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COMMON FIXED POINTS FOR GENERALIZED NONLINEAR CONTRACTIVE MAPPINGS IN METRIC SPACES

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Abstract. The purpose of this paper is to present a common fixed point theorem for generalized nonlinear contractive mappings in complete metric spaces by generalizing and extending some known results. As a consequence, a common fixed point theorem for a Banach operator pair is obtained.

1. Introduction and preliminaries

It is well known that Banach's fixed point theorem for contraction mappings is one of the pivotal result of analysis. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a *contraction* if there exists $0 \le k < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \le kd(x, y). \tag{1.1}$$

If the metric space (X,d) is complete, then the mapping satisfying (1.1) has a unique fixed point.

Let (X,d) be a metric space. A map $T : X \to X$ is called a *Chatterjee's contraction* (see [7, 9]) if there exists $0 \le k < \frac{1}{2}$ such that for each $x, y \in X$,

$$d(Tx, Ty) \le k[d(x, Ty) + d(y, Tx)]. \tag{1.2}$$

A map $T : X \to X$ is called a *weakly contractive mapping* (see [1, 9, 13]) if for each $x, y \in X$,

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)) \tag{1.3}$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$. If we take $\psi(x) = kx$, $0 < k < 1$, then a weakly contractive mapping is a contraction.

A map $T : X \to X$ is called an *f-weakly contractive mapping* (see [10]) if for each $x, y \in X$,

$$d(Tx, Ty) \le d(fx, fy) - \psi(d(fx, fy)) \tag{1.4}$$

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where $f : X \to X$ is a self-mapping, $\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$. If we take $\psi(x) = (1 - k)x$, $0 < k < 1$, then an f-weakly contractive mapping is called an f-contraction. Further, if $f =$ identity mapping and $\psi(x) = (1 - k)x$, $0 < k < 1$, then a f-weakly contractive mapping is a contraction.

A map $T : X \to X$ is called a *generalized weakly contractive mapping* (see [9]) if for each $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)) \tag{1.5}$$

where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If we take $\psi(x, y) = (\frac{1}{2} - k)(x + y)$, $0 < k < \frac{1}{2}$, then inequality (1.5) reduces to (1.2). Choudhury [9] has showed that generalized weakly contractive mapping are generalizations of contractive mappings given by Chatterjee (1.2) and they constitute a strictly larger class of mappings than Chatterjee's contraction.

A map $T : X \to X$ is called a *generalized f-weakly contractive mapping* (see [4]) if for each $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \psi(d(fx, Ty), d(fy, Tx)) \tag{1.6}$$

where $f : X \to X$ is a self-mapping, $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If f is an identity mapping, then a generalized f-weakly contractive mapping is a generalized weakly contractive mapping.

Alber and Guerre-Delabriere [1] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [13] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contain contractions as a special case and he also showed that some results of [1] are true for any Banach space. In fact, weakly contractive mappings are closely related to maps of Boyd and Wong [3] and of Reich types [12].

Let M be a nonempty subset of a metric space (X, d); a point $x \in M$ is a *common fixed (coincidence) point* of f and T if $x = fx = Tx$ ($fx = Tx$). The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f, T)$ (respectively, $C(f, T)$). The mappings $T, f : M \to M$ are called *commuting* if $Tfx = fTx$ for all $x \in M$; *compatible* if $\lim d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim fx_n = t$ for some t in M; *weakly compatible* if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$.

The ordered pair (T, I) of two self-maps of a metric space (X, d) is called a *Banach operator pair* [8], if the set $F(I)$ is T-invariant, i.e. $T(F(I)) \subseteq F(I)$. Obviously, a commuting pair (T, I) is a Banach operator pair but not conversely. If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair. If the self-maps T and I of X satisfy $d(ITx, Tx) \leq kd(Ix, x)$, for all $x \in X$ and $k \geq 0$, then (T, I) is a Banach operator pair. This class of non-commuting mappings is different from the known classes of non-commuting mappings viz. R-weakly commuting, R-subweakly commuting, compatible, weakly compatible and C_q-commuting etc. existing in the literature (see e.g. [5, 6, 8, 10] and references cited therein).

Example 1.1 Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ and $K = [1, \infty)$. Let $T(x) = x^3$ and $I(x) = 2x - 1$, for all $x \in K$. Then $F(I) = \{1\}$. Here (T, I) is a Banach operator pair but T and I are not commuting.

Example 1.2 (see [10]) Consider $X = R^2$ with the usual metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $(x_1, y_1), (x_2, y_2) \in R^2$. Define T and I on X as $T(x, y) = (x^3 + x - 1, \frac{\sqrt[3]{x^2+y^3-1}}{3})$ and $I(x, y) = (x^3 + x - 1, \sqrt[3]{x^2 + y^3 - 1})$. $F(T) = (1, 0)$, $F(I) = \{(1, y) : y \in R\}$ and $C(I, T) = \{(x, y) : y = \sqrt[3]{1 - x^2}, x \in R\}$. $T(F(I)) = \{T(1, y) : y \in R\} = \{(1, \frac{y}{3}) : y \in R\} \subseteq \{(1, y) : y \in R\} = F(I)$. Thus (T, I) is a Banach operator pair, which is not weakly compatible as T and I do not commute on the set $C(I, T)$ and hence it is not compatible.

In this paper, we prove a common fixed point theorem for weakly compatible generalized f-weakly contractive mappings in complete metric spaces by generalizing and extending some known results. As a consequence, a common fixed point theorem for a Banach operator pair is obtained.

2. Main results

Theorem 2.1. *Let M be a subset of a metric space (X, d) and f and T be self-mappings of M such that $\operatorname{cl} T(M) \subseteq f(M)$. If $\operatorname{cl} T(M)$ is complete and T is a generalized f-weakly contractive mapping, then T and f have a unique point of coincidence in M. If, in addition (f, T) is weakly compatible, then $F(T) \cap F(f)$ is a singleton.*

Proof. Let $x_0 \in M$. Since $T(M) \subseteq f(M)$, we can choose $x_1 \in M$ so that $fx_1 = Tx_0$. Since $Tx_1 \in f(M)$, there exists $x_2 \in M$ such that $fx_2 = Tx_1$. By induction, we construct a sequence $\{x_n\}$ in M such that $fx_{n+1} = Tx_n$, for every $n \geq 0$. Consider

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \frac{1}{2}[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})] \\ &\qquad - \psi(d(fx_{n+1}, Tx_n), d(fx_n, Tx_{n+1})) \\ &= \frac{1}{2} d(Tx_{n-1}, Tx_{n+1}) - \psi(0, d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \frac{1}{2} d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \frac{1}{2}[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \qquad (*) \end{aligned}$$

Hence for all $n = 1, 2 \ldots$, we have $d(Tx_{n+1}, Tx_n) \leq d(Tx_n, Tx_{n-1})$. Thus $\{d(Tx_{n+1}, Tx_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent.

Let $d(Tx_{n+1}, Tx_n) \to r$. From inequality $(*)$, we have

$$d(Tx_{n+1}, Tx_n)\frac{1}{2}d(Tx_{n-1}, Tx_{n+1}) \leq \frac{1}{2}[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]$$

Taking $n \to \infty$, we have $r \leq \lim \frac{1}{2}d(Tx_{n-1}, Tx_{n+1}) \leq \frac{1}{2}r + \frac{1}{2}r$. So $\lim d(Tx_{n-1}, Tx_{n+1}) = 2r$. Using the continuity of ψ and inequality $(*)$, we have $r \leq r - \psi(0, 2r)$, and consequently, $\psi(0, 2r) \leq 0$. Thus $r = 0$. Hence $d(Tx_{n+1}, Tx_n) \to 0$.

Now, we show that $\{Tx_n\}$ is a Cauchy sequence. If otherwise, then there exists $\epsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that for every k, $d(Tx_{m(k)}, Tx_{n(k)}) \geq \epsilon$, $d(Tx_{m(k)}, Tx_{n(k)-1}) < \epsilon$. So, we have

$$\begin{aligned}\epsilon \leq d(Tx_{m(k)}, Tx_{n(k)}) &\leq d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)})\\ &< \epsilon + d(Tx_{n(k)-1}, Tx_{n(k)}).\end{aligned}$$

Letting $n \to \infty$ and using $d(Tx_{n+1}, Tx_n) \to 0$, we have

$$\lim d(Tx_{m(k)}, Tx_{n(k)}) = \epsilon = \lim d(Tx_{m(k)}, Tx_{n(k)-1}). \tag{2.1}$$

Again,

$$\begin{aligned}d(Tx_{m(k)}, Tx_{n(k)-1}) &\leq d(Tx_{m(k)}, Tx_{m(k)-1})\\ &\quad + d(Tx_{m(k)-1}, Tx_{n(k)}) + d(Tx_{n(k)}, Tx_{n(k)-1}),\end{aligned}$$

and

$$d(Tx_{m(k)-1}, Tx_{n(k)}) \leq d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)}).$$

Taking $n \to \infty$ in the above two inequalities and using (2.1) we get $\lim d(Tx_{m(k)-1}, Tx_{n(k)}) = \epsilon$.

Also, we have

$$\begin{aligned}\epsilon &\leq d(Tx_{m(k)}, Tx_{n(k)})\\ &\leq \frac{1}{2}[d(fx_{m(k)}, Tx_{n(k)}) + d(fx_{n(k)}, Tx_{m(k)})]\\ &\qquad - \psi(d(fx_{m(k)}, Tx_{n(k)}), d(fx_{n(k)}, Tx_{m(k)}))\\ &= \frac{1}{2}[d(Tx_{m(k)-1}, Tx_{n(k)}) + d(Tx_{n(k)-1}, Tx_{m(k)})]\\ &\qquad - \psi(d(Tx_{m(k)-1}, Tx_{n(k)}), d(Tx_{n(k)-1}, Tx_{m(k)})).\end{aligned}$$

Taking $n \to \infty$, we have $\epsilon \leq \frac{1}{2}[\epsilon + \epsilon] - \psi(\epsilon, \epsilon)$ and consequently $\psi(\epsilon, \epsilon) \leq 0$, which is a contradiction since $\epsilon > 0$. Hence $\{Tx_n\}$ is a Cauchy sequence. By the completeness of $\mathrm{cl}\, T(M)$ there is some $u \in \mathrm{cl}\, T(M)$ such that $u = \lim Tx_n$. As $clT(M) \subseteq f(M)$, there is some $z \in M$ such that $fz = u$. Consider

$$\begin{aligned}d(Tz, fz) &\leq d(Tz, Tx_{n+1}) + d(Tx_{n+1}, fz)\\ &\leq \frac{1}{2}[d(fz, Tx_{n+1}) + d(fx_{n+1}, Tz)]\\ &\qquad - \psi(d(fz, Tx_{n+1}), d(fx_{n+1}, Tz)) + d(Tx_{n+1}, fz)\\ &= \frac{1}{2}[d(u, Tx_{n+1}) + d(Tx_n, Tz)]\\ &\qquad - \psi(d(u, Tx_{n+1}), d(Tx_n, Tz)) + d(Tx_{n+1}, u)\end{aligned}$$

Letting $n \to \infty$, we have

$$d(Tz, u) \leq \frac{1}{2} d(u, Tz) - \psi(0, d(u, Tz))$$

This implies that $d(Tz, fz) = 0$. Hence $Tz = fz = u$ is a point of coincidence of T and f.

Now suppose that T and f are weakly compatible. Then $T(u) = T(f(z)) = f(T(z)) = f(u)$. Consider

$$\begin{aligned} d(T(z), T(u)) &\leq \frac{1}{2}[d(fz, Tu) + d(fu, Tz)] - \psi(d(fz, Tu), d(fu, Tz)) \\ &= \frac{1}{2}[d(Tz, Tu) + d(Tu, Tz)] - \psi(d(Tz, Tu), d(Tu, Tz)) \\ &= d(Tu, Tz) - \psi(d(Tz, Tu), d(Tu, Tz)). \end{aligned}$$

This implies that $d(Tz, Tu) = 0$, by the property of ψ. Therefore, $T(u) = f(u) = u$. Hence u is a common fixed point of f and T. The uniqueness follows from (1.6). ■

If f is an identity mapping of X, then we have:

COROLLARY 2.2. *Let M be a subset of a metric space (X, d) and T be a self-mapping of M such that $\operatorname{cl} T(M) \subseteq M$. If $\operatorname{cl} T(M)$ is complete and T is a generalized weakly contractive mapping, then T has a unique fixed point.*

COROLLARY 2.3. [9] *Let T be a self-mapping of X, where (X, d) is a complete metric space. If T is a generalized weakly contractive mapping, then T has a unique fixed point.*

If $\psi(x, y) = (\frac{1}{2} - k)(x + y)$, $0 < k < \frac{1}{2}$, we have

COROLLARY 2.4. [7] *If $T : X \to X$, where (X, d) is a complete metric space, satisfies*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \tag{2.2}$$

where $0 \leq k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

EXAMPLE 2.5. Let $X = \{p, q, r\}$ and d be a metric defined on X. Let T and f be self-mappings of X such that $Tp = fq$, $Tq = fq$, $Tr = fp$, $d(fp, fq) = 1$, $d(fq, fr) = 2$, $d(fr, fp) = 1.5$ and $\psi(a, b) = \frac{1}{2} \min\{a, b\}$. Then T is a generalized f-weakly contraction and q is a coincidence point of T and f.

If f is an identity mapping of X, this example is given in [9]

REMARK 2.6. Theorem 2.1 extends and generalizes the corresponding results of [2, 5, 7, 9, 11].

As an application of Corollary 2.2, we obtain the following general result for a Banach operator pair.

THEOREM 2.7. *Let M be a subset of a metric space (X, d) and f and T are self-mappings of M such that $\operatorname{cl} T(F(f)) \subseteq F(f)$. If $\operatorname{cl} T(M)$ is complete, $F(f)$ is nonempty and T is a generalized f-weakly contractive mapping for all $x, y \in F(f)$, then $F(T) \cap F(f)$ is a singleton.*

Proof. $\mathrm{cl}\, T(F(f))$ being a subset of $\mathrm{cl}\, T(M)$ is complete and and $\mathrm{cl}\, T(F(f)) \subseteq F(f)$. So for all $x, y \in F(f)$, we have

$$\begin{aligned} d(Tx,Ty) &\leq \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] - \psi(d(fx,Ty),d(fy,Tx)) \\ &= \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \psi(d(x,Ty),d(y,Tx)). \end{aligned}$$

Thus by Corollary 2.2, T has a unique fixed point z in $F(f)$ and consequently, $F(T) \cap F(f)$ is a singleton. ■

COROLLARY 2.8. *Let M be a subset of a metric space (X,d) and f and T are self-mappings of M. If $\mathrm{cl}\, T(M)$ is complete, (T,f) is a Banach operator pair, $F(f)$ is nonempty and closed and T is a generalized f-weakly contractive mapping, then $F(T) \cap F(f)$ is a singleton.*

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