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ON π-IMAGES OF SEPARABLE METRIC SPACES AND A PROBLEM OF SHOU LIN

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Abstract. In this paper, we give some characterizations of images of separable metric spaces under certain π-maps, and give an affirmative answer to the problem posed by Shou Lin in [Point-Countable Covers and Sequence-Covering Mappings, Chinese Science Press, Beijing, 2002].

1. Introduction and preliminaries

In his book [7], S. Lin proved that a T_1 and regular space X is a quotient compact image of a separable metric space iff X is a quotient π-image of a separable metric space, iff X has a countable weak base. But he does not know whether quotient π-images of separable metric spaces and quotient compact images of separable metric spaces are equivalent. So, the following question was posed by S. Lin.

QUESTION 1.1. [8, Question 3.2.12] *Is a quotient π-image of a separable metric space a quotient compact image of a separable metric space?*

In [10], S. Lin and P. Yan proved that a T_1 and regular space X is a compact-covering compact image of a separable metric space if and only if X is a sequentially-quotient compact image of a separable metric space, if and only if X has a countable *sn*-network. And in [3], Y. Ge proved that a T_1 and regular space X is a sequentially-quotient compact image of a separable metric space if and only if X is a sequentially-quotient π-image of a separable metric space, if and only if X has a countable *sn*-network. Thus, we are interested in the following question.

QUESTION 1.2. *Is an image (resp., a sequentially-quotient π-image, a sequence-covering π-image) of a separable metric space a compact image (resp., sequentially-quotient compact image, sequence-covering compact image) of a separable metric space?*

In this paper, we give affirmative answers to the Question 1.1, Question 1.2 and give some characterizations of images of separable metric spaces under certain π-maps.

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Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of X, we denote $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$ and $\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. For a sequence $\{x_n\}$ converging to x and $P \subset X$, we say that $\{x_n\}$ is *eventually* in P if $\{x\} \cup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in P if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1.3. [14] We say that $f : X \to Y$ is a *weak-open* map, if there exists a weak base $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ for Y, and for every $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each open neighbourhood U of x, $B \subset f(U)$ for some $B \in \mathcal{B}_y$.

Definition 1.4. Let d be a d-function on a space X.

(1) For each $x \in X$ and $n \in \mathbb{N}$, let $S_n(x) = \{y \in X : d(x, y) < 1/n\}$.
(2) X is *sn-symmetric* [5], if $\{S_n(x) : n \in \mathbb{N}\}$ is an *sn*-network at x in X for each $x \in X$.
(3) X is *Cauchy sn-symmetric*, if it is *sn*-symmetric and every convergent sequence in X is d-Cauchy.

Definition 1.5. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network for a space X.

(1) $\mathcal{P}$ is a *σ-strong network consisting of covers* (*cs^*-covers*, *cs-covers*, *sn-covers*), if each $\mathcal{P}_n$ is a cover (resp., cs^*-cover, *cs*-cover, *sn*-cover).
(2) $\mathcal{P}$ is a *σ-strong network consisting of finite covers* (*finite cs^*-covers*, *finite cs-covers*, *finite sn-covers*), if each $\mathcal{P}_n$ is a finite cover (resp., finite cs^*-cover, finite *cs*-cover, finite *sn*-cover).

Notation 1.6. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network for a space X. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow Λ_n with the discrete topology. Then,

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\Big\}$$

is a metric space and the point x_α is unique in X for every $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. Let us call $(f, M, X, \mathcal{P}_n)$ a *Ponomarev's system*, following [11].

For some undefined or related concepts, we refer the readers to [6] and [8].

2. Results

Lemma 2.1 *For a space X, the following statements hold.*

(1) If X has a σ-strong network consisting of cs^*-covers, then X is sn-symmetric.
(2) If X has a σ-strong network consisting of cs-covers, then X is Cauchy sn-symmetric.

Proof. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network for X. For each $x, y \in X$ with $x \neq y$, let $\delta(x, y) = \min\{n : y \notin \mathrm{st}(x, \mathcal{P}_n)\}$. Then, we define

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1/\delta(x, y) & \text{if } x \neq y. \end{cases}$$

(1) If each $\mathcal{P}_n$ is a cs^*-cover, then d is a d-function on X and $\mathrm{st}(x, \mathcal{P}_n) = S_n(x)$ for all $n \in \mathbb{N}$. Since $\mathcal{P}$ is a σ-strong network consisting of cs^*-covers, $\{S_n(x) : n \in \mathbb{N}\}$ is an sn-network at x for every $x \in X$. Therefore, X is sn-symmetric.

(2) If each $\mathcal{P}_n$ is a cs-cover, then X is sn-symmetric by (1). Now, we shall show that every convergent sequence in X is d-Cauchy. In fact, let $\{x_i\}$ be a sequence converging to $x \in X$. Then, for any $\varepsilon > 0$, choose $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Since $\mathcal{P}_k$ is a cs-cover, there exist $P \in \mathcal{P}_k$, and $m \in \mathbb{N}$ such that $x_i \in P$ for all $i \geq m$. This implies that $d(x_i, x_j) < \varepsilon$ for all $i, j \geq m$. ∎

Lemma 2.2. *Let X be an sn-symmetric space. Then,*
(1) If P is a sequential neighbourhood at x, then $S_n(x) \subset P$ for some $n \in \mathbb{N}$.
(2) If X has a countable cs*-network, then X has a countable sn-network.

Proof. (1) If not, for each $n \in \mathbb{N}$, there exists $x_n \in S_n(x) - P$. Then, $\{x_n\}$ converges to x. Hence, there exists $m \in \mathbb{N}$ such that $x_n \in P$ for every $n \geq m$. This is a contradiction.

(2) Let $\mathcal{P}$ be a countable cs^*-network for X. We can assume that $\mathcal{P}$ is a countable cs-network, and $\mathcal{P}$ is closed under finite intersections. For each $x \in X$, put $\mathcal{G}_x = \{P \in \mathcal{P} : S_n(x) \subset P \text{ for some } n \in \mathbb{N}\}$, and put $\mathcal{G} = \bigcup\{\mathcal{G}_x : x \in X\}$. Then, each element of $\mathcal{G}_x$ is a sequential neighbourhood at x, and for $P_1, P_2 \in \mathcal{G}_x$, there exists $P \in \mathcal{G}$ such that $P \subset P_1 \cap P_2$. Furthermore, by using the proof in [9, Lemma 7], we get $\mathcal{G}_x$ is a network at x. Thus, (2) holds. ∎

Lemma 2.3. *For a Ponomarev's system $(f, M, X, \mathcal{P}_n)$, where each $\mathcal{P}_n$ is finite. Then, the following statements hold.*
(1) M is separable.
(2) f is compact.
(3) f is pseudo-sequence-covering, if each $\mathcal{P}_n$ is a cs*-cover.
(4) f is 1-sequence-covering compact-covering, if each $\mathcal{P}_n$ is an sn-cover.

Proof. Since each $\mathcal{P}_n$ is finite, (1) holds. For (2), by [11, Lemma 13(1)]. And for (3), see the proof of (d) $\Longrightarrow$ (a) in [6, Theorem 4].

For (4), by using the proof of (e) $\Longrightarrow$ (f) in [6, Theorem 9], we get f is sequence-covering. By (1) and [1, Theorem 2.5], f is 1-sequence-covering. Furthermore, since $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of finite sn-covers, X has a countable cs-network. Thus, each compact subset of X is metrizable. Similar to the proof of [13, Lemma 3.10], each $\mathcal{P}_n$ is a cfp-cover. By [11, Lemma 13(2)], f is compact-covering. ∎

Theorem 2.4. *The following are equivalent for a space X.*
(1) X is an sn-symmetric with a countable cs*-network;
(2) X has a σ-strong network consisting of finite cs*-covers;
(3) X is a pseudo-sequence-covering compact image of a separable metric space;
(4) X is a sequentially-quotient π-image of a separable metric space.

Proof. (1) $\Longrightarrow$ (2). Let X be an sn-symmetric space with a countable cs^*-network. By Lemma 2.2(2), X has a countable sn-network $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\} = \{P_n : n \in \mathbb{N}\}$. For each $m, n \in \mathbb{N}$, put $A_{m,n} = \{x \in X : S_n(x) \subset P_m\}$; $B_{m,n} =$

$X - A_{m,n}$, and $\mathcal{F}_{m,n} = \{P_m, B_{m,n}\}$. Then, each $\mathcal{F}_{m,n}$ is finite. Furthermore, we have

(i) *Each $\mathcal{F}_{m,n}$ is a cs*-cover.* Let $L = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to $x \in X$, then

Case 1. If $x \in A_{m,n}$, then $S_n(x) \subset P_m$. Thus, L is eventually in $P_m \in \mathcal{F}_{m,n}$.

Case 2. If $x \notin A_{m,n}$ and $L \cap B_{m,n}$ is infinite, then L is frequently in $B_{m,n} \in \mathcal{F}_{m,n}$.

Case 3. If $x \notin A_{m,n}$ and $L \cap B_{m,n}$ is finite, then there exists $i_0 \in \mathbb{N}$ such that $\{x_i : i \geq i_0\} \subset L \cap A_{m,n}$. Since $x_i \in A_{m,n}$, $x_i \in S_n(x_i) \subset P_m$ for each $i \geq i_0$. On the other hand, since $\{x_i\}$ converges to x, $\{x_i\}$ is eventually in $S_n(x)$. Thus, there exists $k_0 \geq i_0$ such that $d(x, x_i) < 1/n$ for all $i \geq k_0$. Then, $\{x, x_i\} \subset S_n(x_i) \subset P_m$ for all $i \geq k_0$. This follows that L is eventually in $P_m \in \mathcal{F}_{m,n}$.

Therefore, each $\mathcal{F}_{m,n}$ is a cs^*-cover for X.

(ii) $\{st(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ *is a network at x.* Let $x \in U$ with U open in X. Since $\mathcal{P}_x$ is an sn-network at x, there exists $m_0 \in \mathbb{N}$ such that $P_{m_0} \in \mathcal{P}_x$ and $P_{m_0} \subset U$. By Lemma 2.2(1), there exists $n_0 \in \mathbb{N}$ such that $S_{n_0}(x) \subset P_{m_0}$. This implies that $x \in A_{m_0,n_0}$. Hence, $\mathrm{st}(x, \mathcal{F}_{m_0,n_0}) = P_{m_0} \subset U$.

Next, we write $\{\mathcal{F}_{m,n} : m, n \in \mathbb{N}\} = \{\mathcal{H}_i : i \in \mathbb{N}\}$, and for each $i \in \mathbb{N}$, put $\mathcal{G}_i = \bigwedge\{\mathcal{H}_j : j \leq i\}$. Then, $\mathcal{G} = \bigcup\{\mathcal{G}_i : i \in \mathbb{N}\}$ is a σ-strong network consisting of finite cs^*-covers for X. Thus, (2) holds.

(2) $\Longrightarrow$ (3). Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network consisting of finite cs^*-covers for X. Consider the Ponomarev's system $(f, M, X, \mathcal{P}_n)$. By Lemma 2.3, (3) holds.

(3) $\Longrightarrow$ (4). It is obvious.

(4) $\Longrightarrow$ (1). Assume that (4) holds. Since M is separable, there exists a countable dense subset D of M. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \{f(S_n(x)) : x \in D\}$. Since f is a sequentially-quotient π-map, $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of countable cs^*-covers for X. Thus, X has a countable cs^*-network. On the other hand, by Lemma 2.1(1), X is sn-symmetric. Hence, (1) holds. ∎

The following corollary holds by Theorem 2.4.

Corollary 2.5. *The following are equivalent for a space X.*

(1) X is a symmetric with a countable cs-network;*

(2) X is a sequential space with a σ-strong network consisting of finite cs-covers;*

(3) X is a pseudo-sequence-covering quotient compact image of a separable metric space;

(4) X is a quotient π-image of a separable metric space.

Remark 2.6. By Corollary 2.5, we get an affirmative answer to the Question 1.1.

Theorem 2.7. *The following are equivalent for a space X.*

(1) X is a Cauchy sn-symmetric with a countable cs-network;*

(2) X has a σ-strong network consisting of finite cs-covers;

(3) X has a σ-strong network consisting of finite sn-covers;

(4) X is a 1-sequence-covering compact-covering compact image of a separable metric space;

(5) X is a sequence-covering π-image of a separable metric space.

Proof. (1) $\Longrightarrow$ (2). Let (1) holds. Since every Cauchy *sn*-symmetric is *sn*-symmetric, by using again notations and arguments as in the proof (1) $\Longrightarrow$ (2) of Theorem 2.4, it suffices to prove that each $\mathcal{F}_{m,n}$ is a *cs*-cover for X. Let $x \in X$ and $L = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to x; then

Case 1. If $x \in A_{m,n}$, then $S_n(x) \subset P_m$. Hence, L is eventually in $P_m \in \mathcal{F}_{m,n}$.

Case 2. If $x \notin A_{m,n}$ and $L \cap A_{m,n}$ is finite, then L is eventually in $B_{m,n} \in \mathcal{F}_{m,n}$.

Case 3. If $x \notin A_{m,n}$ and $L \cap A_{m,n}$ is infinite, then we can assume that $L \cap A_{m,n} = \{x_{i_k} : k \in \mathbb{N}\}$. Since X is Cauchy *sn*-symmetric and L converges to x, there exists $n_0 \in \mathbb{N}$ such that $d(x_i, x_j) < 1/n$ and $d(x, x_i) < 1/n$ for all $i, j \geq n_0$. Now, we pick $k_0 \in \mathbb{N}$ such that $i_{k_0} \geq n_0$. Since $d(x_{i_{k_0}}, x) < 1/n$ and $d(x_{i_{k_0}}, x_i) < 1/n$ for every $i \geq n_0$, L is eventually in $S_n(x_{i_{k_0}})$. Furthermore, since $x_{i_{k_0}} \in A_{m,n}$, $S_n(x_{i_{k_0}}) \subset P_m$. Hence, L is eventually in $P_m \in \mathcal{F}_{m,n}$.

Therefore, each $\mathcal{F}_{m,n}$ is a *cs*-cover for X.

(2) $\Longrightarrow$ (3). Let $\bigcup\{\mathcal{F}_i : i \in \mathbb{N}\}$ be a σ-strong network consisting of finite *cs*-covers for X. For each $i \in \mathbb{N}$, put $\mathcal{P}_i = \big\{P \in \mathcal{F}_i :$ there exist $x \in X, k \in \mathbb{N}$ such that $S_k(x) \subset P\big\}$. Then, each $\mathcal{P}_i$ is finite and each $P \in \mathcal{P}_i$ is a sequential neighbourhood of some $x \in X$. Furthermore, by using the proof in [9, Lemma 7], for each $x \in X$, there exist $P \in \mathcal{P}_i$ and $k \in \mathbb{N}$ such that $S_k(x) \subset P$. Thus, for each $x \in X$, there exists $P \in \mathcal{P}_i$ such that P is a sequential neighbourhood at x. Then, $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of finite *sn*-covers for X.

(3) $\Longrightarrow$ (4). Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network consisting of finite *sn*-covers for X. Consider the Ponomarev's system $(f, M, X, \mathcal{P}_n)$. By Lemma 2.3, (4) holds.

(4) $\Longrightarrow$ (5). It is obvious.

(5) $\Longrightarrow$ (1). Assume that (5) holds. Then, X has a countable cs^*-network. Furthermore, by [6, Proposition 16(3b)], X has a σ-strong network consisting of *cs*-covers. It follows from Lemma 2.1(2) that X is Cauchy *sn*-symmetric. ∎

The following corollary holds by [1, Corollary 2.8] and Theorem 2.7.

Corollary 2.8 *The following are equivalent for a space X.*

(1) X is a Cauchy symmetric space with a countable cs^*-network;

(2) X is a sequential space with a σ-strong network consisting of finite cs-covers;

(3) X is a sequential space with a σ-strong network consisting of finite sn-covers;

(4) X is a weak-open compact-covering compact image of a separable metric space;

(5) X is a weak-open π-image of a separable metric space.

Remark 2.9. Using [4, Example 3.1], it is easy to see that X is Hausdorff, non-regular and X has a countable base, but it is not a sequentially-quotient π-image of a metric space. This shows that "*sn*-symmetric" (resp., "Cauchy *sn*-symmetric") cannot be omitted in Theorem 2.4 (resp., Theorem 2.7).

Theorem 2.10. *The following are equivalent for a space X.*

(1) X has a countable network;

(2) X has a σ-strong network consisting of finite covers;
(3) X is a compact image of a separable metric space;
(4) X is a π-image of a separable metric space;
(5) X is an image of a separable metric space.

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ be a countable network for X. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_n, X - P_n\}$ and $\mathcal{G}_n = \bigwedge\{\mathcal{P}_i : i \leq n\}$. Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of finite covers.

(2) $\Longrightarrow$ (3). Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network consisting of finite covers for X. Consider the Ponomarev's system $(f, M, X, \mathcal{P}_n)$. Then, (3) holds by Lemma 2.3.

(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1). It is obvious. ∎

REMARK 2.11. By Theorem 2.4, Theorem 2.7 and Theorem 2.10, we get an affirmative answer to the Question 1.2.

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