NONLINEAR DIFFERENTIAL POLYNOMIALS
SHARING A SMALL FUNCTION

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Abstract. In the paper, we investigate the uniqueness problems on entire and meromorphic functions concerning nonlinear differential polynomials that share a small function and obtain some results which improve and generalize some previous results due to Zhang-Chen-Lin, Banerjee-Bhattacharjee and Xu-Han-Zhang.

1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of Nevanlinna theory as explained in [5, 17, 20]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ $(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions. A meromorphic function $a(z)$ is said to be a small function of f, provided $T(r, a) = S(r, f)$. Let k be a positive integer or infinity. We denote by $E_{k)}(a; f)$ the set of all zeros of $f - a$ with multiplicities not exceeding k, where each zero is counted according to its multiplicity. If for some a, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$ we say that f and g share a CM (counting multiplicities). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o\{T(r)\}$ $(r \to \infty, r \notin E)$. Throughout this paper, we use the following definition. For any $a \in \mathbb{C} \cup \{\infty\}$,

$$\Theta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

In 1999, Lahiri [6] asked the following question.

What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?

During the last couple of years a substantial amount of investigations have been carried out by a number of authors on the uniqueness of meromorphic functions concerning nonlinear diffential polynomials and naturally several elegant results have been obtained in this aspect (see [1, 3, 4, 11–13]). In 2004, Lin and Yi proved the following results.

THEOREM A. [13] *Let f and g be two transcendental entire functions, and let $n \geq 7$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

THEOREM B. [13] *Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty, f) > \frac{2}{n+1}$, and let $n \geq 11$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In 2005, Lahiri and Sahoo [11] proved the following theorems first of which improve Theorem A.

THEOREM C. *Let f and g be two transcendental entire functions, and let $n \geq 7$ be an integer. If $E_{3)}(1; f^n(f-1)f') = E_{3)}(1; g^n(g-1)g')$, then $f \equiv g$.*

THEOREM D. *Let f and g be two transcendental meromorphic functions such that $\Theta(\infty, f) > 0$, $\Theta(\infty, g) > 0$, $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, and let $n \geq 11$ be an integer. If $E_{3)}(1; f^n(f-1)f') = E_{3)}(1; g^n(g-1)g')$, then $f \equiv g$.*

The following example was given in [11] to show that the condition $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$ is sharp in Theorem D.

EXAMPLE 1. Let
$$f = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad g = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})} \quad \text{and} \quad h = \frac{\alpha^2(e^z-1)}{e^z-\alpha},$$
where $\alpha = \exp\left(\frac{2\pi i}{n+2}\right)$ and n is a positive integer.

Then $T(r, f) = (n+1)T(r, h) + O(1)$ and $T(r, g) = (n+1)T(r, h) + O(1)$. Also we see that $h \neq \alpha, \alpha^2$ and a root of $h = 1$ is not a pole of f and g. Hence $\Theta(\infty; f) = \Theta(\infty; g) = 2/(n+1)$. Also $f^{n+1}\left(\frac{f}{n+1} - \frac{1}{n+1}\right) \equiv g^{n+1}\left(\frac{g}{n+1} - \frac{1}{n+1}\right)$ and $f^n(f-1)f' \equiv g^n(g-1)g'$ but $f \not\equiv g$.

In 2008, Zhang-Chen-Lin proved the following theorem for meromorphic functions concerning some general differential polynomials.

THEOREM E. [21] *Let f and g be two nonconstant meromorphic functions, let n and m be two positive integers with $n > \max\{m+10, 3m+3\}$, and $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, where $a_0(\neq 0)$, a_1, ..., a_{m-1}, $a_m(\neq 0)$ are complex constants. If $f^n P(f)f'$ and $g^n P(g)g'$ share 1 CM, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where*

$$R(x, y) = x^{n+1}\left(\frac{a_m}{n+m+1}x^m + \frac{a_{m-1}}{n+m}x^{m-1} + \cdots + \frac{a_0}{n+1}\right)$$
$$- y^{n+1}\left(\frac{a_m}{n+m+1}y^m + \frac{a_{m-1}}{n+m}y^{m-1} + \cdots + \frac{a_0}{n+1}\right).$$

In this direction, Banerjee-Bhattacharjee proved the following theorems.

THEOREM F. [2] *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$. Suppose $E_{k)}(1; f^n(f-1)f') = E_{k)}(1; g^n(g-1)g')$. If $k \geq 3$, $\Theta(\infty, f) > 0$, $\Theta(\infty, g) > 0$ and $n \geq 11$ or if $k = 2$ and $n \geq 14$ or if $k = 1$ and $n \geq 21$, then $f \equiv g$.*

THEOREM G. [2] *Let f and g be two nonconstant entire functions, and let n, k be two positive integers. Suppose $E_{k)}(1; f^n(f-1)f') = E_{k)}(1; g^n(g-1)g')$. If $k \geq 3$ and $n \geq 7$ or if $k = 2$ and $n \geq 9$ or if $k = 1$ and $n \geq 13$, then $f \equiv g$.*

In 2009, Xu-Han-Zhang proved the following results.

THEOREM H. [15] *Let f and g be two nonconstant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $m(\geq 2)$ be three integers and $(n+1, m) = 1$. Suppose $E_{k)}(1; f^n(f^m-1)f') = E_{k)}(1; g^n(g^m-1)g')$. If $k \geq 3$ and $n > m + 10$ or if $k = 2$ and $n > \frac{3m}{2} + 12$ or if $k = 1$ and $n > 3m + 18$, then $f \equiv g$.*

THEOREM I. [15] *Let f and g be two nonconstant entire functions, and let n, k, m be three positive integers. Suppose $E_{k)}(1; f^n(f^m-1)f') = E_{k)}(1; g^n(g^m-1)g')$. If $k \geq 3$ and $n > m + 5$ or if $k = 2$ and $n > \frac{3m+13}{2}$ or if $k = 1$ and $n > 3m + 11$, then $f \equiv g$.*

This paper is motivated by the following question.

What can be said if the sharing value 1 is replaced by a small function in the above results?

In the paper we shall investigate the possible solutions in the above question. In the paper we will prove two theorems first of which not only improve Theorem E, but also improve and supplement Theorems F and H. Our second result will improve and supplement Theorems G and I.

The following theorems are the main results of the paper.

THEOREM 1. *Let f and g be two nonconstant meromorphic functions and $n(\geq 1)$, $k(\geq 1)$, $m(\geq 1)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$. Let $P(z)$ be defined as in Theorem E. Suppose $E_{k)}(\alpha; f^n P(f)f') = E_{k)}(\alpha; g^n P(g)g')$ where $\alpha(\not\equiv 0, \infty)$ be a small function of f and g and one of the following holds:*

(i) $k \geq 3$, $\Theta(\infty, f) > 0$, $\Theta(\infty, g) > 0$ and $n > \max\{3m+1, m+9\}$;

(ii) $k = 2$ and $n > \max\{3m+1, \frac{3m}{2} + 12\}$;

(iii) $k = 1$ and $n > 3m + 17$.

Then the conclusion of Theorem E holds.

REMARK 1. In Theorem 1, if we take $n > \max\{3m+1, m+10\}$ for $k = 3$, then the conditions $\Theta(\infty, f) > 0$, $\Theta(\infty, g) > 0$ can be removed.

REMARK 2. Theorem 1 is an improvement of Theorem E.

Taking $a_m = 1$, $a_0 = -1$ and $a_{m-i} = 0$ for $i = 1, 2, \ldots, m - 1$ in $P(z)$ in Theorem 1, we obtain the following corollary.

COROLLARY 1. *Let f and g be two nonconstant meromorphic functions and $n(\geq 1)$, $k(\geq 1)$, and $m(\geq 2)$ be three integers. Suppose $E_{k)}(\alpha; f^n(f^m - 1)f') = E_{k)}(\alpha; g^n(g^m - 1)g')$ where $\alpha(\not\equiv 0, \infty)$ be a small function of f and g and one of the following holds:*

(i) $k \geq 3$ and $n > m + 10$;

(ii) $k = 2$ and $n > \frac{3m}{2} + 12$;

(iii) $k = 1$ and $n > 3m + 17$.

Then either $f \equiv g$ or $f \equiv -g$. The possibility $f \equiv -g$ arise only when n is odd and m is even.

REMARK 3. Since Theorems F and H can be obtained as special cases of Theorem 1, Theorem 1 improves and supplements them.

THEOREM 2. *Let f and g be two nonconstant entire functions, and let $n(\geq 1)$, $m(\geq 1)$, $k(\geq 1)$ be three integers. Suppose $E_{k)}(\alpha; f^n P(f)f') = E_{k)}(\alpha; g^n P(g)g')$ where $P(z)$ and α be defined as in Theorem E and Theorem 1 respectively and one of the following holds:*

(i) $k \geq 3$ and $n > \max\{3m + 1, m + 5\}$;

(ii) $k = 2$ and $n > \max\{3m + 1, \frac{3m}{2} + 6\}$;

(iii) $k = 1$ and $n > 3m + 9$.

Then the conclusion of Theorem E holds.

COROLLARY 2. *Let f and g be two nonconstant entire functions, and let $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 1)$ be three integers. Suppose $E_{k)}(\alpha; f^n(f^m - 1)f') = E_{k)}(\alpha; g^n(g^m - 1)g')$ where $\alpha(\not\equiv 0, \infty)$ be a small function of f and g and one of the following holds:*

(i) $k \geq 3$ and $n > m + 5$;

(ii) $k = 2$ and $n > \frac{3m}{2} + 6$;

(iii) $k = 1$ and $n > 3m + 9$.

Then the conclusion of Corollary 1 holds.

REMARK 4. Since Theorems G and I are special cases of Theorem 2, Theorem 2 improves and supplements them.

We now explain some definitions and notations which are used in the paper.

DEFINITION 1. [11] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid = 1)$ the counting functions of simple a-points of f. For a positive integer p we denote by $N(r, a; f \mid \leq p)$ $(N(r, a; f \mid \geq p))$ the counting function of those a-points of f whose multiplicities are not greater (less) than p, where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid \leq p)$ and $\overline{N}(r, a; f \mid \geq p)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r, a; f \mid < p)$ and $N(r, a; f \mid > p)$ are defined analogously.

DEFINITION 2. For an integer $k(\geq 2)$, $\overline{N}(r, a; f \mid = k)$ denotes the reduced counting function of those a-points of f whose multiplicities are exactly k.

DEFINITION 3. [7] Let p be a positive integer or infinity. We denote by $N_p(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \cdots + \overline{N}(r, a; f \mid \geq p).$$

DEFINITION 4. [2] Let m be a positive integer and $E_{m)}(a; f) = E_{m)}(a; g)$ for some $a \in \mathbb{C}$. Let z_0 be a zero of $f - a$ with multiplicity p and a zero of $g - a$ with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g for which $p > q \geq m + 1$, by $\overline{N}_E^{(m+1)}(r, a; f)$ the reduced counting function of those a-points of f and g for which $p = q \geq m + 1$, and by $\overline{N}_{f>m+1}(r, 1; g)$ the reduced counting function of f and g for which $p \geq m + 2$ and $q = m + 1$. Also by $\overline{N}_{f \geq m+1}(r, a; f \mid g \neq a)$ we denote the reduced counting functions of those a-points of f and g for which $p \geq m + 1$ and $q = 0$. Analogously we can define $\overline{N}_L(r, a; g)$, $\overline{N}_E^{(m+1)}(r, a; g)$ and $\overline{N}_{g \geq m+1}(r, a; g \mid f \neq a)$.

DEFINITION 5. [9] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ $(N(r, a; f \mid g \neq b))$ the counting function of those a-points of f, counted according to multiplicity, which are the b-points (not the b-points) of g.

DEFINITION 6. [2] Let $a, b \in \mathbb{C} \cup \{\infty\}$ and p be a positive integer. Then we denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ $(\overline{N}(r, a; f \mid \geq p \mid g \neq b))$ the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

LEMMA 1. [14, 16] *Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

LEMMA 2. [18] *Let f be a nonconstant meromorphic function. Then*

$$N\left(r, 0; f^{(k)}\right) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

LEMMA 3. [19] *Let f and g be two nonconstant meromorphic functions. If*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \equiv \frac{g''}{g'} - \frac{2g'}{g-1}$$

and

$$\limsup_{r \to \infty, r \notin E} \frac{\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)}{T(r)} < 1$$

then $f \equiv g$ or $fg \equiv 1$.

LEMMA 4. [2] *Let f and g be two nonconstant meromorphic functions. If $E_{k)}(1;f) = E_{k)}(1;g)$ and $2 \leq k < \infty$, then*

$$\overline{N}(r,1;f \mid= 2) + 2\overline{N}(r,1;f \mid= 3) + \cdots + (k-1)\overline{N}(r,1;f \mid= k) + k\overline{N}_E^{(k+1}(r,1;f)$$
$$+ k\overline{N}_L(r,1;f) + (k+1)\overline{N}_L(r,1;g) + k\overline{N}_{g \geq k+1}(r,1;g \mid f \neq 1)$$
$$\leq N(r,1;g) - \overline{N}(r,1;g).$$

LEMMA 5. [2] *Suppose that f, g be two nonconstant meromorphic functions and $E_{2)}(1;f) = E_{2)}(1;g)$. Then*

$$\overline{N}_{f \geq 3}(r,1;f \mid g \neq 1) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f),$$

where $N_0(r,0;f')$ denotes the counting function of those zeros of f' which are not the zeros of $f(f-1)$, each point is counted according to its multiplicity.

LEMMA 6. [2] *Let f and g be two nonconstant meromorphic functions. If $E_{1)}(1;f) = E_{1)}(1;g)$, then*

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) + \overline{N}_{g \geq 2}(r,1;g \mid f \neq 1) - \overline{N}_{f>2}(r,1;g)$$
$$\leq N(r,1;g) - \overline{N}(r,1;g).$$

LEMMA 7. [2] *Let f and g be two nonconstant meromorphic functions. If $E_{1)}(1;f) = E_{1)}(1;g)$, then*

$$\overline{N}_{f \geq 2}(r,1;f \mid g \neq 1) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f).$$

LEMMA 8. [2] *Suppose that f, g be two nonconstant meromorphic functions and $E_{1)}(1;f) = E_{1)}(1;g)$. Then*

$$\overline{N}_{f>2}(r,1;g) + \overline{N}_{f \geq 2}(r,1;f \mid g \neq 1) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f).$$

LEMMA 9. *Let f and g be two nonconstant meromorphic functions and $\alpha(\not\equiv 0, \infty)$ be a small function of f and g. Let n and m be two positive integers such that $n > 3m + 1$. Then*

$$f^n P(f) f' g^n P(g) g' \not\equiv \alpha^2,$$

where $P(z)$ is defined as in Theorem E.

Proof. We suppose that

$$f^n P(f) f' g^n P(g) g' \equiv \alpha^2. \tag{2.1}$$

We write

$$P(z) = a_m (z - b_1)^{l_1} (z - b_2)^{l_2} \cdots (z - b_i)^{l_i} \cdots (z - b_s)^{l_s},$$

where $\sum_{i=1}^{s} l_i = m$, $1 \leq s \leq m$; $b_i \neq b_j$, $i \neq j$, $1 \leq i, j \leq s$; b_i is nonzero constant and l_i is positive integer, $i = 1, 2, \ldots, s$. Let z_0 $(\alpha(z_0) \neq 0, \infty)$ be a zero of f with multiplicity p. Then z_0 is a pole of g with multiplicity q, say. From (2.1) we get $np + p - 1 = nq + mq + q + 1$ and so

$$mq + 2 = (n + 1)(p - q). \tag{2.2}$$

From (2.2) we get $q \geq \frac{n-1}{m}$ and from (2.2) we obtain

$$p \geq \frac{1}{n+1} \left[\frac{(n + m + 1)(n - 1)}{m} + 2 \right] = \frac{n + m - 1}{m}.$$

Let z_1 $(\alpha(z_1) \neq 0, \infty)$ be a zero of $P(f)$ of order p and be a zero of $f - b_i$ of order q_i for $i = 1, 2, \ldots, s$. Then $p = l_i q_i$ for $i = 1, 2, \ldots, s$. Then z_1 is a pole of g with multiplicity q, say. So from (2.1) we get

$$q_i l_i + q_i - 1 = (n + m + 1)q + 1 \geq n + m + 2$$

i.e., $q_i \geq \dfrac{n + m + 3}{l_i + 1}$ for $i = 1, 2, \ldots, s$. Since a pole of f (which is not a pole of α) is either a zero of $g^n P(g)$ or a zero of g', we have

$$\overline{N}(r, \infty; f) \leq \overline{N}(r, 0; g) + \sum_{i=1}^{s} \overline{N}(r, b_i; g) + \overline{N}_0(r, 0; g') + S(r, f) + S(r, g)$$

$$\leq \left(\frac{m}{n + m - 1} + \frac{m + s}{n + m + 3} \right) T(r, g) + \overline{N}_0(r, 0; g') + S(r, f) + S(r, g),$$

where $\overline{N}_0(r, 0; g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $gP(g)$.

Then by the second fundamental theorem of Nevanlinna we get

$$sT(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \sum_{i=1}^{s} \overline{N}(r, b_i; f) - \overline{N}_0(r, 0; f') + S(r, f)$$

$$\leq \left(\frac{m}{n + m - 1} + \frac{m + s}{n + m + 3} \right) \{ T(r, f) + T(r, g) \}$$

$$+ \overline{N}_0(r, 0; g') - \overline{N}_0(r, 0; f') + S(r, f) + S(r, g). \tag{2.3}$$

Similarly

$$sT(r,g) \le \left(\frac{m}{n+m-1} + \frac{m+s}{n+m+3} \right) \{T(r,f)+T(r,g)\}$$
$$+ \overline{N}_0(r,0;f') - \overline{N}_0(r,0;g') + S(r,f) + S(r,g).$$

Adding (2.3) and (2.4) we obtain

$$\left(s - \frac{2m}{n+m-1} - \frac{2(m+s)}{n+m+3} \right) \{T(r,f)+T(r,g)\} \le S(r,f) + S(r,g),$$

which is a contradiction as $n > 3m + 1$. This proves the lemma. ∎

NOTE 1. If $P(z) = a_0 z + a_1$, for any two nonzero constants a_0 and a_1, the lemma holds for $n \ge 5$.

NOTE 2. If $P(z)$ is a polynomial of degree ≥ 2 and all the zeros are simple, then the lemma is true for $n \ge 4$.

LEMMA 10. *Let f and g be two nonconstant meromorphic functions and*

$$F = f^{n+1} \left[\frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + \cdots + \frac{a_0}{n+1} \right]$$

and

$$G = g^{n+1} \left[\frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + \cdots + \frac{a_0}{n+1} \right],$$

where $a_0(\ne 0)$, a_1, ... , a_{m-1}, $a_m(\ne 0)$ are complex constants. Further let $F_0 = \frac{F'}{\alpha}$ and $G_0 = \frac{G'}{\alpha}$, where $\alpha \ (\not\equiv 0, \infty)$ is a small function of f and g. Then $S(r,F_0)$ and $S(r,G_0)$ are replaceable by $S(r,f)$ and $S(r,g)$ respectively.

Proof. By Lemma 1

$$T(r,F_0) \le T(r,F') + S(r,f) \le 2T(r,F) + S(r,f)$$
$$= 2(n+m+1)T(r,f) + S(r,f)$$

and similarly $T(r,G_0) \le 2(n+m+1)T(r,g) + S(r,g)$. This proves the lemma. ∎

LEMMA 11. *Let F, G, F_0 and G_0 be defined as in Lemma 10. We define $F = f^{n+1}F^*$ and $G = g^{n+1}G^*$ where*

$$F^* = \left[\frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + \cdots + \frac{a_0}{n+1} \right]$$

and

$$G^* = \left[\frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + \cdots + \frac{a_0}{n+1} \right].$$

Then

(i) $T(r,F) \le T(r,F_0) + N(r,0;f) + \sum\limits_{i=1}^{m} N(r,c_i;f) - \sum\limits_{j=1}^{m} N(r,d_j;f) - N(r,0;f') + S(r,f),$

(ii) $T(r, G) \leq T(r, G_0) + N(r, 0; g) + \sum\limits_{i=1}^{m} N(r, c_i; g) - \sum\limits_{j=1}^{m} N(r, d_j; g) - N(r, 0; g') + S(r, g)$,

where $c_1, c_2, \ldots, c_m$ are the roots of the equation

$$\frac{a_m}{n+m+1} z^m + \frac{a_{m-1}}{n+m} z^{m-1} + \cdots + \frac{a_0}{n+1} = 0,$$

and $d_1, d_2, \ldots, d_m$ are the roots of the equation $P(z) = 0$.

Proof. We prove (i) only, as the proof of (ii) is similar. Using Nevanlinna's first fundamental theorem and Lemma 1 we get

$$T(r, F) = T(r, \frac{1}{F}) + O(1) = N(r, 0; F) + m(r, \frac{1}{F}) + O(1)$$

$$\leq N(r, 0; F) + m(r, \frac{F_0}{F}) + m(r, 0; F_0) + O(1)$$

$$= N(r, 0; F) + T(r, F_0) - N(r, 0; F_0) + S(r, F)$$

$$= T(r, F_0) + N(r, 0; f) + N(r, 0; F^*) - N(r, 0; P(f)) - N(r, 0; f') + S(r, f)$$

$$= T(r, F_0) + N(r, 0; f) + \sum\limits_{i=1}^{m} N(r, c_i; f) - \sum\limits_{j=1}^{m} N(r, d_j; f)$$

$$- N(r, 0; f') + S(r, f).$$

This proves the lemma. ∎

Thed following lemma can be proved in the line of [10, Lemma 2.10].

LEMMA 12. *Let F and G be defined as in Lemma 10, where m and $n(> m+2)$ are positive integers. Then $F' \equiv G'$ implies $F \equiv G$.*

3. Proofs of the theorems

Proof of Theorem 1. Let F, G, F_0 and G_0 be defined as in Lemma 10. Then $E_{k)}(1; F_0) = E_{k)}(1; G_0)$. Let

$$H = \left(\frac{F_0''}{F_0'} - \frac{2F_0'}{F_0 - 1} \right) - \left(\frac{G_0''}{G_0'} - \frac{2G_0'}{G_0 - 1} \right). \tag{3.1}$$

We assume that $H \not\equiv 0$. Suppose that z_0 be a simple 1-point of F_0. Then z_0 is a simple 1-point of G_0. So from (3.1) we see that z_0 is a zero of H. Thus

$$N(r, 1; F_0 \mid = 1) \leq N(r, 0; H) \leq T(r, H) + O(1)$$
$$\leq N(r, \infty; H) + S(r, F) + S(r, G). \tag{3.2}$$

From (3.1) it is clear that

$$N(r, \infty; H) \leq \overline{N}(r, \infty; F_0) + \overline{N}(r, \infty; G_0) + \overline{N}(r, 0; F_0 \mid \geq 2) + \overline{N}(r, 0; G_0 \mid \geq 2)$$

$$+ \overline{N}_L(r, 1; F_0) + \overline{N}_L(r, 1; G_0) + \overline{N}_{F_0 \geq k+1}(r, 1; F_0 \mid G_0 \neq 1)$$

$$+ \overline{N}_{G_0 \geq k+1}(r, 1; G_0 \mid F_0 \neq 1) + \overline{N}_0\left(r, 0; F_0'\right) + \overline{N}_0\left(r, 0; G_0'\right), \tag{3.3}$$

where $\overline{N}_0(r,0;F_0')$ $\left(\overline{N}_0(r,0;G_0')\right)$ denotes the reduced counting function of those zeros of F_0' (G_0') which are not the zeros of $F_0(F_0-1)$ $(G_0(G_0-1))$. Now we discuss the following two cases.

CASE 1. Let $k \geq 2$. By Lemma 4, (3.2) and (3.2) we obtain

$$\overline{N}(r,1;F_0) + \overline{N}(r,1;G_0)$$
$$\leq N(r,1;F_0 \mid= 1) + \overline{N}(r,1;F_0 \mid= 2) + \cdots + \overline{N}(r,1;F_0 \mid= k)$$
$$+ \overline{N}_L(r,1;F_0) + \overline{N}_L(r,1;G_0) + \overline{N}_{F_0 \geq k+1}(r,1;F_0 \mid G_0 \neq 1)$$
$$+ \overline{N}_E^{(k+1)}(r,1;G_0) + \overline{N}(r,1;G_0)$$
$$\leq \overline{N}(r,\infty;F_0) + \overline{N}(r,\infty;G_0) + \overline{N}(r,0;F_0 \mid\geq 2) + \overline{N}(r,0;G_0 \mid\geq 2)$$
$$+ 2\overline{N}_{F_0 \geq k+1}(r,1;F_0 \mid G_0 \neq 1) - (k-1)\overline{N}_{G_0 \geq k+1}(r,1;G_0 \mid F_0 \neq 1)$$
$$+ T(r,G_0) + \overline{N}_0\left(r,0;F_0'\right) + \overline{N}_0\left(r,0;G_0'\right) + S(r,F_0) + S(r,G_0). \tag{3.4}$$

From (3.4) and Nevanlinna's second fundamental theorem we obtain

$$T(r,F_0) + T(r,G_0)$$
$$\leq 2\overline{N}(r,\infty;F_0) + 2\overline{N}(r,\infty;G_0) + N_2(r,0;F_0) + N_2(r,0;G_0)$$
$$+ 2\overline{N}_{F_0 \geq k+1}(r,1;F_0 \mid G_0 \neq 1) - (k-1)\overline{N}_{G_0 \geq k+1}(r,1;G_0 \mid F_0 \neq 1)$$
$$+ T(r,G_0) + S(r,F_0) + S(r,G_0). \tag{3.5}$$

This gives

$$T(r,F_0) \leq 2\overline{N}(r,\infty;F_0) + 2\overline{N}(r,\infty;G_0) + N_2(r,0;F_0) + N_2(r,0;G_0)$$
$$+ 2\overline{N}_{F_0 \geq k+1}(r,1;F_0 \mid G_0 \neq 1) - (k-1)\overline{N}_{G_0 \geq k+1}(r,1;G_0 \mid F_0 \neq 1)$$
$$+ S(r,F_0) + S(r,G_0). \tag{3.6}$$

Similarly

$$T(r,G_0) \leq 2\overline{N}(r,\infty;F_0) + 2\overline{N}(r,\infty;G_0) + N_2(r,0;F_0) + N_2(r,0;G_0)$$
$$+ 2\overline{N}_{G_0 \geq k+1}(r,1;G_0 \mid F_0 \neq 1) - (k-1)\overline{N}_{F_0 \geq k+1}(r,1;F_0 \mid G_0 \neq 1)$$
$$+ S(r,F_0) + S(r,G_0). \tag{3.7}$$

Suppose $k \geq 3$. Adding (3.6) and (3.7) we get

$$T(r,F_0) + T(r,G_0) \leq 4\overline{N}(r,\infty;F_0) + 4\overline{N}(r,\infty;G_0) + 2N_2(r,0;F_0)$$
$$+ 2N_2(r,0;G_0) + S(r,F_0) + S(r,G_0).$$

Using Lemmas 10 and 11 we obtain

$$T(r,F) + T(r,G)$$
$$\leq 4\overline{N}(r,\infty;F_0) + 4\overline{N}(r,\infty;G_0) + 2N_2(r,0;F_0) + 2N_2(r,0;G_0) + N(r,0;f)$$

$$+ N(r,0;g) + \sum_{i=1}^{m} N(r,c_i;f) + \sum_{i=1}^{m} N(r,c_i;g) - \sum_{j=1}^{m} N(r,d_j;f)$$

$$- \sum_{j=1}^{m} N(r,d_j;g) - N(r,0;f') - N(r,0;g') + S(r,f) + S(r,g)$$

$$\leq 4\overline{N}(r,\infty;f) + 4\overline{N}(r,\infty;g) + 4\overline{N}(r,0;f) + 4\overline{N}(r,0;g) + N(r,0;f)$$

$$+ N(r,0;g) + \sum_{i=1}^{m} N(r,c_i;f) + \sum_{i=1}^{m} N(r,c_i;g) + \sum_{j=1}^{m} N(r,d_j;f)$$

$$+ \sum_{j=1}^{m} N(r,d_j;g) + N(r,0;f') + N(r,0;g') + S(r,f) + S(r,g).$$

Applying Lemmas 1 and 2 we obtain

$$(n+m+1)\{T(r,f) + T(r,g)\} \leq [2m + 11 - 5\Theta(\infty,f) + \epsilon]T(r,f)$$
$$+ [2m + 11 - 5\Theta(\infty,g) + \epsilon]T(r,g) + S(r,f) + S(r,g),$$

where $\epsilon(>0)$ is arbitrary. This implies

$$[n - m - 10 + 5\Theta(\infty,f) - \epsilon]T(r,f) + [n - m - 10 + 5\Theta(\infty,g) - \epsilon]T(r,g)$$
$$\leq S(r,f) + S(r,g).$$

Since $n > m + 9$, choosing $0 < \epsilon < \min\{\Theta(\infty,f), \Theta(\infty,g)\}$, we arrive at a contradiction.

Now we assume that $k = 2$. Adding (3.6), (3.7) and using Lemma 5 we obtain

$$T(r,F_0) + T(r,G_0)$$
$$\leq 4\overline{N}(r,\infty;F_0) + 4\overline{N}(r,\infty;G_0) + 2N_2(r,0;F_0) + 2N_2(r,0;G_0)$$
$$+ \overline{N}_{F_0 \geq 3}(r,1;F_0 \mid G_0 \neq 1) + \overline{N}_{G_0 \geq 3}(r,1;G_0 \mid F_0 \neq 1)$$
$$+ S(r,F_0) + S(r,G_0)$$
$$\leq \frac{9}{2}\overline{N}(r,\infty;F_0) + \frac{9}{2}\overline{N}(r,\infty;G_0) + 2N_2(r,0;F_0) + 2N_2(r,0;G_0)$$
$$+ \frac{1}{2}\overline{N}(r,0;F_0) + \frac{1}{2}\overline{N}(r,0;G_0) + S(r,F_0) + S(r,G_0).$$

Using Lemmas 10 and 11 we obtain

$$T(r,F) + T(r,G)$$
$$\leq \frac{9}{2}\overline{N}(r,\infty;F_0) + \frac{9}{2}\overline{N}(r,\infty;G_0) + 2N_2(r,0;F_0) + 2N_2(r,0;G_0)$$
$$+ \frac{1}{2}\overline{N}(r,0;F_0) + \frac{1}{2}\overline{N}(r,0;G_0) + N(r,0;f) + N(r,0;g)$$
$$+ \sum_{i=1}^{m} N(r,c_i;f) + \sum_{i=1}^{m} N(r,c_i;g) - \sum_{j=1}^{m} N(r,d_j;f) - \sum_{j=1}^{m} N(r,d_j;g)$$
$$- N(r,0;f') - N(r,0;g') + S(r,f) + S(r,g)$$

$$\leq \frac{9}{2}\overline{N}(r,\infty;f) + \frac{9}{2}\overline{N}(r,\infty;g) + \frac{9}{2}\overline{N}(r,0;f) + \frac{9}{2}\overline{N}(r,0;g) + N(r,0;f)$$

$$+ N(r,0;g) + \sum_{i=1}^{m} N(r,c_i;f) + \sum_{i=1}^{m} N(r,c_i;g) + \frac{3}{2}\sum_{j=1}^{m} N(r,d_j;f)$$

$$+ \frac{3}{2}\sum_{j=1}^{m} N(r,d_j;g) + \frac{3}{2}N(r,0;f') + \frac{3}{2}N(r,0;g') + S(r,f) + S(r,g).$$

From Lemmas 1 and 2 we obtain

$$(n+m+1)\{T(r,f)+T(r,g)\}$$
$$\leq \left(\frac{5m}{2}+13\right)T(r,f) + \left(\frac{5m}{2}+13\right)T(r,g) + S(r,f) + S(r,g),$$

which is a contradiction since $n > \frac{3m}{2} + 12$.

CASE 2. Let $k = 1$. In view of Lemmas 6–8, (3.2) and (3.3) we obtain

$$\overline{N}(r,1;F_0) + \overline{N}(r,1;G_0)$$
$$\leq N(r,1;F_0 \mid = 1) + \overline{N}_L(r,1;F_0) + \overline{N}_L(r,1;G_0)$$

$$+ \overline{N}_{F_0 \geq 2}(r,1;F_0 \mid G_0 \neq 1) + \overline{N}_E^{(2}(r,1;F_0) + \overline{N}(r,1;G_0)$$
$$\leq \overline{N}(r,\infty;F_0) + \overline{N}(r,\infty;G_0) + \overline{N}(r,0;F_0 \mid \geq 2) + \overline{N}(r,0;G_0 \mid \geq 2)$$
$$+ 2\overline{N}_{F_0 \geq 2}(r,1;F_0 \mid G_0 \neq 1) + \overline{N}_{F_0 > 2}(r,1;G_0) + T(r,G_0)$$
$$+ \overline{N}_0(r,0;F_0') + \overline{N}_0(r,0;G_0') + S(r,F_0) + S(r,G_0)$$
$$\leq 3\overline{N}(r,\infty;F_0) + \overline{N}(r,\infty;G_0) + \overline{N}(r,0;F_0 \mid \geq 2) + \overline{N}(r,0;G_0 \mid \geq 2)$$
$$+ 2\overline{N}(r,0;F_0) + T(r,G_0) + \overline{N}_0(r,0;F_0') + \overline{N}_0(r,0;G_0')$$
$$+ S(r,F_0) + S(r,G_0). \tag{3.8}$$

In view of second fundamental theorem of Nevanlinna we obtain

$$T(r,F_0) + T(r,G_0) \leq 4\overline{N}(r,\infty;F_0) + 2\overline{N}(r,\infty;G_0) + N_2(r,0;F_0) + N_2(r,0;G_0)$$
$$+ 2\overline{N}(r,0;F_0) + T(r,G_0) + S(r,F_0) + S(r,G_0). \tag{3.9}$$

This gives

$$T(r,F_0) \leq 4\overline{N}(r,\infty;F_0) + 2\overline{N}(r,\infty;G_0) + N_2(r,0;F_0) + N_2(r,0;G_0)$$
$$+ 2\overline{N}(r,0;F_0) + S(r,F_0) + S(r,G_0). \tag{3.10}$$

By Lemmas 10 and 11 we have

$$T(r,F) \leq 4\overline{N}(r,\infty;F_0) + 2\overline{N}(r,\infty;G_0) + N_2(r,0;F_0) + N_2(r,0;G_0)$$
$$+ 2\overline{N}(r,0;F_0) + N(r,0;f) + \sum_{i=1}^{m} N(r,c_i;f) - \sum_{j=1}^{m} N(r,d_j;f)$$
$$- N(r,0;f') + S(r,f) + S(r,g)$$

$$\leq 4\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + 4\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;f)$$
$$+ \sum_{i=1}^{m} N(r,c_i;f) + 2\sum_{j=1}^{m} N(r,d_j;f) + \sum_{j=1}^{m} N(r,d_j;g)$$
$$+ 2\overline{N}(r,0;f') + N(r,0;g') + S(r,f) + S(r,g).$$

Hence by Lemmas 1 and 2 we obtain

$$(n+m+1)T(r,f) \leq [3m+13-6\Theta(\infty,f)+\epsilon]T(r,f)$$
$$+ [m+6-3\Theta(\infty,g)+\epsilon]T(r,g) + S(r,f) + S(r,g)$$
$$\leq [4m+19-6\Theta(\infty,f)-3\Theta(\infty,g)+2\epsilon]T(r)$$
$$+ S(r,f) + S(r,g), \qquad (3.11)$$

where $\epsilon(>0)$ is arbitrary. Similarly

$$(n+m+1)T(r,g) \leq [4m+19-3\Theta(\infty,f)-6\Theta(\infty,g)+2\epsilon]T(r)$$
$$+ S(r,f) + S(r,g). \qquad (3.12)$$

From (3.11) and (3.12) we get

$$[n-3m-18+3\Theta(\infty,f)+3\Theta(\infty,g)+3\min\{\Theta(\infty,f),\Theta(\infty,g)\}-2\epsilon]T(r) \leq S(r).$$

Since $n > 3m+17$ and $\Theta(\infty,f)+\Theta(\infty,g) > \frac{4}{n+1}$, we arrive at a contradiction.

We now assume that $H \equiv 0$. By Lemma 1 we get

$$(n+m)T(r,f) = T(r,f^n P(f)) + S(r,f)$$
$$\leq T(r,F') + T(r,f') + S(r,f)$$
$$\leq T(r,F_0) + 2T(r,f) + S(r,f)$$

and so $T(r,F_0) \geq (n+m-2)T(r,f) + S(r,f)$. Similarly

$$T(r,G_0) \geq (n+m-2)T(r,g) + S(r,g).$$

Also from Lemma 2 we have

$$\overline{N}(r,0;F_0) + \overline{N}(r,\infty;F_0) + \overline{N}(r,0;G_0) + \overline{N}(r,\infty;G_0)$$
$$\leq \overline{N}(r,0;f) + \sum_{j=1}^{m} N(r,d_j;f) + \overline{N}(r,0;f') + \overline{N}(r,\infty;f) + \overline{N}(r,0;g)$$
$$+ \sum_{j=1}^{m} N(r,d_j;g) + \overline{N}(r,0;g') + \overline{N}(r,\infty;g) + S(r,f) + S(r,g)$$
$$\leq \{m+4-2\Theta(\infty;f)+\epsilon\}T(r,f) + \{m+4-2\Theta(\infty;g)+\epsilon\}T(r,g)$$
$$+ S(r,f) + S(r,g)$$
$$\leq \frac{2m+8-2\Theta(\infty;f)-2\Theta(\infty;g)+2\epsilon}{n+m-2}T(r) + S(r),$$

where $\epsilon(>0)$ is sufficiently small.

In view of the hypothesis we get from above

$$\limsup_{r\to\infty, r\notin E} \frac{\overline{N}(r,0;F_0) + \overline{N}(r,\infty;F_0) + \overline{N}(r,0;G_0) + \overline{N}(r,\infty;G_0)}{T(r)} < 1.$$

Applying Lemma 3 we obtain either $F_0 G_0 \equiv 1$ or $F_0 \equiv G_0$. Since by Lemma 9, $F_0 G_0 \not\equiv 1$, we get by Lemma 12 that $F \equiv G$. This gives

$$f^{n+1} \left[\frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + \cdots + \frac{a_0}{n+1} \right]$$
$$= g^{n+1} \left[\frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + \cdots + \frac{a_0}{n+1} \right]. \quad (3.13)$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in (3.13) we get

$$\frac{a_m}{m+n+1} g^m(h^{n+m+1} - 1) + \frac{a_{m-1}}{m+n} g^{m-1}(h^{n+m} - 1) + \cdots + \frac{a_0}{n+1}(h^{n+1} - 1) = 0,$$

which implies $h^d = 1$, where $d = (n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, $d = (n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is not a constant, then from (3.13) we can say that f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(x,y) = x^{n+1} \left(\frac{a_m}{n+m+1} x^m + \frac{a_{m-1}}{n+m} x^{m-1} + \cdots + \frac{a_0}{n+1} \right)$$
$$- y^{n+1} \left(\frac{a_m}{n+m+1} y^m + \frac{a_{m-1}}{n+m} y^{m-1} + \cdots + \frac{a_0}{n+1} \right).$$

This completes the proof of the theorem. ∎

Proof of Theorem 2. We omit the proof since proceeding in the same way the proof can be carried out in the line of the proof of Theorem 1. ∎

Proof of Corollary 1. Proceeding in the like manner as in the proof of Theorem 1 we get

$$\frac{1}{n+m+1} f^{n+m+1} - \frac{1}{n+1} f^{n+1} \equiv \frac{1}{n+m+1} g^{n+m+1} - \frac{1}{n+1} g^{n+1}.$$

Then using Note 2 of Lemma 9 and [12, Lemma 10] we obtain the conclusions of the Corollary. ∎

REFERENCES

[1] A. Banerjee, *A uniqueness result on some differential polynomials sharing* 1-*points*, Hiroshima Math. J. **37** (2007), 397–408.

[**2**] A. Banerjee, P. Bhattacharjee, *Some uniqueness results related to certain nonlinear differential polynomials sharing the same 1-points*, Vietnam J. Math. **37** (2009), 97–112.

[**3**] S.S. Bhoosnurmath, R.S. Dyavanal, *Uniqueness and value sharing of meromorphic functions*, Comput. Math. Appl. **53** (2007), 1191–1205.

[**4**] M.L. Fang, *Uniqueness and value sharing of entire functions*, Comput. Math. Appl. **44** (2002), 823–831.

[**5**] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[**6**] I. Lahiri, *Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points*, Ann. Polon. Math. **71** (1999), 113–128.

[**7**] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. **46** (2001), 241–253.

[**8**] I. Lahiri, *On a question of Hong Xun Yi*, Arch. Math. (Brno) **38** (2002), 119–128.

[**9**] I. Lahiri, A. Banerjee, *Weighted sharing of two sets*, Kyungpook Math. J. **46** (2006), 79–87.

[**10**] I. Lahiri, A. Sarkar, *Nonlinear differential polynomials sharing 1-points with weight two*, Chinese J. Contemp. Math. **25** (2004), 325–334.

[**11**] I. Lahiri, P. Sahoo, *Uniqueness of nonlinear differential polynomials sharing 1-points*, Georgian Math. J. **12** (2005), 131–138.

[**12**] I. Lahiri, P. Sahoo, *Uniqueness of meromorphic functions when two nonlinear differential polynomials share a small function*, Arch. Math. (Brno) **44** (2008), 201–210.

[**13**] W.C. Lin, H.X. Yi, *Uniqueness theorems for meromorphic functions*, Indian J. Pure Appl. Math. **35** (2004), 121–132.

[**14**] A.Z. Mohonko, *On the Nevanlinna characteristics of some meromorphic functions*, The Theory of Functions, Functional Analysis and Their Applications, Izd-vo Khar'kovsk. Un-ta, **14** (1971), 83–87.

[**15**] J. Xu, Q. Han, J.L. Zhang, *Uniqueness theorems of meromorphic functions of a certain form*, Bull. Korean Math. Soc. **46** (2009), 1079–1089.

[**16**] C.C. Yang, *On deficiencies of differential polynomials II*, Math. Z. **125** (1972), 107–112.

[**17**] L.Yang, *Value Distribution Theory*, Springer- Verlag, Berlin, 1993.

[**18**] H.X. Yi, *Uniqueness of meromorphic functions and a question of C. C. Yang*, Complex Var. Theory Appl. **14** (1990), 169–176.

[**19**] H.X. Yi, *Meromorphic functions that share one or two values*, Complex Var. Theory Appl. **28** (1995), 1–11.

[**20**] H.X.Yi, C.C.Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, 1995.

[**21**] X.Y. Zhang, J.F. Chen and W.C. Lin, *Entire or meromorphic functions sharing one value*, Comput. Math. Appl. **56** (2008), 1876–1883.

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