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# GROWTH OF POLYNOMIALS WITH PRESCRIBED ZEROS

**M. S. Pukhta**

**Abstract.** In this paper we study the growth of polynomials of degree $n$ having all their zeros on $|z| = k$, $k \le 1$. Using the notation $M(p,t) = \max_{|z|=t} |p(z)|$, we measure the growth of $p$ by estimating $\left\{\frac{M(p,t)}{M(p,1)}\right\}^s$ from above for any $t \ge 1$, $s$ being an arbitrary positive integer. Also in this paper we improve the results recently proved by K. K. Dewan and Arty Ahuja [*Growth of polynomials with prescribed zeros*, J. Math. Ineq. **5** (2011), 355–361].

## 1. Introduction and statement of results

For an arbitrary entire function $f(z)$, let $M(f,r) = \max_{|z|=r} |f(z)|$ and $m(f,k) = \min_{|z|=k} |f(z)|$. Then for a polynomial $p(z)$ of degree $n$, it is a simple consequence of maximum modulus principle (for reference see [4, Vol. I, p. 137, Problem III, 269]) that

$$M(p,R) \le R^n M(p,1), \quad \text{for} \quad R \ge 1. \tag{1.1}$$

The result is best possible and equality holds for $p(z) = \lambda z^n$, where $|\lambda| = 1$.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.1) can be sharpened. In fact it was shown by Ankeny and Rivlin [1] that if $p(z) \ne 0$ in $|z| < 1$, then (1.1) can be replaced by

$$M(p,R) \le \Big(\frac{R^n + 1}{2}\Big) M(p,1), \quad R \ge 1. \tag{1.2}$$

The result is sharp and equality holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

For the class of polynomials not vanishing in the disk $|z| < k$, $k \ge 1$, Shah [6] proved that if $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \ge 1$, then for every real number $R > K$,

$$M(p,R) \le \Big(\frac{R^n + k}{1+k}\Big) M(p,1) - \Big(\frac{R^n - 1}{1+k}\Big) m(p,k).$$

The result is best possible in case $k = 1$ and equality holds for the polynomial $p(z) = z^n + 1$.

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Recently Dewan and Arty [3] proved that if $p(z)$ is a polynomial of degree $n$ having all its zeros on $|z| = k$, $k \leq 1$ then for every positive integer $s$

$$\{M(p,R)\}^s \leq \Big(\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\Big)\{M(p,1)\}^s, \quad R \geq 1\,.$$

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \leq 1$, we have been able to prove the following results.

Theorem 1. *If $p(z) = c_n z^n + \sum_{v=\mu}^{n} c_{n-v} z^{n-v}$, $1 \leq \mu < n$ is a polynomial of degree $n$ having all its zeros on $|z| = k, k \leq 1$, then for every positive integer $s$*

$$\{M(p,R)\}^s \leq \Big(\frac{k^{n-\mu}(k^{1-\mu}+k)+(R^{ns}-1)}{k^{n-2\mu+1}+k^{n-\mu+1}}\Big)\{M(p,1)\}^s, \quad R \geq 1. \tag{1.3}$$

If we take $k = 1$ in Theorem 1, we get the following result.

Corollary 1. *If $p(z) = \sum_{v=0}^{n} c_v z^v$ is a polynomial of degree $n$ having all its zeros on $|z| = k, k \leq 1$, then for every positive integer $s$*

$$\{M(p,R)\}^s \leq \Big(\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\Big)\{M(p,1)\}^s, \quad R \geq 1.$$

The following corollary immediately follows from inequality (1.6) by taking $s = 1$.

Corollary 2. *If $p(z) = \sum_{v=0}^{n} c_v z^v$ is a polynomial of degree $n$ having all its zeros on $|z| = k, k \leq 1$, then*

$$M(p,R) \leq \Big(\frac{k^{n-1}(1+k)+(R^{n}-1)}{k^{n-1}+k^n}\Big)M(p,1), \quad R \geq 1.$$

If we take $\mu = 1$ in inequality (1.3), we get the following corollary

Corollary 3. *If $p(z) = \sum_{v=0}^{n} c_v z^v$ is a polynomial of degree $n$ having all its zeros on $|z| = 1$, then*

$$M(p,R) \leq \Big(\frac{R^n+1}{2}\Big)M(p,1), \quad R \geq 1.$$

If we involve the coefficients of $p(z)$ also, then we are able to obtain a bound which is better than the bound obtained in Theorem 1. More precisely, we prove

Theorem 2. *If $p(z) = c_n z^n + \sum_{v=\mu}^{n} c_{n-v} z^{n-v}, 1 \leq \mu < n$ is a polynomial of degree $n$ having all its zeros on $|z| = k, k \leq 1$, then for every positive integer $s$*

$$\begin{aligned}&\{M(p,R)\}^s \leq \frac{1}{k^{n-\mu+1}}\\ &\times\left[\frac{n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^{ns}-1)\}+\mu|c_{n-\mu}|\{k^n(1+k^{1-\mu})+k^{\mu-1}(R^{ns}-1)\}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right]\\ &\times\{M(p,1)\}^s, \quad R \geq 1.\end{aligned}$$

To prove that the bound obtained in the above theorem is, in general, better than the bound obtained in Theorem 1, we show that

$$\begin{aligned}
&\frac{1}{k^{n-\mu+1}} \\
&\times\left[\frac{n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^{ns}-1)\}+\mu|c_{n-\mu}|\{k^n(1+k^{1-\mu})+k^{\mu-1}(R^{ns}-1)\}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right] \\
&\qquad\le\left(\frac{k^{n-\mu}(k^{1-\mu}+k)+(R^{ns}-1)}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&n|c_n|\{k^n(1+k^{\mu+1})(k^{-\mu}+1)+k^{2\mu}(k^{-\mu}+1)(R^{ns}-1)\} \\
&\quad+\mu|c_{n-\mu}|\{k^n(1+k^{1-\mu})(k^{-\mu}+1)+k^{\mu-1}(k^{-\mu}+1)(R^{ns}-1)\} \\
&\le n|c_n|k^{\mu-1}(k^{\mu+1}+1)\{k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1\} \\
&\quad+\mu|c_{n-\mu}|(1+k^{\mu-1})\{k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1\}
\end{aligned}$$

and therefore

$$\begin{aligned}
&n|c_n|\{-k^\mu+k^\mu R^{ns}\}+\mu|c_{n-\mu}|\{k^n-k^{-1}+k^{-1}R^{ns}\} \\
&\le n|c_n|\{-k^{\mu-1}+k^{\mu-1}R^{ns}\}+\mu|c_{n-\mu}|\{k^n-1+R^{ns}\}
\end{aligned}$$

or

$$\begin{aligned}
n|c_n|\{k^\mu(R^{ns}-1)\}&+\mu|c_{n-\mu}|\{k^{-1}(R^{ns}-1)\} \\
&\le n|c_n|\{k^{\mu-1}(R^{ns}-1)\}+\mu|c_{n-\mu}|\{(R^{ns}-1)\} \\
\mu|c_{n-\mu}|(k^{-1}-1)&\le n|c_n|(k^{-1}-1)k^\mu \\
\frac{\mu}{n}\frac{|c_{n-\mu}|}{|c_n|}&\le k^\mu,
\end{aligned}$$

which is always true (see Lemma 4).

Example 1. Let $p(z)=z^4-\frac{1}{50}z^2+(\frac{1}{100})^2$ and $k=\frac{1}{10}$, $R=1.5$, $\mu=1$ and $s=2$.

Then by Theorem 1, we have $\{M(p,R)\}^s\le 22390.909\{M(p,1)\}^s$, while by Theorem 2, we get $\{M(p,R)\}^s\le 2439.505\{M(p,1)\}^s$.

If we take $\mu=1$ in Theorem 2, we get the following corollary.

Corollary 4. *If $p(z)=\sum_{v=0}^n c_v z^v$ is a polynomial of degree $n$ having all its zeros on $|z|=k$, $k\le 1$, then for every positive integer $s$*

$$\begin{aligned}
\{M(p,R)\}^s\le\frac{1}{k^n}&\left[\frac{n|c_n|\{k^n(1+k^2)+k^2(R^{ns}-1)\}+|c_{n-1}|\{2k^n+(R^{ns}-1)\}}{2|c_{n-1}|+c_n|(1+k^2)}\right] \\
&\times\{M(p,1)\}^s,\quad R\ge 1.
\end{aligned}$$

In the above inequality, if we take $s = 1$, we get the following result.

Corollary 5. *If $p(z) = \sum_{v=0}^{n} c_v z^v$ is a polynomial of degree $n$ having all its zeros on $|z| = k$, $k \le 1$, then*

$$M(p,R) \le \frac{1}{k^n}\left[\frac{n|c_n|\{k^n(1+k^2)+k^2(R^n-1)\}+|c_{n-1}|\{2k^n+(R^n-1)\}}{2|c_{n-1}|+c_n|(1+k^2)}\right] \times M(p,1), \quad R \ge 1\,.$$

## 2. Lemmas

For the proof of these theorems, we need the following lemmas.

Lemma 1. [7] *If $p(z) = c_n z^n + \sum_{v=\mu}^{n} c_{n-v} z^{n-v}, 1 \le \mu < n$ is a polynomial of degree $n$ having all its zeros on $|z| = k, k \le 1$, then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-2\mu+1}+k^{n-\mu+1}} \max_{|z|=1} |p(z)|.$$

Lemma 2. [2] *If $p(z) = c_n z^n + \sum_{v=\mu}^{n} c_{n-v} z^{n-v}, 1 \le \mu < n$ is a polynomial of degree $n$ having all its zeros on $|z| = k, k \le 1$, then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-\mu+1}}\left[\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|(k^{2\mu}+k^{\mu-1})+\mu|c_{n-\mu}|(k^{\mu-1}+1)}\right] \max_{|z|=1} |p(z)|.$$

Lmma 3. [5, Remark 1] *If $p(z) = c_0 + \sum_{v=\mu}^{n} c_v z^v$, $1 \le \mu \le n$ is a polynomial of degree $n$ having no zeros in the disk $|z| < k$, $k \ge 1$, then for $|z| = 1$,*

$$\frac{\mu}{n}\left|\frac{c_\mu}{c_0}\right| k^\mu \le 1.$$

Lemma 4. *If $p(z) = c_n z^n + \sum_{v=\mu}^{n} c_{n-v} z^{n-v}$, $1 \le \mu < n$ is a polynomial of degree $n$ having all its zeros on $|z| = k, k \le 1$, then*

$$\frac{\mu}{n}\left|\frac{c_{n-\mu}}{c_n}\right| \le k^\mu.$$

*Proof.* If $p(z)$ has all its zeros on $|z| = k, k \le 1$, then $q(z) = z^n p(1/z)$ has all its zeros on $|z| \ge 1/k$, $1/k \le 1$. Now apply Lemma 3 to the polynomial $q(z)$, and Lemma 4 follows.

## 3. Proof of the theorems

*Proof of Theorem 1.* Let $M(p,1) = \max_{|z|=1} |p(z)|$. Since $p(z)$ is a polynomial of degree $n$ having all its zeros $|z| = k$, $k \le 1$, therefore, by Lemma 1, we have

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-2\mu+1}+k^{n-\mu+1}} M(p,1) \quad \text{for} \quad |z| = 1. \tag{3.1}$$

Now applying inequality (1.1) to the polynomial $p'(z)$ which is of degree $n-1$ and noting (3.1), it follows that for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq \frac{nr^{n-1}}{k^{n-2\mu+1}+k^{n-\mu+1}} M(p,1). \tag{3.2}$$

Also for each $\theta$, $0 \leq \theta < 2\pi$ and $R \geq 1$, we obtain

$$\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s = \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s\,dt = \int_1^R s\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta}\,dt.$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s\int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})|\,dt$$

which on combining with inequality (3.2) and (1.1), gives

$$\begin{aligned}|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\leq \frac{ns}{k^{n-2\mu+1}+k^{n-\mu+1}}\{M(p,1)\}^s \int_1^R t^{ns-1}\,dt \\ &= \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s\end{aligned}$$

and therefore,

$$\begin{aligned}|p(Re^{i\theta})|^s &\leq |p(e^{i\theta})|^s + \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s \\ &\leq \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s. \end{aligned} \tag{3.3}$$

Hence from (3.3) we conclude that

$$\{M(p,R)\}^s \leq \left(\frac{k^{n-\mu}(k^{1-\mu}+k)+(R^{ns}-1)}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s.$$

This completes the proof of Theorem 1. ∎

*Proof of Theorem 2.* The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using Lemma 2 instead of Lemma 1. But for the sake of completeness we give a brief outline of the proof. Since $p(z)$ is a polynomial of degree $n$ having all its zeros on $|z| = k$, $k \leq 1$, therefore, by Lemma 2, we have

$$|p'(z) \leq \frac{n}{k^{n-\mu+1}}\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)M(p,1) \text{ for } |z|=1.$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq \frac{nr^{n-1}}{k^{n-\mu+1}}\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)M(p,1). \tag{3.4}$$

Also for each $\theta$, $0 \leq \theta < 2\pi$ and $R \geq 1$ we obtain

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s\int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})|\,dt$$

which on combining with inequalities (1.1) and (3.4), gives

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq \left(\frac{R^{ns}-1}{k^{n-\mu+1}}\right)\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)\{M(p,1)\}^s$$

and therefore

$$|p(Re^{i\theta})^s \leq \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^{n-\mu+1}}\right) \times \left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)\{M(p,1)\}^s. \tag{3.5}$$

Hence, from (3.5), we conclude that

$$\{M(p,R)\}^s \leq \frac{1}{k^{n-\mu+1}} \times\left(\frac{n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^{ns}-1)\}+\mu|c_{n-\mu}|\{k^n(1+k^{1-\mu})+k^{\mu-1}(R^{ns}-1)\}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right) \times \{M,(p,1)\}^s$$

This completes the proof of Theorem 2. ■

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Division of Agri. Engineering, Sher-e-Kashmir University of Agricultural Sciences and Technology of Kashmir, Srinagar 191121 (India)

*E-mail*: mspukhta_67@yahoo.co.in