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FINITE DIMENSIONS DEFINED BY MEANS OF m-COVERINGS

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Abstract. We introduce and investigate finite dimensions (m,n)-dim defined by means of m-coverings. These dimensions generalize the Lebesgue dimension: dim $= (2,1)$-dim. If $n < m$ and (m,n)-dim$X < \infty$, then X is weakly infinite-dimensional in the sense of Smirnov.

Introduction

In [7] there were introduced classes of $\mathcal{G}$-C-*spaces* and m-$\mathcal{G}$-C-space, where $\mathcal{G}$ is a class of simplicial complexes and $m \geq 2$ is an integer. Partial cases of these classes were considered in [8], where (m,n)-C-spaces were defined ($m \geq n \geq 1$). Let (m,n)-C be the class of all (m,n)-C-spaces. Then all classes (m,n)-C are intermediate between the class wid $= (2,1)$-$C = (n+1,n)$-C of all *weakly infinite-dimensional spaces* in the sense of Smirnov and the class C of all C-spaces in the sense of Haver [9], Addis and Gresham [1]. For example,

$$\text{wid} = (2,1)\text{-}C \supset (3,1)\text{-}C \supset \cdots \supset (m,1)\text{-}C \supset \cdots \supset C.$$

Here we define new dimension functions: (m,n)-dim (Definition 2.8). From definitions it follows that

$$(m,n)\text{-dim}X < \infty \Rightarrow X \in (m,n)\text{-}C. \tag{0.1}$$

For every normal space X we have

$$(2,1)\text{-dim}X = \text{dim}X \tag{0.2}$$

in view of the partition theorem.

For every metrizable space we have (Theorem 3.7)

$$(m,n)\text{-dim}X \leq \text{dim}X \tag{0.3}$$

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and (Theorem 3.9)

$$(m,1)\text{-dim}X = \text{dim}X. \tag{0.4}$$

One of the main results is

THEOREM 3.4. *If $n < m$, then for every space X we have*

$$(m,n)\text{-dim}X \le 0 \iff \text{dim}X \le n-1.$$

This theorem gives us a lot of spaces X with (m,n)-dim$X <$ dim$X$.

In § 2 we study general properties of dimension (m,n)-dim. This dimension satisfies the addition property for hereditarily normal spaces (Theorem 2.17):

$$X = X_1 \cup X_2 \Rightarrow (m,n)\text{-dim}X \le (m,n)\text{-dim}X_1 + (m,n)\text{-dim}X_2 + 1. \tag{0.5}$$

Theorem 2.21 states that if X is the limit of an inverse system $\{X_\alpha, \pi^\alpha_\beta, A\}$ of compact spaces, then

$$(m,n)\text{-dim}X \le \sup\{(m,n)\text{-dim}X_\alpha : \alpha \in A\}. \tag{0.6}$$

§ 1 has an auxiliary character. It contains necessary definitions and facts. One can find an additional information on dimension theory in [3] and [6].

§ 1. Preliminaries

All spaces are assumed to be normal T_1. All mappings are continuous. The symbol $|A|$ stands for the cardinality of a set A. If A is a subset of a space X, then $\text{Cl}(A) = \text{Cl}_X(A)$ denotes the closure of A in X.

By a cover we mean an open cover of a space. By $\text{cov}(X)$ we denote the set of all covers of X. The set of all finite covers of X is denoted by $\text{cov}_\infty(X)$ and $\text{cov}_m(X)$ stands for the set of all covers of X consisting of $\le m$ members.

Let u and v be families of subsets of a set X. They say that v *refines* u (v is a *refinement* of u) if each $V \in v$ is contained in some $U \in u$. A family v *combinatorially refines* u (v is a *combinatorially refinement* of u) if there exists an injection $i : v \to u$ such that $V \subset i(V)$ for each $V \in v$. If v refines u we write $u \prec v$.

For a simplicial complex K by $v(K)$ we denote the set of all its vertices. By Fin$S$ we denote the set of all non-empty finite subsets of S. Let u be a family of arbitrary sets and let $u_0 = \{U \in u : U \ne \emptyset\}$. The *nerve* $N(u)$ of the family u is a simplicial complex such that $v\big(N(u)\big) = \{a_U : U \in u_0\}$ and a set $\triangle \in \text{Fin}v\big(N(u)\big)$ is a simplex of $N(u)$ if and only if $\bigcap\{U : a_U \in \triangle\} \ne \emptyset$.

By the *order* of a family u of sets we mean the largest n such that u contains n sets with a non-empty intersection. If no such integer exist, we say that u has order ∞. The order of u is denoted by ord$u$. Clearly,

$$\text{ord}u \le \iff \text{dim}N(u) \le n-1;$$
$$\text{ord}u \le 1 \iff u \quad \text{is a disjoint family.}$$

By $\mathbb{N}$ we denote the set of all positive integers.

Let u be a family of subsets of a set X and let $M \subset X$. Then

$$u|M = \{U \cap M : U \in u\}.$$

1.1. Open swelling lemma. *If $\Phi = \{F_1, \ldots, F_m\}$ is a sequence of closed subsets of a space X, then there exists a family $v = (V_1, \ldots, V_m)$ of open subsets of X such that*

$$F_j \subset V_j, \quad j = 1, \ldots, m;$$
$$N(v) = N(\Phi). \quad \blacksquare$$

The Urysohn lemma and Lemma 1.1 yield

1.2. Lemma. *Let $u = (U_1, \ldots, U_m)$ be a sequence of open subsets of a space X and let $\Phi = (F_1, \ldots, F_m)$ be a sequence of closed subsets of X such that*

$$F_j \subset U_j, \quad j = 1, \ldots, m.$$

Then there exists a sequence $v = (V_1, \ldots, V_m)$ of open subsets of X such that

$$F_j \subset V_j \subset \mathrm{Cl}(V_j) \subset U_j, \quad j = 1, \ldots, m;$$
$$N(v) = N(\Phi). \quad \blacksquare$$

1.3. Lemma [5]. *Let X be a hereditarily normal space and let $M \subset X$. Then for every sequence $v = (V_1, \ldots, V_m)$ of open subsets of M there exists a sequence $w = (W_1, \ldots, W_m)$ of open subsets of X such that $w|M = v$ and $N(w) = N(v)$.* $\blacksquare$

1.4. Definition. Let $u = (U_1, \ldots, U_m)$ be a cover of a space X. A sequence φ of functions $f_j : X \to [0; 1]$, $j = 1, \ldots, m$, is said to be a *partition of unity subordinated to the cover u* if

$$f_1(x) + \cdots + f_m(x) = 1 \quad \text{for every} \quad x \in X;$$
$$f_j^{-1}(0; 1] \subset U_j, \quad j = 1, \ldots, m.$$

1.5. Closed shrinking lemma. *Let $u = (U_1, \ldots, U_m) \in \mathrm{cov}_m(X)$. Then there exists a family $\Phi = (F_1, \ldots, F_m)$ of closed subsets of X such that*

$$F_j \subset U_j, \quad j = 1, \ldots, m;$$
$$F_1 \cup \cdots \cup F_m = X. \quad \blacksquare$$

The Urysohn lemma and Lemma 1.5 imply

1.6. Partition of unity lemma. *For every finite cover u of a space X there exists a partition of unity subordinated to u.* $\blacksquare$

1.7. Theorem on partitions [10]. *A space X satisfies the inequality* $\dim X \le n \ge 0$ *if and only if for every sequence* (A_i, B_i), $i = 1, \dots, n+1$, *of pairs of disjoint closed subsets of X there exist partitions $P_1, \dots, P_{n+1}$ between A_i and B_i such that $P_1 \cap \dots \cap P_{n+1} = \emptyset$.* ■

1.8. Definition. A mapping $f : X \to \Delta_n$ to the n-dimensional simplex Δ_n is said to be *inessential*, if the mapping $g = f|f^{-1}S^{n-1} : f^{-1}S^{n-1} \to S^{n-1}$, where S^{n-1} is the combinatorial boundary of Δ_n, can be extended over X.

1.9. Theorem [2]. *A space X satisfies the inequality* $\dim X \le n \ge 0$ *if and only if each mapping $f : X \to \Delta_{n+1}$ is inessential.* ■

1.10. Theorem [11]. *Let X be a metrizable space with* $\dim X \le n \ge 0$. *Then X can be represented as the union of $n+1$ its subspaces X_i, $i = 1, \dots, n$, so that* $\dim X_i \le 0$. ■

1.11. Borsuk's theorem on extension of homotopy [12, 13]. *If F is a closed subspace of X, then each mapping $f : (X \times \{0\}) \cup (F \times I) \to R$ into ANR-compactum R extends over $X \times I$.* ■

1.12. Theorem [4]. *Let $f : X \to K$ and $g : X \to K$ be mappings to a simplicial complex K satisfying the following condition:*

$$\text{if} \quad f(x) \in Oa_j, \quad \text{then} \quad g(x) \in Oa_j,$$

where Oa_j is the star of a vertex $a_j \in K$ in K.

Then f and g are homotopically equivalent. ■

1.13. Definition. Let $u = (U_1, \dots, U_m)$ be a finite sequence of sets and let $u \prec v$. An *integration* of the family v with respect to u is the following sequence

$I(v, u) = (W_1, \dots, W_m)$:

$W_1 = \bigcup\{V \in v : V \subset U_1\}, \quad W_j = \bigcup\{V \in v : V \subset U_j; \quad V \not\subset U_k, \quad k < j\}.$

1.14. Proposition. 1) $\cup I(v, u) = \cup v$, 2) $u \prec I(v, u)$, 3) $\operatorname{ord} I(v, u) \le \operatorname{ord} v$. ■

1.15. Lemma. *Let $\alpha = (A_1, \dots, A_m)$ and $\beta = (B_1 \dots, B_m)$ be sequences of sets and let $\alpha \vee \beta = (A_1 \cup B_1, \dots, A_m \cup B_m)$. Assume that*

1) $\operatorname{ord}\beta \le 1$;

2) $B_j \cap A_k = \emptyset$ for all $k \ne j$.

Then $N(\alpha \vee \beta) = N(\alpha)$.

Proof. We have to show that for every family $j_1, \dots, j_k$,

$$\bigcap\{A_{j_1} \cup B_{j_1} : i = 1, \dots, k\} = \emptyset \iff \bigcap\{A_{j_i} : i = 1, \dots, k\} = \emptyset.$$

Implication $\Rightarrow$ is obvious. Now let $A_{j_1} \cap \dots \cap A_{j_k} = \emptyset$. Then by virtue of Newton binom we have

$$\left(A_{j_1} \cup B_{j_1}\right) \cap \left(A_{j_2} \cup B_{j_2}\right) \cap \dots \cap \left(A_{j_k} \cup B_{j_k}\right) = \sum\nolimits_{\mu \subset \{1, \dots, k\}} C_\mu$$

where $\nu = \{1, \dots, k\} \setminus \mu$ and $C_\mu = \left(\bigcap\{A_{j_i} : i \in \mu\}\right) \cap \left(\bigcap\{B_{j_i} : i \in \nu\}\right)$.

If $|\mu| = k$, then $C_\mu = A_{i_1} \cap \dots \cap A_{i_k} = \emptyset$ according to our assumption. If $|\mu| = k-1$, then $C_\mu = \emptyset$ in view of condition 2). At last, if $|\mu| \le k-2$, then $C_\mu = \emptyset$ by virtue of 1). ■

§ 2. Basic properties of finite (m,n)-dimensions

2.1. Definition. Let $u = (U_1, \dots, U_m) \in \mathrm{cov}_m(X)$ and let $\Phi = (F_1, \dots, F_m)$ be a family of closed subsets of X such that

$$F_j \subset U_j,\ j = 1, \dots, m;$$
$$\mathrm{ord}\Phi \le 1.$$

Then (u, Φ) is said to be an *m-pair in X*. The set of all m-pairs in X is denoted by $m(X)$.

2.2. Definition. Let $m, n \in \mathbb{N},\ n \le m,\ (u, \Phi)$ be an m-pair in X and let $v = (V_1, \dots, V_m)$ be a family of open subsets of X such that

$$F_j \subset V_j \subset U_j,\ j = 1, \dots, m;$$
$$\mathrm{ord} v \le n.$$

Then (u, v, Φ) is called an *(m,n)-triple in X*.

2.3. Lemma. *Let $n_1 \le n_2$ and let (u, v, Φ) be an (m, n_1)-triple in X. Then (u, v, Φ) is an (m, n_2)-triple in X.* ■

Lemma 1.2 yields

2.4. Lemma. *Every m-pair (u, Φ) in X can be included in $(m,1)$-triple (u, v, Φ) in X.* ■

2.5. Definition. Let $(u, \Phi) \in m(X)$. A closed set $P \subset X$ is said to be an *n-partition* of (u, Φ) (notation: $P \in \mathrm{Part}(u, \Phi, n)$) if there exists an (m,n)-triple (u, v, Φ) in X such that $P = X \setminus \bigcup v$.

Lemma 2.4 yields

2.6. Proposition. *Every m-pair (u, Φ) in X has an n-partition P.* ■

2.7. Definition. Let $(u_i, \Phi_i) \in m(X),\ i = 1, \dots, r$. The sequence $\big((u_1, \Phi_1), \dots, (u_r, \Phi_r)\big)$ is called *n-inessential in X* if there exist partitions $P_i \in \mathrm{Part}(u_i, \Phi_i, n)$ such that $P_1 \cap \dots \cap P_r = \emptyset$.

2.8. Definition. Let $m, n \in \mathbb{N}, n \le m$. To every space X one assigns the *dimension* (m,n)-$\dim X$, which is an integer ≥ -1 or ∞. The dimension function (m,n)-dim is defined in the following way:

(1) (m,n)-$\dim X = -1$ if and only if $X = \emptyset$;

(2) (m,n)-dim$X \le k$, where $k = 0, 1, \dots,$ if every sequence $\big((u_1, \Phi_1), \dots, (u_{k+1}, \Phi_{k+1})\big)$, $(u_i, \Phi_i) \in m(X)$, is n-inessential in X;

(3) (m,n)-dim$X = \infty$, if (m,n)-dim$X > k$ for each $k \in \mathbb{N}$.

2.9. Theorem. *For every space X we have*

$$(2,1)\text{-dim}X = \text{dim}X.$$

Proof. We start with inequality $(2,1)$-dim$X \le$ dim$X$. Let dim$X = n$ and let $(u_i, \Phi_i) \in 2(X)$, $i = 1, \dots, n+1$. Let $u_i = (U_1^i, U_2^i)$ and $\Phi_i = (F_1^i, F_2^i)$. Put

$$G_1^i = F_1^i \cup (X \setminus U_2^i), \quad G_2^i = F_2^i \cup (X \setminus U_1^i).$$

Then the family $\Gamma_i = (G_1^i, G_2^i)$ is disjoint, $i = 1, \dots, n+1$. Since dim$X \le n$, from Theorem 1.7 it follows that there exist partitions P_i in X between G_1^i and G_2^i such that $P_1 \cap \dots \cap P_{n+1} = \emptyset$. From definitions of the sets G_j^i we get $P_i \in$ Part$(u_i, \Phi_i,1)$. Hence the sequence $\big((u_1, \Phi_1), \dots, (u_{n+1}, \Phi_{n+1})\big)$ is 1-inessential in X and, consequently, (2,1)-dim$X \le n$.

Now let (2,1)-dim$X \le n$. Let $\Phi_i = (F_1^i, F_2^i)$, $i = 1, \dots, n+1$, be pairs of disjoint closed subsets of X. Put

$$U_1^i = X \setminus F_2^i, \quad U_2^i = X \setminus F_1^i, \quad i = 1, \dots, n+1.$$

Then

$$u_i = (U_1^i, U_2^i) \in \text{cov}_2(X), \quad i = 1, \dots, n+1.$$

Moreover, $(u_i, \Phi_i) \in 2(X)$, $i = 1, \dots, n+1$. Since $(2,1)$-dim$X \le n$, there exist partitions $P_i \in$ Part$(u_i, \Phi_i, 1)$ such that $P_1 \cap \dots \cap P_{n+1} = \emptyset$. Since $P_i \in$ Part$(u_i, \Phi_i, 1)$, there exist pairs $v_i = (V_1^i, V_2^i)$ of disjoint open subsets of X such that

$$F_j^i \subset V_j^i \subset U_j^i, \quad j = 1, 2; \quad i = 1, \dots, n+1;$$
$$P_i = X \setminus V_1^i \cup V_2^i.$$

Hence P_i are partitions of pairs Φ_i. By virtue of Theorem 1.7 we have dim$X \le n$. ■

2.10. Proposition. *Let M be a closed subset of X. Then*

$$(m,n)\text{-dim}M \le (m,n)\text{-dim}X.$$

Proof. The theorem is obvious if (m,n)-dim$X = -1$ or (m,n)-dim$X = \infty$, so that we can assume that (m,n)-dim$X = k$, $0 \le k < \infty$. Let

$$(u_i, \Phi_i) \in m(M), \quad i = 1, \dots, k+1;$$
$$u_i = (U_1^i, \dots, U_m^i), \quad \Phi_i = (F_1^i, \dots, F_m^i).$$

Put $W_j^i = U_j^i \cup (X \setminus M)$ and $w_i = (W_1^i, \dots, W_m^i)$. Then $(w_i, \Phi_i) \in m(X)$. Since (m,n)-dim$X = k$, the sequence $(w_1, \Phi_1), \dots, (w_{k+1}, \Phi_{k+1})$ is n-inessential in X. Clearly, the sequence $(w_1|M, \Phi_1), \dots, (w_{k+1}|M, \Phi_{k+1})$ is n-inessential in M. But $w_i|M = u_i$. ■

2.11. Proposition. *If a space X can be represented as the union of a discrete family X_α, $\alpha \in A$, of closed subspaces such that (m,n)-dim$X_\alpha \le k$ for $\alpha \in A$, then (m,n)-dim$X \le k$.* ■

2.12. Lemma. *Let X be a hereditarily normal space and let Y be its subspace. Let $F, F_1, F_2, \ldots, F_k$ be a disjoint family of closed subsets of X, V be a an open subset of Y, OF be a neighbourhood of F in X such that*

$$Y \cap \mathrm{Cl}(OF) \subset V; \tag{2.1}$$

$$(V \cup OF) \cap F_j = \emptyset, \quad j = 1, \ldots, m. \tag{2.2}$$

Then $V \cup F$ is open in $Y_1 = Y \cup F \cup F_1 \cup \cdots \cup F_k$.

Proof. From (2.1) it follows that $(Y \setminus V) \cap \mathrm{Cl}(OF) = \emptyset$ and, consequently, $\mathrm{Cl}(Y \setminus V) \cap OF = \emptyset$. Hence

$$OF \subset X \setminus \mathrm{Cl}(Y \setminus V) = W. \tag{2.3}$$

On the other hand,

$$V \subset W. \tag{2.4}$$

In fact, since V is open in Y, we have

$$V \cap \mathrm{Cl}(Y \setminus V) = V \cap \mathrm{Cl}_Y(Y \setminus V) = \emptyset. \tag{2.5}$$

Then $y \in V \Rightarrow (2.5) \Rightarrow y \notin \mathrm{Cl}(Y \setminus V) \Rightarrow y \in X \setminus \mathrm{Cl}(Y \setminus V) = W$.

Conditions (2.3) and (2.4) yield $V \cup OF \subset W$. Consequently, $V \cup F \subset W$ and, in accordance with (2.2), we have

$$V \cup F \subset W \setminus \bigcup \{F_j : j = 1, \ldots, m\}. \tag{2.6}$$

To prove our lemma it suffices to check that

$$V \cap F = Y_1 \cap \big(W \setminus \bigcup \{F_j : j = 1, \ldots, m\}\big).$$

By virtue of (2.6) it remains to show that

$$Y_1 \cap \big(W \setminus \bigcup \{F_j : j = 1, \ldots, m\}\big) \subset V \cup F. \tag{2.7}$$

Since $Y_1 \setminus \bigcup \{F_j : j = 1, \ldots, m\} = Y \cup F$, we have

$$Y_1 \cap \big(W \setminus \bigcup \{F_j : j = 1, \ldots, m\}\big) = W \cap (Y \cup F).$$

Consequently, to prove (2.7), it suffices to check that $W \cap Y \subset V$. But $W \cap Y = Y \setminus \mathrm{Cl}(Y \setminus V)$ according to (2.3). Let $y \in Y \setminus \mathrm{Cl}(Y \setminus V)$. Then there exists a neighbourhood Oy such that $Oy \cap (Y \setminus V) = \emptyset$. Consequently, $Y \cap Oy \subset V$. ■

2.13. Definition. For a subspace M of a space X, the *relative* (m,n)-*dimension* of M is defined by the formula

$$r\text{-}(m,n)\text{-}d_X M = \sup\{\,(m,n)\text{-dim}F : F \subset M \ \text{ and } \ F \ \text{ is closed in } \ X\,\}.$$

Proposition 2.10 implies

2.14. Proposition. *For every normal subspace M of a space X we have*

$$r\text{-}(m,n)\text{-}d_X M \le (m,n)\text{-}\dim M. \quad \blacksquare$$

2.15. Lemma. *Let $(u,\Phi) \in m(X)$, where $u = (U_1,\dots,U_m)$, $\Phi = (F_1,\dots,F_m)$. Then there exist a cover $u_1 = (U_1^1,\dots,U_m^1) \in \mathrm{cov}_m(X)$ and neighbourhoods OF_j such that*

$$OF_j \subset \mathrm{Cl}(OF_j) \subset U_j, \quad j = 1,\dots,m; \tag{2.8}$$

$$\mathrm{ord}\big(\mathrm{Cl}(OF_1),\dots,\mathrm{Cl}(OF_m)\big) \le 1; \tag{2.9}$$

$$\mathrm{Cl}(OF_j) \subset U_j^1 \subset U_j, \quad j = 1,\dots,m; \tag{2.10}$$

$$j_1 \ne j_2 \Rightarrow \mathrm{Cl}\big(OF_{j_1}\big) \cap U_{j_2}^1 = \emptyset. \tag{2.11}$$

Proof. By virtue of Lemma 1.2 there exist neighbourhoods OF_j satisfying conditions (2.8) and (2.9). Put

$$U_j^1 = U_j \setminus \bigcup\{\mathrm{Cl}(OF_k) : k \ne j\}. \tag{2.12}$$

Then (2.9) and (2.12) yield (2.10) and (2.11). It remains to show that $u_1 = (U_1^1,\dots,U_m^1) \in \mathrm{cov}(X)$.

Let $x \in U_j \setminus U_j^1$. Then $x \in \mathrm{Cl}(OF_k)$ for some $k \ne j$. Consequently, from (2.10) it follows that $x \in U_k^1$. $\blacksquare$

2.16. Proposition. *If a hereditarily normal space X can be represented as the union of two subspaces Y and Z such that*

$$(m,n)\text{-}\dim Y \le k, \quad r\text{-}(m,n)\text{-}d_X Z \le l,$$

then

$$(m,n)\text{-}\dim X \le k + l + 1. \tag{2.13}$$

Proof. We can assume that $0 \le k < \infty$, $0 \le l < \infty$. To prove (2.13), we have to show that every sequence $(u_i,\Phi_i) \in m(X)$, $i = 1,\dots,k+l+2$, is n-inessential in X (see Definition 2.8). Let

$$u_i = (U_1^i,\dots,U_m^i), \quad \Phi_1 = \big(F_1^i,\dots,F_m^i\big), \quad i = 1,\dots,k+l+2.$$

By virtue of Lemma 2.15 we may assume that there exist neighbourhoods OF_j^i such that

$$F_j^i \subset OF_j^i \subset \mathrm{Cl}(OF_j^i) \subset U_j^i; \tag{2.14}$$

$$l \ne j \implies U_l^i \cap \mathrm{Cl}(O\Phi_j^i) = \emptyset, \ \ i = 1,\dots,k+1. \tag{2.15}$$

From (2.14) and (2.15) it follows that

$$(u_i,\Omega_i) \in m(X), \quad \text{where } \Omega_i = (\mathrm{Cl}(O\Phi_1^i),\dots,\mathrm{Cl}(O\Phi_{k+1}^i)).$$

Since (m,n)-dim$Y \le k$, the sequence $(u_i|Y, \Omega_i|Y)$, $i = 1,\dots,k+1$, is n-inessential in Y. Hence there exist sequences $v_i = (V_1^i,\dots,V_m^i)$, $i = 1,\dots,k+1$, of open subsets of Y such that

$$Y \cap \mathrm{Cl}\big(OF_j^i\big) \subset V_j^i \subset U_j^i, \quad i = 1,\dots,k+1; \quad j = 1,\dots,m;$$
$$\mathrm{ord} v_i \le n, \quad i = 1,\dots,k+1;$$
$$v_1 \cup \dots \cup v_{k+1} \in \mathrm{cov}(Y).$$

Put $Y_1^i = Y \cup F_1^i \cup \dots \cup F_m^i$ and $\varphi_i = \big(V_1^i \cup F_1^i,\dots,V_m^i \cup F_m^i\big)$, $i = 1,\dots,k+1$. By virtue of (2.15) and Lemma 1.15 we have

$$\mathrm{ord}\varphi_i = \mathrm{ord} v_i \le n. \tag{2.16}$$

The pair $\big(V_j^i, F_j^i\big)$ satisfies conditions of Lemma 2.12. Hence members of φ_i are open in Y_1^i. Since X is hereditarily normal, according to Lemma 1.3 there exist families

$$w_i = \big(W_1^i,\dots,W_m^i\big), \quad i = 1,\dots,k+1,$$

of open subsets of X such that

$$V_j^i \cup F_j^i \subset W_j^i \subset U_j^i, \quad j = 1,\dots,m; \tag{2.17}$$
$$\mathrm{ord} w_i \le n. \tag{2.18}$$

Put $W_i = W_1^i \cup \dots \cup W_m^i$ and $W = W_1 \cup \dots \cup W_{k+1}$. By definition we have

$$w_1 \cup \dots \cup w_{k+1} \in \mathrm{cov}(W). \tag{2.19}$$

Let $F = X \setminus W$. By virtue of (2.17) we have $F \subset Z$. Since r-(m,n)-$d_X Z \le l$, we have (m,n)-dim$F \le l$. Hence the sequence $(u_i|F,\ \Phi_i|F)$, $i = k+2,\dots,k+l+2$, is n-inessential in F. Following to the first part of the proof we can find families

$$w_i = \big(W_1^i,\dots,W_m^i\big), \quad i = k+2,\dots,k+l+2,$$

of open subsets of X such that $\mathrm{ord} w_i \le n$,

$$F_j^i \subset W_j^i \subset U_j^i, \quad i = k+2,\dots,k+l+2; \quad j = 1,\dots,m;$$

and

$$F \subset \bigcup\{W_j^i : \quad i = k+2,\dots,k+l+2; \quad j = 1,\dots,m\}.$$

Thus the sequence $w_1,\dots,w_{k+l+2}$ realizes the conditions of an n-inessentialitness of the sequence (u_i, Φ_i), $i = 1,\dots,k+l+2$. ■

Proposition 2.16 implies

2.17. The addition theorem for (m,n)-dim. *If a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then*

$$(m,n)\text{-dim}X \le (m,n)\text{-dim}X_1 + (m,n)\text{-dim}X_2 + 1. \quad ■$$

Theorem 2.17 yields

2.18. Corollary. *If a hereditarily normal space X can be represented as the union of $k+1$ subspaces $X_0, X_1, \dots, X_k$ such that (m,n)-dim$X_i \le 0$ for $i = 0,1,\dots,k$, then (m,n)-dim$X \le k$.* ■

2.19. Proposition. *Let $f : X \to Y$ be a mapping and let a sequence $(u_i, \Phi_i) \in m(Y)$ be n-inessential in Y. Then the sequence $(f^{-1}u_i,\ f^{-1}\Phi_i)$ is n-inessential in X.* ■

2.20. Proposition. *Let $\left(u_i^l, \Phi_i^l\right) \in m(X)$, $u_i^l = \left({}^lU_1^i, \dots, {}^lU_m^i\right)$, $\Phi_i^l = \left({}^lF_1^i, \dots, {}^lF_m^i\right)$, $i = 1,\dots,r$; $l = 1,2$. Assume that*

$$ {}^1F_j^i \subset {}^2F_j^i \subset {}^2U_j^i \subset {}^1U_j^i, \quad i = 1,\dots,r; \quad j = 1,\dots,m. $$

Let the sequence (u_i^2, Φ_i^2), $i = 1,\dots,r$, be n-inessential in X. Then the sequence (u_i^1, Φ_i^1), $i = 1,\dots,r$, is n-inessential in X. ■

2.21. Theorem. *Let $S = \{X_\alpha, \pi_\beta^\alpha, A\}$ be an inverse system of compact spaces X_α with (m,n)-dim$X_\alpha \le k$, and let $X = \lim S$. Then (m,n)-dim$X \le k$.*

Proof. We have to verify that an arbitrary sequence $(u_i, \Phi_i) \in m(X)$, $i = 1,\dots,k+1$, is n-inessential in X. Let $u_i = (U_1^i,\dots,U_m^i)$, $\Phi_i = (F_1^i,\dots,F_m^i)$. Since X is a compact space, by definition of the inverse limit topology, for each $i = 1,\dots,k+1$ there exists $\alpha_i \in A$ and

$$ u_i^i = \left({}^iU_1^i, \dots, {}^iU_m^i\right) \in \mathrm{cov}_m(X_{\alpha_i}) \tag{2.20} $$

such that

$$ \pi_{\alpha_i}^{-1}\left({}^iU_j^i\right) \subset U_j^i, \quad j = 1,\dots,m; \tag{2.21} $$

$$ \mathrm{ord}\left(\pi_{\alpha_i}(\Phi_i)\right) \le 1, \tag{2.22} $$

where $\pi_\alpha : X \to X_\alpha$ are the limit projections of the system S and $\pi_\alpha(\Phi_i) = \Big(\pi_\alpha\left(F_1^i\right), \dots, \pi_\alpha\left(F_m^i\right)\Big)$. Since A is a directed set, there exists $\alpha_0 \in A$ such that

$$ \alpha_i \le \alpha_o, \quad i = 1,\dots,k+1. $$

Put

$$ {}^0U_j^i = \left(\pi_{\alpha_i}^{\alpha_0}\right)^{-1}\left({}^iU_j^i\right), \quad j = 1,\dots,m; \tag{2.23} $$

$$ {}^0F_j^i = \left(\pi_{\alpha_i}^{\alpha_0}\right)^{-1}\left(\pi_\alpha(F_j^i)\right), \quad j = 1,\dots,m; \tag{2.24} $$

$$ u_i^0 = \left({}^0U_1^i, \dots, {}^0U_m^i\right), \quad i = 1,\dots,k+1; \tag{2.25} $$

$$ \Phi_i^0 = \left({}^0F_1^i, \dots, \dots, {}^0F_m^i\right), \quad i = 1,\dots,k+1. \tag{2.26} $$

By virtue of (2.20)–(2.22) we have

$$ \left(u_i^0, \Phi_i^0\right) \in m\left(X_{\alpha_0}\right), \quad i = 1,\dots,k+1. \tag{2.27} $$

Since (m,n)-dim$X_{\alpha_0} \leq k$, the sequence (2.27) is n-inessential in X_{α_0}. Then the sequence

$$\left(\pi_{\alpha_0}^{-1}(u_i^0),\ \pi_{\alpha_0}^{-1}(\Phi_i^0)\right),\quad i = 1,\dots,k+1,$$

is n-inessential in X according to Proposition 2.19. On the other hand, from (2.21), (2.23)–(2.25) it follows that

$$\Phi_i \quad \text{refines} \quad \pi_{\alpha_0}^{-1}(\Phi_i^0) \quad \text{and} \quad \pi_\alpha^{-1}(u_i^0) \quad \text{refines} \quad u_i, \quad i = 1,\dots,k+1.$$

Consequently, Proposition 2.20 implies that the sequence (u_i, Φ_i), $i = 1,\dots,k+1$, is n-inessential in X. ∎

§ 3. Comparison of dimensions

3.1. Proposition. *If $n \geq m$, then (m,n)-dim$X \leq 0$ for every space X.* ∎

The condition

$$n_1 \leq n_2 \Rightarrow \mathrm{Part}(u,\Phi,n_2) \subset \mathrm{Part}(u,\Phi,n_1) \tag{3.1}$$

implies

3.2. Proposition. *If $n_1 \leq n_2$, then*

$$(m,n_1)\text{-dim}X \geq (m,n_2)\text{-dim}X$$

for every space X. ∎

The condition

$$m_1 \leq m_2 \Rightarrow \mathrm{cov}_{m_1}(X) \subset \mathrm{cov}_{m_2}(X) \tag{3.2}$$

yields

3.3. Proposition. *If $m_1 \leq m_2$, then*

$$(m_1,n)\text{-dim}X \leq (m_2,n)\text{-dim}X$$

for every space X. ∎

3.4. Theorem. *If $n < m$, then for every space X we have*

$$(m,n)\text{-dim}X \leq 0 \iff \mathrm{dim}X \leq n-1.$$

Proof. Let (m,n)-dim$X \leq 0$. We have to show that

$$\mathrm{dim}X \leq n-1. \tag{3.3}$$

According to Theorem 1.9 condition (3.3) is equivalent to the condition

$$\text{every mapping} \quad f : X \to \Delta_n \quad \text{is inessential.} \tag{3.4}$$

Let a_j, $j = 1,\dots,n+1$, be the vertices of the simplex Δ_n and let O_j be the stars of Δ_n with respect to a_j. Put

$$U_j = f^{-1}O_j, \quad j = 1,\dots,n+1. \tag{3.5}$$

Since $n < m$, we have $u = (U_1, \ldots, U_{n+1}) \in \text{cov}_m(X)$. Consider a pair (u, Φ), where $\Phi = (F_1, \ldots, F_{n+1})$ and $F_j = \emptyset$, $j = 1, \ldots, n+1$. Then $(u, \Phi) \in m(X)$. In view of (m,n)-dim$X \leq 0$ there exists a cover $v = (V_1, \ldots, V_{n+1})$ of X such that

$$V_j \subset U_j, \quad j = 1, \ldots, n+1; \tag{3.6}$$

$$\text{ord} v \leq n. \tag{3.7}$$

Consider a partition of unity $(\varphi_1, \ldots, \varphi_{n+1})$ subordinated to the cover v. Let

$$\varphi = \varphi_1 \triangle \ldots \triangle \varphi_{n+1} \to \Delta_n$$

be the barycentric mapping defined by $(\varphi_1, \ldots, \varphi_{n+1})$, that is

$$\varphi(x) = \big(\varphi_1(x), \ldots, \varphi_{n+1}(x)\big),$$

where $\varphi_j(x)$ is the barycentric coordinate of the point $\varphi(x)$ corresponding to the vertex $a_j \in \Delta_n$. We have

$$\varphi^{-1} O_j = \{x \in X : \varphi_j(x) > 0\} \subset V_j \subset U_j. \tag{3.8}$$

From (3.7) it follows that

$$\varphi(X) \subset \Delta_n^{n-1} = S^{n-1}, \tag{3.9}$$

where $\Delta_n^{n-1} = S^{n-1}$ is the $(n-1)$-dimensional skeleton of the simplex Δ_n. Let $F = f^{-1} S^{n-1}$. Conditions (3.5) and (3.8) imply that

$$\varphi(x) \in O_j \Rightarrow f(x) \in O_j.$$

Hence the mappings $\varphi : F \to S^{n-1}$ and $f_0 = f|_F : F \to S^{n-1}$ are homotopically equivalent by Theorem 1.12. Consequently, from (3.9) it follows that the mapping f_0 is extended over X by virtue of Theorem 1.11. Thus f is inessential. Inequality (3.3) is proved.

Now let dim$X \leq n-1$. We have to check that

$$(m,n)\text{-dim} X \leq 0. \tag{3.10}$$

If $m = n$, then (3.10) is a corollary of Proposition 3.1, so that we assume that $m - n \geq 1$. Let (u, Φ), $u = (U_1, \ldots, U_m)$, $\Phi = (F_1, \ldots, F_m)$, be an m-pair in X. To prove (3.10), we have to find a cover $v = (V_1, \ldots, V_m) \in \text{cov}_m(X)$ such that

$$F_j \subset V_j \subset U_j, \quad j = 1, \ldots, m; \tag{3.11}$$

$$\text{ord} v \leq n. \tag{3.12}$$

Let us take a cover $u_1 = (U_1^1, \ldots, U_m^1)$ and neighbourhoods OF_j from Lemma 2.15. Since dim$X \leq n-1$, there exist a cover $w_1 \in \text{cov}(X)$ such that w_1 refines u_1 and ord$w_1 \leq n$. Let $w = (W_1, \ldots, W_m)$ be an integration of w_1 with respect to u_1. In accordance with Definition 1.13 and Proposition 1.14 w is a cover of order $\leq n$ such that

$$W_j \subset U_j^1, \quad j = 1, \ldots, m. \tag{3.13}$$

Put $V_j = W_j \cup OF_j$ and $v = (V_1, \ldots, V_m)$. From Lemma 1.15 (for $A_j = W_j$ and $B_j = OF_j$), (2.10), and (3.13) it follows that v is a cover satisfying conditions (2.11) and (2,12). ∎

Theorem 3.4 implies

3.5. Theorem. *Let $m \geq n+2$. Then* $\dim X \leq n$ *if and only if for every cover* $u = (U_1, \ldots, U_m) \in \mathrm{cov}_m(X)$ *and for every disjoint family* $\Phi = (F_1, \ldots, F_m)$ *of closed subsets of X such that $F_j \subset U_j$ there exists a cover* $v = (V_1, \ldots, V_m) \in \mathrm{cov}_m(X)$ *such that*

$$F_j \subset V_j \subset U_j, \quad j = 1, \ldots, m;$$
$$\mathrm{ord} v \leq n+1. \quad ∎$$

Another corollary of Theorem 3.4 is

3.6. Theorem. *For every space X we have*

$$\dim X \leq 0 \Rightarrow (m,n)\text{-}\dim X \leq 0.$$

Proof. Theorem 3.4 implies that $(m,1)$-$\dim X \leq 0$. Applying Proposition 3.2 we get the required property. ∎

3.7. Theorem. *For every metrizable space X we have*

$$(m,n)\text{-}\dim X \leq \dim X. \tag{3.14}$$

Proof. The assertion is obvious if $\dim X = -1$ or $\dim X = \infty$. Assume that $\dim X = k$, $0 \leq k < \infty$. By virtue of Katetov theorem (Theorem 1.10) there exist subspaces $X_i \subset X$, $0 \leq i \leq k$, such that $\dim X_i \leq 0$ and $X = X_0 \cup X_1 \cup \cdots \cup X_k$. According to Theorem 3.6 we have (m,n)-$\dim X \leq 0$. It remains to apply Corollary 2.18. ∎

3.8. Question. Does equality (3.14) hold for an arbitrary space X?

3.9. Theorem. *If $m \geq 2$, then*

$$(m,1)\text{-}\dim X = \dim X \tag{3.15}$$

for every metrizable space X.

Proof. By virtue of Theorem 2.9

$$(2,1)\text{-}\dim X = \dim X. \tag{3.16}$$

From (3.16) and Proposition 3.3 it follows that $(m,1)$-$\dim X \leq \dim X$. At last, Theorem 3.7 yields

$$(m,1)\text{-}\dim X \geq \dim X. \quad ∎$$

3.10. Question. Does equality (3.15) hold for an arbitrary space X?

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