc}(G)$. This implies that $\gamma_{cc}(G) = n - 1$. It follows from Theorem 1 that $\delta(G) = 1$ which is a contradiction. This completes the proof. ∎

A consequence of Theorem 6 is that $\mathrm{sd}_{\gamma_{cc}}(G)$ is defined for every connected graph G of order $n \geq 2$.

THEOREM 7. *For any connected graph G of order $n \geq 2$,*

$$\mathrm{sd}_{\gamma_{cc}}(G) \leq \alpha'(G).$$

Proof. If $\alpha'(G) = 1$, then clearly $\delta(G) \leq 2$ and the statement is true by Theorem 6. Assume $\alpha'(G) \geq 2$. Let $M = \{u_1v_1, \dots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of G, and let X be the independent set of M-unsaturated vertices. First let $|X| \geq 2$. If y and z are vertices of X and $yu_i \in E(G)$, then since the matching

M is maximum, $zv_i \notin E(G)$. Therefore, for all $i \in \{1, 2, \ldots, \alpha'\}$ there are at most two edges between the sets $\{u_i, v_i\}$ and $\{y, z\}$. So $\deg(y) + \deg(z) \leq 2\alpha'$ for every pair of distinct vertices y and z in X. Let $y, z \in X$. Then $\min\{\deg(y), \deg(z)\} \leq \alpha'$. Thus $\delta(G) \leq \alpha'(G)$ and the result follows from Theorem 6.

Now let $|X| \leq 1$. If $\delta(G) \leq \alpha'(G)+1$, then the result follows from Theorem 6. Thus we may assume $\delta(G) \geq \alpha'(G)+2$. Let G' be obtained from G by subdividing the edges of $u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}$ and let S be the set of all subdivision vertices. Suppose D is a γ_{cc}-set of G'. If $D \cap S \neq \emptyset$, then obviously $D \setminus S$ is a doubly connected dominating set of G smaller than $\gamma_{cc}(G')$. Let $D \cap S = \emptyset$. Since D is a dominating set of G' and since $G'[V(G') - D]$ is connected, we deduce that $|D \cap \{u_i, v_i\}| = 1$ for each i. We may assume without loss of generality that $D \cap \{u_1, \ldots, u_{\alpha'}\} = \{u_1, \ldots, u_s\}$ and $D \cap \{v_1, \ldots, v_{\alpha'}\} = \{v_{s+1}, \ldots, v_{\alpha'}\}$ if $s < \alpha'$. Assume T is a spanning tree of $G[D]$. Since T has at least two leaves, we may assume without loss of generality that u_1 is a leaf of T. We claim that $D - \{u_1\}$ is a doubly connected dominating set of G. Obviously $G[D - \{u_1\}]$ is connected. Since $|V(G)| = 2\alpha'(G) + 1$, $\alpha'(G) - 1 \leq |D - \{u_1\}| \leq \alpha'(G)$ and $\deg(v) \geq \alpha'(G) + 2$ for each $v \in V(G)$, we deduce that $D - \{u_1\}$ is a dominating set of G. It remains to show that $G[V(G) \setminus (D - \{u_1\})]$ is connected. Since D is a $\gamma_{cc}(G')$-set, the subgraph induced by $V(G') - D$ is connected. On the other hand, since each subdivision vertex has degree two and has neighbors in D and $V(G') - D$, the subdivision vertices are not end vertices of G'. Since $u_1v_1 \in E(G)$, we deduce that $G[V(G) \setminus (D - \{u_1\})]$ is connected and the proof is complete. ■

Theorem 8. *If G contains a matching M such that $\gamma_{cc}(G) < |M|$, then $sd_{\gamma_{cc}}(G) \leq |M|$. In particular, if $\alpha'(G) > \gamma_{cc}(G)$, then $sd_{\gamma_{cc}}(G) \leq \gamma_{cc}(G) + 1$.*

Proof. Let G' be obtain by subdividing every edge of M. Each doubly connected dominating set of G' has order at least $|M|$. Hence $\gamma_{cc}(G') > \gamma_{cc}(G)$ and thus $\mathrm{sd}_{\gamma_{cc}}(G) \leq |M|$. If $\alpha'(G) > \gamma_{cc}(G)$, then G contains a matching M of size $\gamma_{cc}(G) + 1$, which leads to the result. ■

Next result is an immediate consequence of Theorems 7 and 8.

Corollary 9. *For every connected graph G of order $n \geq 2$,*

$$\mathrm{sd}_{\gamma_{cc}}(G) \leq \gamma_{cc}(G) + 1.$$

Corollary 10. *For any connected graph G of order $n \geq 3$, $sd_{\gamma_{cc}}(G) \leq \lceil \frac{n-1}{2} \rceil$.*

Proof. The statement is true if $\delta \leq \lceil \frac{n+1}{2} \rceil$ by Theorem 6. Let $\delta \geq \lceil \frac{n+1}{2} \rceil + 1$. It follows from Corollary 9 and Theorem 1 that

$$\mathrm{sd}_{\gamma_{cc}}(G) \leq \gamma_{cc}(G) + 1 \leq n - \delta(G) + 1 \leq n - \lceil \frac{n+1}{2} \rceil \leq \lceil \frac{n-1}{2} \rceil,$$

as desired. ■

For a path or a cycle of order 3, the previous bound is attained.

Theorem 11. *Let G be a connected graph containing an odd cycle. Then*

$$\mathrm{sd}_{\gamma_{cc}}(G) \leq \min\{\ell \mid \text{there is an odd cycle in } G \text{ of length } \ell\}.$$

Proof. Let $C = (v_1 v_2 \ldots v_k)$ be an odd cycle of G and let G' be obtained from G by subdividing the edges $v_1v_2, \ldots, v_{k-1}v_k, v_kv_1$ with subdivision vertices $z_1, \ldots, z_k$, respectively. Assume D is a $\gamma_{cc}(G')$-set and S is the set of all subdivision vertices. We claim that $D \cap S \neq \emptyset$. In this case, obviously $D - S$ is a doubly connected dominating set of G smaller than $\gamma_{cc}(G')$. Suppose to the contrary that $D \cap S = \emptyset$. To dominate z_1, we may assume without loss of generality that $v_1 \in D$. Since D is a DCDS of G' and since $z_1 \notin D$, we have $v_2 \notin D$. Now to dominate z_2 we must have $v_3 \in D$. By repeating this process we deduce that $\{v_1, v_3, \ldots, v_k\} \in D$ which implies that z_k is an isolated vertex in $G'[V(G') - D]$ which is a contradiction. This completes the proof. ■

Next two results are immediate consequences of Theorems A, B and 11.

Corollary 12. *For any graph G of order $n \geq 2$ and size $m \geq \frac{n^2}{4}$, $\mathrm{sd}_{\gamma_{cc}}(G) \leq 3$.*

Corollary 13. *Let G be a connected planar graph of order $n \geq 2$. Then*

$$\mathrm{sd}_{\gamma_{cc}}(G) \leq 3.$$

Proof. If G contains a triangle, then the statement is true by Theorem 11. Let G be triangle-free graph. Then by Theorem A, $|E(G)| \leq 2n - 4$ which implies that $\delta(G) \leq 3$. Hence $\mathrm{sd}_{\gamma_{cc}}(G) \leq 2$ by Theorem 6. This completes the proof. ■

Next result gives an upper bound on the sum $\mathrm{sd}_{\gamma_{cc}}(G) + \gamma_{cc}(G)$ and determines all the extremal graphs.

Theorem 14. *Every connected graph G of order $n \geq 2$ satisfies $\gamma_{cc}(G) + \mathrm{sd}_{\gamma_{cc}}(G) \leq n$. Moreover $\gamma_{cc}(G) + \mathrm{sd}_{\gamma_{cc}}(G) = n$ if and only if either $G = K_2$ or for each pair of adjacent non-cut vertices $u, v \in V(G)$, $G[V(G) - \{u, v\}]$ is disconnected.*

Proof. Let G be a connected graph of order $n \geq 2$. If $\delta(G) = 1$, then it follows from Theorems 1 and 6 that

$$\gamma_{cc}(G) + \mathrm{sd}_{\gamma_{cc}}(G) \leq (n - 1) + 1 = n. \tag{1}$$

Let $\delta(G) \geq 2$. It follows from Theorems 1 and 6 that

$$\gamma_{cc}(G) + \mathrm{sd}_{\gamma_{cc}}(G) \leq (n - \delta(G)) + (\delta(G) - 1) = n - 1.$$

If $\gamma_{cc}(G) + \mathrm{sd}_{\gamma_{cc}}(G) = n$, then the two inequalities occurring in (1) become equalities. Hence, $\gamma_{cc}(G) = n - 1$ and the result follows by Theorem C.

If either $G = K_2$ or for each pair of adjacent non-cut vertices $u, v \in V(G)$, $G[V(G) - \{u, v\}]$ is disconnected, then $\gamma_{cc}(G) = n - 1$ by Theorem C. It follows from Theorem 1 that $\delta(G) = 1$ and hence $\mathrm{sd}_{\gamma_{cc}}(G) = 1$ by Theorem 6. Thus $\gamma_{cc}(G) + \mathrm{sd}_{\gamma_{cc}}(G) = n$. This completes the proof. ■

3. Graphs with doubly connected domination subdivision number 3

Our aim in this section is to demonstrate an infinite family of graphs with the doubly connected domination subdivision number three. The following graph was introduced by Haynes et al. in [4]. Let $X = \{1, 2, \ldots, 3(k-1)\}$ and let $\mathcal{Y} = \{Y \subset X \mid |Y| = k\}$. Thus, $\mathcal{Y}$ consists of all k-subsets of X, and so $|\mathcal{Y}| = \binom{3(k-1)}{k}$. Let H be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$, add an edge joining x and Y if and only if $x \in Y$. Then, H is a connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. The set X induces a clique in H, while the set $\mathcal{Y}$ is an independent set each vertex of which has degree k in H. Therefore $\delta = k$. Favaron et al. [3] proved the following results.

Theorem E. *For any integer $k \geq 2$, $\gamma_c(H) = 2(k-1)$ and $sd_{\gamma_c}(H) = k$.*

Theorem 15. (1) *For any integer $k \geq 2$, $\gamma_{cc}(H) = 2(k-1) + \binom{2(k-1)}{k}$.*
(2) *For any integer $k \geq 4$, $sd_{\gamma_{cc}}(H) = 3$.*

Proof. (1) Let D be a $\gamma_{cc}(H)$-set. If $|D \cap X| \leq 2k-3$, then let S be a k-subset of $X - D$. Then either S is an isolated vertex in $H[D]$ when $S \in D$ or S is not dominated by D when $S \notin D$ which is a contradiction. Thus $|D \cap X| \geq 2(k-1)$. If S' be a k-subset of $D \cap X$, then $S' \in D$, for otherwise S' is an isolated vertex in $H[V(H) - D]$, a contradiction. Thus every k-subset of $D \cap X$ belongs to D. Hence $|D \cap \mathcal{Y}| \geq \binom{2(k-1)}{k}$. This implies that $\gamma_{cc}(H) \geq 2(k-1) + \binom{2(k-1)}{k}$. Now let X_1 be a $2(k-1)$-subset of X and let $\mathcal{Y}_1$ consist of all k-subsets of X_1. It is easy to see that $A = X_1 \cup \mathcal{Y}_1$ is a DCDS of H and so $\gamma_{cc}(H) = 2(k-1) + \binom{2(k-1)}{k}$.

(2) Since H has triangle, it follows from Theorem 11 that $\mathrm{sd}_{\gamma_{cc}}(H) \leq 3$. We show next that $\mathrm{sd}_{\gamma_{cc}}(G) \geq 3$. Let $F = \{e_1, e_2\}$ be an arbitrary subset of 2 edges of H. Let H' be obtain from H by subdividing each edge in F. We show that $\gamma_{cc}(H) = \gamma_{cc}(H')$. Let $e_i = u_i v_i$ for each i. Since every edge of H is incident with at least one vertex of X, we may assume $u_i \in X$ for each i. If $v_1, v_2 \in \mathcal{Y}$, then let $r_i, z_i \in v_i$ and let A_1 be a $2(k-1)$-subset of $X - \{z_1, z_2\}$ containing u_1, u_2, r_1, r_2. If $v_1, v_2 \in X$, then let A_2 be a $2(k-1)$-subset of $X - \{v_1, v_2\}$ containing u_1, u_2. Finally, If $v_1 \in X$ and $v_2 \in \mathcal{Y}$, then let $r_2, z_2 \in v_2$ and let A_3 be a $2(k-1)$-subset of $X - \{v_1, z_2\}$ containing u_1, u_2, r_1. Assume $B_i = \{Y \subset A_i \mid |Y| = k\}$ and let $D_i = A_i \cup B_i$ for $1 \leq i \leq 3$. It is easy to see that D_i is a DCDS of H' in each case. Thus $\gamma_{cc}(H) = \gamma_{cc}(H')$, whence $\mathrm{sd}_{\gamma_{cc}}(H) \geq 3$. Consequently, $\mathrm{sd}_{\gamma_{cc}}(H) = 3$.

This completes the proof. ■

Note that since H contains triangle, H is an example of equality in Theorem 11. The following corollary is immediate consequences of Theorems E and 15.

Corollary 16. *The difference of $\mathrm{sd}_{\gamma_c}(G) - \mathrm{sd}_{\gamma_{cc}}(G)$ can be arbitrarily large.*

We conclude this paper with two open problems.

Problem 1. Prove or disprove: For any connected simple graph G of order $n \geq 2$,

$$\mathrm{sd}_{\gamma_{cc}}(G) \leq \max\{1, \kappa(G) - 1\}.$$

Conjecture 1. For any connected planar graph G of order $n \geq 2$, $\mathrm{sd}_{\gamma_{cc}}(G) \leq 2$.

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