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THE EULER THEOREM AND DUPIN INDICATRIX FOR SURFACES AT A CONSTANT DISTANCE FROM EDGE OF REGRESSION ON A SURFACE IN E_1^3

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Abstract. In this paper we give the Euler theorem and Dupin indicatrix for surfaces at a constant distance from edge of regression on a surface in E_1^3.

1. Introduction

Let k_1, k_2 denote principal curvature functions and e_1, e_2 be principal directions of a surface M, respectively. Then the normal curvature $k_n(v_p)$ of M in the direction $v_p = (\cos\theta)e_1 + (\sin\theta)e_2$ is

$$k_n(v_p) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

This equation is called Euler's formulae (Leonhard Euler, 1707–1783). The generalized Euler theorem for hypersurfaces in Euclidean space E^{n+1} can be found in [8]. In 1984, A. Kılıç and H.H. Hacısalihoğlu gave the Euler theorem and Dupin indicatrix for parallel hypersurfaces in E^n [12]. Also the Euler theorem and Dupin indicatrix are obtained for the parallel hypersurfaces in pseudo-Euclidean spaces E_1^{n+1} and E_ν^{n+1} in the papers [4, 6, 7].

In 2005 H.H. Hacısalihoğlu and Ö. Tarakçı introduced surfaces at a constant distance from edge of regression on a surface. These surfaces are a generalization of parallel surfaces in E^3. Because the authors took any vector instead of normal vector [15]. Euler theorem and Dupin indicatrix for these surfaces are given [2]. In 2010 we obtained the surfaces at a constant distance from edge of regression on a surface in E_1^3 [14].

In this paper we give the Euler theorem and Dupin indicatrix for surfaces at a constant distance from edge of regression on a surface in E_1^3.

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Definition 1.1. [3, 9, 10, 11, 13] (i) Hyperbolic angle: Let x and y be timelike vectors in the same timecone of Minkowski space. Then there is a unique real number $\theta \geq 0$, called the hyperbolic angle between x and y, such that

$$\langle x, y\rangle = -\left\|x\right\|\left\|y\right\|\cosh\theta.$$

(ii) Central angle: Let x and y be spacelike vectors in Minkowski space that span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$, called the central angle between x and y, such that

$$|\langle x, y\rangle| = \left\|x\right\|\left\|y\right\|\cosh\theta.$$

(iii) Spacelike angle: Let x and y be spacelike vectors in Minkowski space that span a spacelike vector subspace. Then there is a unique real number θ between 0 and π called the spacelike angle between x and y, such that

$$\langle x, y\rangle = \left\|x\right\|\left\|y\right\|\cos\theta.$$

(iv) Lorentzian timelike angle: Let x be a spacelike vector and y be a timelike vector in Minkowski space. Then there is a unique real number $\theta \geq 0$, called the Lorentzian timelike angle between x and y, such that

$$|\langle x, y\rangle| = \left\|x\right\|\left\|y\right\|\sinh\theta.$$

Definition 1.2. Let M and M^f be two surfaces in E_1^3 and N_p be a unit normal vector of M at the point $P \in M$. Let $T_p(M)$ be tanjant space at $P \in M$ and $\{X_p, Y_p\}$ be an orthonormal bases of $T_p(M)$. Let $Z_p = d_1X_p + d_2Y_p + d_3N_p$ be a unit vector, where $d_1, d_2, d_3 \in R$ are constant numbers and $\varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1\varepsilon_2 d_3^2 = \pm 1$. If a function f exists and satisfies the condition $f : M \to M^f$, $f(P) = P + rZ_p$, r constant, M^f is called as the surface at a constant distance from the edge of regression on M and M^f denoted by the pair (M, M^f).

If $d_1 = d_2 = 0$, then we have $Z_p = N_p$ and $f(P) = P + rN_p$. In this case M and M^f are parallel surfaces [14].

Theorem 1.3. [14] *Let the pair* (M, M^f) *be given in* E_1^3. *For any* $W \in \chi(M)$, *we have* $f_*(W) = \overline{W} + r\overline{D_W Z}$, *where* $W = \sum_{i=1}^3 w_i \dfrac{\partial}{\partial x_i}$, $\overline{W} = \sum_{i=1}^3 \overline{w_i}\dfrac{\partial}{\partial x_i}$ *and* $\forall P \in M$, $w_i(P) = \overline{w_i}(f(p))$, $1 \leq i \leq 3$.

Let (ϕ, U) be a parametrization of M, so we can write that

$$\phi : \underset{(u,v)}{U} \subset E_1^3 \to \underset{P=\phi(u,v)}{M}.$$

In this case $\{\phi_u|_p, \phi_v|_p\}$ is a basis of $T_M(P)$. Let N_p is a unit normal vector at $P \in M$ and $d_1, d_2, d_3 \in R$ be a constant numbers then we may write that $Z_p = d_1\phi_u|_p + d_2\phi_v|_p + d_3N_p$. Since $M^f = \{f(P) \mid f(P) = P + rZ_p\}$, a parametric representation of M^f is $\psi(u, v) = \phi(u, v) + rZ(u, v)$. Thus we may write

$$M^f = \{\psi(u,v) \mid \psi(u,v) = \phi(u,v) + r(d_1\phi_u(u,v) + d_2\phi_v(u,v) + d_3N(u,v)),$$
$$d_1,\ d_2,\ d_3, r \text{ are constant}, \quad \varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1\varepsilon_2 d_3^2 = \pm 1, \}$$

If we take $rd_1 = \lambda_1,\ rd_2 = \lambda_2,\ rd_3 = \lambda_3$ then we have

$$M^f = \{\psi(u,v)|\psi(u,v) = \phi(u,v) + \lambda_1\phi_u(u,v) + \lambda_2\phi_v(u,v) + \lambda_3 N(u,v),\\ \lambda_1, \lambda_2, \lambda_3 \text{ are constant}\}.$$

Let $\{\phi_u, \phi_v\}$ is basis of $\chi(M^f)$. If we take $\langle\phi_u, \phi_u\rangle = \varepsilon_1$, $\langle\phi_v, \phi_v\rangle = \varepsilon_2$ and $\langle N, N\rangle = -\varepsilon_1\varepsilon_2$, then

$$\psi_u = (1 + \lambda_3 k_1)\phi_u + \varepsilon_2\lambda_1 k_1 N,$$
$$\psi_v = (1 + \lambda_3 k_2)\phi_v + \varepsilon_1\lambda_2 k_2 N$$

is a basis of $\chi(M^f)$, where N is unit normal vector field on M and k_1, k_2 are principal of M [14].

THEOREM 1.4. [14] *Let the pair (M, M^f) be given. Let $\{\phi_u, \phi_v\}$ (orthonormal and principal vector fields on M) be basis of $\chi(M)$ and k_1, k_2 be principal curvatures of M. The matrix of the shape operator of M^f with respect to the basis $\{\psi_u = (1 + \lambda_3 k_1)\phi_u + \varepsilon_2\lambda_1 k_1 N,\ \psi_v = (1 + \lambda_3 k_2)\phi_v + \varepsilon_1\lambda_2 k_2 N\}$ of $\chi(M^f)$ is*

$$S^f = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix}$$

where

$$\mu_1 = \frac{(1 + \lambda_3 k_2)}{A^3}\left\{\varepsilon\lambda_1\frac{\partial k_1}{\partial u}(\lambda_2^2 k_2^2 - \varepsilon_1(1 + \lambda_3 k_2)^2) + k_1 A^2\right\}$$
$$\mu_2 = \frac{\varepsilon\lambda_1^2\lambda_2 k_1 k_2(1 + \lambda_3 k_2)}{A^3}\frac{\partial k_1}{\partial u}$$
$$\mu_3 = \frac{-\varepsilon\lambda_1\lambda_2^2 k_1 k_2(1 + \lambda_3 k_1)}{A^3}\frac{\partial k_2}{\partial v}$$
$$\mu_4 = \frac{(1 + \lambda_3 k_1)}{A^3}\left\{-\varepsilon\lambda_2\frac{\partial k_2}{\partial v}(\lambda_1^2 k_1^2 - \varepsilon_2(1 + \lambda_3 k_1)^2) + k_2 A^2\right\}$$

and $A = \sqrt{\varepsilon\,(\varepsilon_1\lambda_1^2 k_1^2(1 + \lambda_3 k_2)^2 + \varepsilon_2\lambda_2^2 k_2^2(1 + \lambda_3 k_1)^2 - \varepsilon_1\varepsilon_2(1 + \lambda_3 k_1)^2(1 + \lambda_3 k_2)^2)}$.

DEFINITION 1.5. [6] Let M be a pseudo-Euclidean surface in E_1^3 and p is nonumbilic point in M. A function k_n which is defined in the following form

$$k_n : T_pM \to R, \quad k_n(X_p) = \frac{1}{\|X_p\|^2}\langle S(X_p), X_p\rangle$$

is called a normal curvature function of M at p.

DEFINITION 1.6. [7] Let M be a pseudo-Euclidean surface in E_1^3 and S be shape operator of M. Then the Dupin indicatrix of M at the point p is

$$\mathcal{D}_p = \{\, X_p \mid \langle S(X_p), X_p\rangle = \pm 1,\ X_p \in T_pM \,\}.$$

2. The Euler theorem for surfaces at a constant distance from edge of regression on a surface in E_1^3

THEOREM 2.1. *Let M^f be a surface at a constant distance from edge of regression on a M in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let $Y_p \in T_pM$ and we denote the normal curvature by $k_n^f(f_*(Y_p))$ of M^f in the direction $f_*(Y_p)$. Thus*

$$k_n^f(f_*(Y_p)) = \frac{\mu_1^* y_1^2 + \varepsilon_1\varepsilon_2\mu_2^* y_1 y_2 + \mu_3^* y_2^2}{|\lambda_1^* y_1^2 - 2\varepsilon_1\varepsilon_2\lambda_1\lambda_2 k_1 k_2 y_1 y_2 + \lambda_2^* y_2^2|} \tag{2.1}$$

where

$$\begin{aligned}
&y_1 = \langle Y_p, \phi_u\rangle\,, \quad y_2 = \langle Y_p, \phi_v\rangle\,,\\
&\lambda_i^* = \varepsilon_i(1+\lambda_3 k_i)^2 - \varepsilon_1\varepsilon_2\lambda_i^2 k_i^2,\ (i = 1,2),\\
&\mu_1^* = \varepsilon_1\mu_1(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_1\lambda_1 k_1 + \mu_2\lambda_2 k_2),\\
&\mu_2^* = \varepsilon_2\mu_2(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_1\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_2\lambda_2 k_2)\\
&\qquad + \varepsilon_1\mu_3(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_3\lambda_1 k_1 + \mu_4\lambda_2 k_2),\\
&\mu_3^* = \varepsilon_2\mu_4(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_3\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_4\lambda_2 k_2).
\end{aligned} \tag{2.2}$$

Proof. Let $f_*(Y_p) \in T_{f(p)}M^f$. Then

$$k_n^f(f_*(Y_p)) = \frac{1}{\|f_*(Y_p)\|^2}\left\langle S^f(f_*(Y_p)), f_*(Y_p)\right\rangle \tag{2.3}$$

Let us calculate $f_*(Y_p)$ and $S^f(f_*(Y_p))$. Since ϕ_u and ϕ_v are orthonormal we have

$$Y_p = \varepsilon_1\langle Y_p, \phi_u\rangle\phi_u + \varepsilon_2\langle Y_p, \phi_v\rangle\phi_v = \varepsilon_1 y_1\phi_u + \varepsilon_2 y_2\phi_v$$

Further without lost of generality, we suppose that Y_p is a unit vector. Then

$$f_*(Y_p) = \varepsilon_1 y_1 f_*(\phi_u) + \varepsilon_2 y_2 f_*(\phi_v) = \varepsilon_1 y_1\psi_u + \varepsilon_2 y_2\psi_v. \tag{2.4}$$

On the other hand we find that

$$\begin{aligned}
S^f(f_*(Y_p)) &= \varepsilon_1 y_1 S^f(\psi_u) + \varepsilon_2 y_2 S^f(\psi_v)\\
&= \varepsilon_1 y_1\left(\mu_1(1+\lambda_3 k_1)\phi_u + \mu_2(1+\lambda_3 k_2)\phi_v + (\mu_1\varepsilon_2\lambda_1 k_1 + \mu_2\varepsilon_1\lambda_2 k_2)N\right)\\
&\quad + \varepsilon_2 y_2\left(\mu_3(1+\lambda_3 k_1)\phi_u + \mu_4(1+\lambda_3 k_2)\phi_v + (\mu_3\varepsilon_2\lambda_1 k_1 + \mu_4\varepsilon_1\lambda_2 k_2)N\right)
\end{aligned} \tag{2.5}$$

Thus using equations (2.4) and (2.5) in equation (2.3) we obtain (2.1). ∎

COROLLARY 2.2. *Let M^f be a surface at a constant distance from edge of regression on M in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1 and θ_2 respectively. Thus the normal curvature of M^f in the direction $f_*(Y_p)$*

(a) *Let N_p be a timelike vector then*

$$k_n^f(f_*(Y_p)) = \frac{\mu_1^*\cos^2\theta_1 + \mu_2^*\cos\theta_1\cos\theta_2 + \mu_3^*\cos^2\theta_2}{|\lambda_1^*\cos^2\theta_1 + \lambda_2^*\cos^2\theta_2 - 2\lambda_1\lambda_2 k_1 k_2\cos\theta_1\cos\theta_2|}$$

(b) *Let N_p be a spacelike vector.*

(b.1) *If Y_p and ϕ_u are timelike vectors in the same timecone then*

$$k_n^f(f_*(Y_p)) = \frac{\mu_1^*\cosh^2\theta_1 + \delta_2\mu_2^*\cosh\theta_1\sinh\theta_2 + \mu_3^*\sinh^2\theta_2}{\left|\lambda_1^*\cosh^2\theta_1 + \lambda_2^*\sinh^2\theta_2 - 2\delta_2\lambda_1\lambda_2 k_1 k_2\cosh\theta_1\sinh\theta_2\right|}$$

(b.2) *If Y_p and ϕ_v are timelike vectors in the same timecone then*

$$k_n^f(f_*(Y_p)) = \frac{\mu_1^*\sinh^2\theta_1 + \delta_1\mu_2^*\sinh\theta_1\cosh\theta_2 + \mu_3^*\cosh^2\theta_2}{\left|\lambda_1^*\sinh^2\theta_1 + \lambda_2^*\cosh^2\theta_2 - 2\delta_1\lambda_1\lambda_2 k_1 k_2\sinh\theta_1\cosh\theta_2\right|}$$

(b.3) *If $Y_p \in T_pM$ is a spacelike vector and ϕ_u is timelike vector then*

$$k_n^f(f_*(Y_p)) = \frac{\mu_1^*\sinh^2\theta_1 - \delta_1\delta_2\mu_2^*\sinh\theta_1\cosh\theta_2 + \mu_3^*\cosh^2\theta_2}{\left|\lambda_1^*\sinh^2\theta_1 + \lambda_2^*\cosh^2\theta_2 + 2\delta_1\delta_2\lambda_1\lambda_2 k_1 k_2\sinh\theta_1\cosh\theta_2\right|}$$

(b.4) *If $Y_p \in T_pM$ is a spacelike vector and ϕ_v is timelike vector then*

$$k_n^f(f_*(Y_p)) = \frac{\mu_1^*\cosh^2\theta_1 - \delta_1\delta_2\mu_2^*\cosh\theta_1\sinh\theta_2 + \mu_3^*\sinh^2\theta_2}{\left|\lambda_1^*\cosh^2\theta_1 + \lambda_2^*\sinh^2\theta_2 + 2\delta_1\delta_2\lambda_1\lambda_2 k_1 k_2\cosh\theta_1\sinh\theta_2\right|}$$

where $\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*$ and μ_3^* are given in (2.2) and δ_i, $(i = 1, 2)$ is 1 or -1 depending on y_i is positive or negative, respectively.

Proof. (a) Let N_p be a timelike vector. In this case θ_1 and θ_2 are spacelike angle then

$$y_1 = \langle Y_p, \phi_u\rangle = \cos\theta_1$$
$$y_2 = \langle Y_p, \phi_v\rangle = \cos\theta_2.$$

Substituting these equations in (2.1), we get $k_n^f(f_*(Y_p))$.

(b) Let N_p be a spacelike vector.

(b.1) If Y_p and ϕ_u are timelike vectors in the same timecone then there is a hyperbolic angle θ_1 and a Lorentzian timelike angle θ_2. Since

$$y_1 = -\cosh\theta_1 \quad \text{and} \quad y_2 = \delta_2\sinh\theta_2$$

the proof is obvious.

(b.2) If Y_p and ϕ_v are timelike vectors in the same timecone then there is a Lorentzian timelike angle θ_1 and a hyperbolic angle θ_2. Thus

$$y_1 = \delta_1\sinh\theta_1 \quad \text{and} \quad y_2 = -\cosh\theta_2.$$

(b.3) If $Y_p \in T_pM$ is a spacelike vector and ϕ_u is timelike vector then there is a Lorentzian timelike angle θ_1 and a central angle θ_2. Thus

$$y_1 = \delta_1 \sinh\theta_1 \quad \text{and} \quad y_2 = \delta_2 \cosh\theta_2.$$

(b.4) If $Y_p \in T_pM$ is a spacelike vector and ϕ_v is timelike vector then there is a central angle θ_1 and a Lorentzian timelike angle θ_2. Thus

$$y_1 = \delta_1 \cosh\theta_1 \quad \text{and} \quad y_2 = \delta_2 \sinh\theta_2. \quad \blacksquare$$

As a special case if we take $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = r =$ constant, then we obtain that M and M^f are parallel surfaces. The following corollary is known the Euler theorem for parallel surfaces in E_1^3.

Corollary 2.3. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let $Y_p \in T_pM$ and we denote the normal curvature by $k_n^r(f_*(Y_p))$ of M_r, in the direction $f_*(Y_p)$. Thus*

$$k_n^r(f_*(Y_p)) = \frac{\varepsilon_1 k_1(1+rk_1)y_1^2 + \varepsilon_2 k_2(1+rk_2)y_2^2}{|\varepsilon_1(1+rk_1)^2y_1^2 + \varepsilon_2(1+rk_2)^2y_2^2|}.$$

Proof. Since

$$\begin{aligned}
\lambda_i^* &= \varepsilon_i(1+rk_i)^2, \ (i=1,2),\\
\mu_1^* &= \varepsilon_1 k_1(1+rk_1),\\
\mu_2^* &= 0,\\
\mu_3^* &= \varepsilon_2 k_2(1+rk_2),
\end{aligned}$$

from (2.1) we find $k_n^r(f_*(Y_p))$. $\blacksquare$

Corollary 2.4. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1 and θ_2 respectively. Thus the normal curvature of M^f in the direction $f_*(Y_p)$*

(a) *Let N_p be a timelike vector then*

$$k_n^r(f_*(Y_p)) = \frac{k_1(1+rk_1)\cos^2\theta_1 + k_2(1+rk_2)\cos^2\theta_2}{(1+rk_1)^2\cos^2\theta_1 + (1+rk_2)^2\cos^2\theta_2}.$$

(b) *Let N_p be a spacelike vector.*

(b.1) *If Y_p and ϕ_u are timelike vectors in the same timecone then*

$$k_n^r(f_*(Y_p)) = \frac{-k_1(1+rk_1)\cosh^2\theta_1 + k_2(1+rk_2)\sinh^2\theta_2}{(1+rk_1)^2\cosh^2\theta_1 - (1+rk_2)^2\sinh^2\theta_2}.$$

(b.2) *If Y_p and ϕ_v are timelike vectors in the same timecone then*

$$k_n^r(f_*(Y_p)) = \frac{k_1(1+rk_1)\sinh^2\theta_1 - k_2(1+rk_2)\cosh^2\theta_2}{-(1+rk_1)^2\sinh^2\theta_1 + (1+rk_2)^2\cosh^2\theta_2}.$$

(b.3) *If $Y_p \in T_pM$ is a spacelike vector and ϕ_u is timelike vector then*

$$k_n^r(f_*(Y_p)) = \frac{-k_1(1+rk_1)\sinh^2\theta_1 + k_2(1+rk_2)\cosh^2\theta_2}{-(1+rk_1)^2\sinh^2\theta_1 + (1+rk_2)^2\cosh^2\theta_2}.$$

(b.4) *If $Y_p \in T_pM$ is a spacelike vector and ϕ_v is timelike vector then*

$$k_n^r(f_*(Y_p)) = \frac{k_1(1+rk_1)\cosh^2\theta_1 - k_2(1+rk_2)\sinh^2\theta_2}{(1+rk_1)^2\cosh^2\theta_1 - (1+rk_2)^2\sinh^2\theta_2}.$$

3. The Dupin indicatrix for surfaces at a constant distance from edge of regression on surfaces in $\boldsymbol{E}_1^3$

THEOREM 3.1. *Let M^f be a surface at a constant distance from edge of regression on M in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and $\{\phi_u, \phi_v\}$ be orthonormal bases such that ϕ_u and ϕ_v are principal directions on M. Thus*

$$D_{f(p)}^f = \left\{ f_*(Y_p) \in T_{f(p)}M^f \mid c_1 y_1^2 + \varepsilon_1\varepsilon_2 c_2 y_1 y_2 + c_3 y_2^2 = \pm 1 \right\},$$

where

$$\begin{aligned}
f_*(Y_p) &= \varepsilon_1 y_1(1+\lambda_3 k_1)\phi_u + \varepsilon_2 y_2(1+\lambda_3 k_2)\phi_v + \varepsilon_1\varepsilon_2(y_1\lambda_1 k_1 + y_2\lambda_2 k_2)N \\
c_1 &= \varepsilon_1\mu_1(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_1\lambda_1 k_1 + \mu_2\lambda_2 k_2), \\
c_2 &= \varepsilon_2\mu_2(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_1\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_2\lambda_2 k_2) \\
&\quad + \varepsilon_1\mu_3(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_3\lambda_1 k_1 + \mu_4\lambda_2 k_2), \\
c_3 &= \varepsilon_2\mu_4(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_3\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_4\lambda_2 k_2).
\end{aligned}$$

Proof. Let $f_*(Y_p) \in T_{f(p)}M^f$. Since

$$D_{f(p)}^f = \left\{ f_*(Y_p) \mid \left\langle S^f(f_*(Y_p)), f_*(Y_p) \right\rangle = \pm 1 \right\}$$

the proof is clear. ∎

According to this theorem the Dupin indicatrix of M^f at the point $f(p)$ in general will be a conic section of the following type:

COROLLARY 3.2. *Let M^f be a surface at a constant distance from edge of regression on M in E_1^3. The Dupin indicatrix of M^f at the point $f(p)$ is:*

(a) *an ellipse, if $c_2^2 - 4c_1c_3 < 0$,*

(b) *two conjugate hyperbolas, if $c_2^2 - 4c_1c_3 > 0$,*

(c) *parallel two lines, if $c_2^2 - 4c_1c_3 = 0$.*

COROLLARY 3.3. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and $\{\phi_u, \phi_v\}$ be orthonormal bases such that ϕ_u and ϕ_v are principal directions on M. In this case*

$$D^r_{f(p)} = \left\{ f_*(Y_p) \in T_{f(p)} M_r \;\middle|\; \varepsilon_1 k_1 (1 + r k_1) y_1^2 + \varepsilon_2 k_2 (1 + r k_2) y_2^2 = \pm 1 \right\}.$$

Hence the point $f(p)$ of M_r is:

(a) *an elliptic point, if $\varepsilon_1 \varepsilon_2 k_1 k_2 (1 + r k_1)(1 + r k_2) > 0$,*

(b) *a hyperbolic point, if $\varepsilon_1 \varepsilon_2 k_1 k_2 (1 + r k_1)(1 + r k_2) < 0$,*

(c) *a parabolic point, if $k_1 k_2 (1 + r k_1)(1 + r k_2) = 0$.*

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