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ON RIGHT IDEALS AND DERIVATIONS IN PRIME RINGS WITH ENGEL CONDITION

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Abstract. Let R be an associative ring with center $Z(R)$ and d a nonzero derivation of R. The main object in this paper is to study the situation $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t]^m \in Z(R)$ for all x, y in some appropriate subset of R, where $n \geq 0$, $s \geq 0$, $t \geq 0$, $m \geq 1$, $r \geq 1$ are fixed integers and R is a prime or semiprime ring.

1. Introduction

Throughout this paper, unless specifically stated, R denotes a prime ring with center $Z(R)$, with extended centroid C, and two-sided Martindale quotient ring Q. Given $x, y \in R$, we set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and inductively $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. By d, we mean a derivation of R.

In [12], Herstein proved that if char $(R) \neq 2$ and a derivation d is nonzero such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Chang and Lin [5] proved that if ρ is a nonzero right ideal of R such that $d(x)x^n = 0$ for all $x \in \rho$, $n \geq 1$ a fixed integer, then $d(\rho)\rho = 0$. Recently, De Filippis [10] proved that if char $(R) \neq 2$ and ρ a nonzero right ideal of R such that $[d(x)x^n, d(y)] = 0$ for all $x, y \in \rho$, then either R is commutative or $d(\rho)\rho = 0$. In another paper, De Filippis [11] proved that if char $(R) \neq 2$, d is nonzero and ρ is a nonzero right ideal of R such that $[[d(x), x], [d(y), y]] = 0$ for all $x, y \in \rho$, then either $[\rho, \rho]\rho = 0$ or $d(\rho)\rho = 0$. In [8], the first author of this paper extended the result of De Filippis by considering Engel conditions. The result of [8] states that if char $(R) \neq 2$ and ρ a non-zero right ideal of R such that $[[d(x), x]_n, [y, d(y)]_m]^t = 0$ for all $x, y \in \rho$, where $n \geq 0, m \geq 0, t \geq 1$ are fixed integers and $[\rho, \rho]\rho \neq 0$, then $d(\rho)\rho = 0$.

On the other hand, a well known result of Posner [22] states that if $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. In [18], Lee considered any constant power values of x and proved that if R be a prime ring and λ a nonzero left ideal of R such that $[d(x^n), x^n]_k = 0$ for all $x \in \lambda$, then either $d = 0$

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or R is commutative. Lee and Shiue [20] proved that if R is noncommutative and λ a nonzero left ideal of R then: (i) if $[d(x^m)x^n, x^r]_k = 0$ for all $x \in \lambda$, then $d = 0$, except when $R \cong M_2(GF(2))$; (ii) if $[x^n d(x^m), x^r]_k = 0$ for all $x \in \lambda$, then either $d = ad(b)$ with $\lambda b = 0$ for some $b \in Q$ or $\lambda[\lambda, \lambda] = 0$ and $d(\lambda) \subseteq \lambda C$.

From the results above, it is natural to consider the situation when $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t]^m \in Z(R)$ for all x, y in some appropriate subset of R, where $n \geq 0$, $s \geq 0$, $t \geq 0$, $m \geq 1$, $r \geq 1$ are fixed integers. As a particular case, we obtain results, when $[x, d(x)]_t = 0$ for all x in some right ideal of a prime ring R or for all x in a semiprime ring R.

Let R be a prime ring and Q its two-sided Martindale quotient ring. Then Q is also a prime ring with center $C = Z(Q)$, a field, which is the extended centroid of R. It is well known that any derivation of R can be uniquely extended to a derivation of Q, and hence any derivation of R can be defined on the whole of Q. We refer to [2, 19] for more details.

Denote by $Q *_C C\{x, y, z\}$ the free product of the C-algebra Q and $C\{x, y, z\}$, the free C-algebra in noncommuting indeterminates x, y, z.

2. The case: R a prime ring

We need the following lemma.

Lemma 2.1. *Let I be a nonzero right ideal of R and d a derivation of R. Then the following conditions are equivalent: (i) d is an inner derivation induced by some $b \in Q$ such that $bI = 0$; (ii) $d(I)I = 0$.*

For its proof we refer to [13] or [4, Lemma].

Theorem 2.2. *Let R be a prime ring of char $(R) \neq 2$ and d a non-zero derivation of R such that $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t]^m = 0$ for all $x, y \in R$, where $n, s, t \geq 0$ and $m, r \geq 1$ are fixed integers, then R is commutative.*

Proof. Assume that R is noncommutative, otherwise we are done. Assume next that d is Q-inner derivation i.e., $d(x) = [a, x]$ for all $x \in R$ and for some $a \in Q$. Then we have

$$[[ax^n, x^r]_{s+1}, [y, [a, y]]_t]^m = 0$$

for all $x, y \in R$. Since $d \neq 0$, $a \notin C$ and hence R satisfies a nontrivial generalized polynomial identity (GPI). Since Q and R satisfy the same generalized polynomial identities with coefficients in Q (see [7]), $[[ax^n, x^r]_{s+1}, [y, [a, y]]_t]^m$ is also satisfied by Q. Since Q is prime, we may replace R by Q and then assume that $a \in R$ and $C = Z(R)$. In this case R is centrally closed (i.e. $RC = R$) prime C-algebra [9]. Then by Martindale's theorem [21], R is a primitive ring. By Jacobson's theorem [15, p. 75] R is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. Since R is noncommutative, $\dim_D V \geq 2$. We assume that for some $v \in V$, $\{av, v\}$ is linearly D-independent. If $a^2v \notin \mathrm{span}_D\{v, av\}$, then $\{v, av, a^2v\}$ is linearly D-independent. By density there exist $x, y \in R$ such that

$$xv = v, \quad xav = 0, \quad xa^2v = 0;$$
$$yv = 0, \quad yav = v, \quad ya^2v = 0$$

for which we have $[a, y]v = -v$, $[a, y]av = av$, $[ax^n, x^r]_{s+1}v = av$ and hence

$$[y, [a, y]]_t v = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y [a, y]^{t-j} v = 0$$

and

$$[y, [a, y]]_t av = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y [a, y]^{t-j} av = \sum_{j=0}^{t} \binom{t}{j} v = 2^t v.$$

Thus

$$\begin{aligned} 0 &= [[ax^n, x^r]_{s+1}, [y, [a, y]]_t]v \\ &= [ax^n, x^r]_{s+1}[y, [a, y]]_t v - [y, [a, y]]_t [ax^n, x^r]_{s+1} v \\ &= 0 - 2^t v = -2^t v \end{aligned}$$

and hence

$$0 = [[ax^n, x^r]_{s+1}, [y, [a, y]]_t]^m v = (-1)^m 2^{mt} v,$$

which is a contradiction, since char $(R) \neq 2$.

If $a^2v \in \text{span}_D\{v, av\}$, then $a^2v = \alpha v + \beta av$ for some $\alpha, \beta \in D$. Then again by density there exist $x, y \in R$ such that $xv = v, xav = 0; yv = 0, yav = v$ for which we get $[a, y]v = -v$, $[a, y]^n av = av$ or $av - \beta v$ according as n is even or odd, $[ax^n, x^r]_{s+1}v = av$ and hence $[y, [a, y]]_t v = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y [a, y]^{t-j} v = 0$ and $[y, [a, y]]_t av = \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, y]^j y [a, y]^{t-j} av = \sum_{j=0}^{t} \binom{t}{j} v = 2^t v$. Therefore,

$$[[ax^n, x^r]_{s+1}, [y, [a, y]]_t]v = -2^t v$$

and hence

$$0 = [[ax^n, x^r]_{s+1}, [y, [a, y]]_t]^m v = (-1)^m 2^{mt} v,$$

which is a contradiction, since char $(R) \neq 2$. Thus we conclude that v and av are linearly D-dependent for all $v \in V$. Let $av = \alpha_v v$ for all $v \in V$, where $\alpha_v \in D$. It is very easy to prove that α_v is independent of choice of $v \in V$. Hence $av = \alpha v$ for all $v \in V$, where $\alpha \in D$ is fixed. Then for all $r \in R$ and $v \in V$, we have $[a, r]v = a(rv) - r(av) = \alpha(rv) - r(\alpha v) = 0$ that is $[a, r]V = 0$. Since V is a left faithful irreducible R-modulo, $[a, r] = 0$ for all $r \in R$, that is $a \in Z(R)$. This leads $d = 0$, a contradiction.

Assume next that d is not a Q-inner derivation in R. By assumption, we have

$$[[(\sum_{i=0}^{r-1} x^i d(x) x^{r-i-1}) x^n, x^r]_s, [y, d(y)]_t]^m = 0$$

for all $x, y \in R$. Then by Kharchenko's theorem [16], we have

$$[[(\sum_{i=0}^{r-1} x^i u x^{r-i-1})x^n, x^r]_s, [y, v]_t]^m = 0$$

for all $x, y, u, v \in R$. This is a polynomial identity for R and hence there exists a field F such that $R \subseteq M_k(F)$ with $k > 1$ and $M_k(F)$ satisfies the same polynomial identity [17, Lemma 1]. But by choosing $u = e_{21}, v = e_{22}, x = e_{11}$, $y = e_{12}$, we get

$$0 = [[(\sum_{i=0}^{r-1} x^i u x^{r-i-1})x^n, x^r]_s, [y, v]_t]^m = e_{22} + (-1)^m e_{11},$$

a contradiction. ∎

Our next theorem is to study the central case.

Theorem 2.3. *Let R be a prime ring of char $(R) \neq 2$ and d a nonzero derivation of R such that $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R)$ for all $x, y \in R$, where $n, s, t \geq 0$ and $r \geq 1$ are fixed integers, then R is commutative.*

Proof. If R is commutative, we are done. So, let R be noncommutative. We have that R satisfies

$$[[d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R). \tag{1}$$

If for all $x, y \in R$, $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t] = 0$, then we are done by Theorem 2.2. So, let there exist $x_1, x_2 \in R$, such that $0 \neq [[d(x_1^r)x_1^n, x_1^r]_s, [x_2, d(x_2)]_t] \in Z(R)$. Then (1) is a central differential identity for R. It follows from [6, Theorem 1] that R is a prime PI-ring and so $RC = Q$ is a finite-dimensional central simple C-algebra by Posner's theorem for prime PI-ring.

Let d be an inner derivation of Q induced by $a \in Q$. Since R and Q satisfy same GPIs [7], we have

$$[[[ax^n, x^r]_{s+1}, [y, [a, y]]_t], z] = 0 \tag{2}$$

for all $x, y \in Q$. Since there exist $x_1, x_2 \in R$, such that $[[ax_1^n, x_1^r]_{s+1}, [x_2, [a, x_2]]_t] \neq 0$, (2) is a nontrivial GPI for Q. Since Q is a finite-dimensional central simple C-algebra, it follows from Lemma 2 in [17] that there exists a suitable field F such that $Q \subseteq M_k(F), k > 1$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies (2), that is,

$$[[[ax^n, x^r]_{s+1}, [y, [a, y]]_t], z] = 0 \tag{3}$$

for all $x, y, z \in M_k(F)$. Let e and f be any two orthogonal idempotent elements in $M_k(F)$. Now, we replace x with e, y with exf and z with exf in (3) and let $Y = [[ae^n, e]_{s+1}, [exf, [a, exf]]_t]$. Then we compute

$$\begin{aligned} Ye &= [[ae^n, e]_{s+1}, [exf, [a, exf]]_t]e \\ &= [ae^n, e]_{s+1}[exf, [a, exf]]_t e - [exf, [a, exf]]_t [ae^n, e]_{s+1} e \\ &= [ae^n, e]_{s+1} \sum_{j=0}^{t} (-1)^j \binom{t}{j} [a, exf]^j exf [a, exf]^{t-j} e \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{j=0}^{t}(-1)^j\binom{t}{j}[a, exf]^j exf[a, exf]^{t-j}[ae^n, e]_{s+1}e \\
&= 0 - \sum_{j=0}^{t}(-1)^j\binom{t}{j}(-exfa)^j exf(aexf)^{t-j}ae \\
&= -2^t(exfa)^{t+1}e. \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
fY &= f[[ae^n, e]_{s+1}, [exf, [a, exf]]_t] \\
&= f[ae^n, e]_{s+1}[exf, [a, exf]]_t - f[exf, [a, exf]]_t[ae^n, e]_{s+1} \\
&= f[ae^n, e]_{s+1}\sum_{j=0}^{t}(-1)^j\binom{t}{j}[a, exf]^j exf[a, exf]^{t-j} \\
&\quad - f\sum_{j=0}^{t}(-1)^j\binom{t}{j}[a, exf]^j exf[a, exf]^{t-j}[ae^n, e]_{s+1} \\
&= fae\sum_{j=0}^{t}(-1)^j\binom{t}{j}(-exfa)^j exf(aexf)^{t-j} - 0 \\
&= 2^t(faex)^{t+1}f. \qquad (5)
\end{aligned}
$$

Hence

$$
\begin{aligned}
0 &= [[[ae^n, e]_{s+1}, [exf, [a, exf]]_t], exf] \\
&= [Y, exf] \\
&= \{-2^t(exfa)^{t+1}exf - 2^t ex(faex)^{t+1}f\} \\
&= -2^{t+1}(exfa)^{t+1}exf. \qquad (6)
\end{aligned}
$$

Since char $(R) \neq 2$, this implies $(faex)^{t+3} = 0$ for all $x \in M_k(F)$. By Levitzki's lemma [14, Lemma 1.1], $faex = 0$ for all $x \in M_k(F)$ and so $fae = 0$. Since f and e are any two orthogonal idempotent elements in $M_k(F)$, we have for any idempotent e in $M_k(F)$, $(1-e)ae = 0 = ea(1-e)$ which implies $[a, e] = 0$. Since a commutes with all idempotents in $M_k(F)$, $a \in C$ and hence $d = 0$.

If d is not Q-inner derivation of R, then by Kharchenko's Theorem [16], we have $0 = [[[(\sum_{i=0}^{r-1} x^i u x^{r-i-1})x^n, x^r]_s, [y, v]_t], z]$ for all $x, y, z, u, v \in R$. Since this is a polynomial identity for R, there exists a field F such that $R \subseteq M_k(F)$ with $k > 1$ and R and $M_k(F)$ satisfy the same polynomial identity [17, Lemma 1]. But by choosing $u = e_{21}, v = e_{22}, x = e_{11}$, $y = e_{12}$, we get

$$
[[(\sum_{i=0}^{r-1} x^i u x^{r-i-1})x^n, x^r]_s, [y, v]_t] = e_{22} - e_{11} \in Z(M_k(F)),
$$

a contradiction, since char $(F) \neq 2$. ■

Theorem 2.4. *Let R be a prime ring of char $(R) \neq 2$, d a nonzero derivation of R and I a nonzero right ideal of R such that $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R)$ for all $x, y \in I$, where $n \geq 0$, $s \geq 0$, $t \geq 0$, $r \geq 1$ are fixed integers. If $[I, I]I \neq 0$, then $d = ad(b)$ with $bI = 0$ for some $b \in Q$.*

We begin with the following lemma.

Lemma 2.5. *If $d(I)I \neq 0$ and $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R)$ for all $x, y \in I$, then R satisfies a non-trivial generalized polynomial identity (GPI).*

Proof. Suppose on the contrary that R does not satisfy any non-trivial GPI. We may assume that R is noncommutative, otherwise R satisfies trivially a non-trivial GPI.

Case I. Suppose that d is a Q-inner derivation induced by an element $a \in Q$. Then for any $u \in I$

$$[[[a(ux)^n, (ux)^r]_{s+1}, [uy, [a, uy]]_t], uz]$$

is a GPI for R, so it is the zero element in $Q *_C C\{x, y, z\}$. Expanding this we get,

$$\begin{aligned}\Big\{\Big(\sum_{j=0}^{s+1}(-1)^j\binom{s+1}{j}(ux)^{rj}a(ux)^n(ux)^{r(s+1-j)}\Big)[uy,[a,uy]]_t\\ -\Big(\sum_{j=0}^{t}(-1)^j\binom{t}{j}(auy-uya)^j uy[a,uy]^{t-j}\Big)[a(ux)^n,(ux)^r]_{s+1}\Big\}uz\\ -uz[[a(ux)^n,(ux)^r]_{s+1},[uy,[a,uy]]_t]=0. \quad (7)\end{aligned}$$

If au and u are linearly C-independent for some $u \in I$ then

$$\begin{aligned}a(ux)^n(ux)^{r(s+1)}[uy,[a,uy]]_t uz\\ -auy\sum_{j=1}^{t}(-1)^j\binom{t}{j}(auy-uya)^{j-1}uy[a,uy]^{t-j}[a(ux)^n,(ux)^r]_{s+1}uz=0. \quad (8)\end{aligned}$$

This implies

$$a(ux)^n(ux)^{r(s+1)}[uy,[a,uy]]_t uz = 0 \quad (9)$$

in $Q *_C C\{x, y, z\}$. Expanding this we write

$$a(ux)^n(ux)^{r(s+1)}\sum_{j=0}^{t}(-1)^j\binom{t}{j}(auy-uya)^j uy(auy-uya)^{t-j}uz = 0.$$

Again, since au and u are linearly C-independent, in the above expression we see that $a(ux)^n(ux)^{r(s+1)}uy(auy)^t uz$ appears nontrivially, a contradiction. Thus for any $u \in I$, au and u are C-dependent. Then $(a-\alpha)I = 0$ for some $\alpha \in C$. Replacing a with $a - \alpha$, we may assume that $aI = 0$. But then by Lemma 2.1, $d(I)I = 0$, contradiction.

Case II. Suppose that d is not a Q-inner derivation of R. If for all $u \in I$, $d(u) \in uC$, then $[d(u), u] = 0$ which implies R to be commutative (see [3]), a contradiction. Therefore there exists $u \in I$ such that $d(u) \notin uC$ i.e., u and $d(u)$ are linearly C-independent.

By our assumption we have that R satisfies

$$[[[d((ux)^r)(ux)^n, (ux)^r]_s, [d(uy), uy]_t], uz] = 0$$

that is

$$[[[(\sum_{i=0}^{r-1}(ux)^i(d(u)x+ud(x))(ux)^{r-1-i})(ux)^n,(ux)^r]_s,[uy,d(u)y+ud(y)]_t],uz]=0.$$

By Kharchenko's theorem [16],

$$[[[\sum_{i=0}^{r-1}(ux)^i(d(u)x+ux_1)(ux)^{n+r-1-i},ux]_s,[uy,d(u)y+uy_1]_t],uz]=0 \tag{10}$$

for all $x,y,z,x_1,y_1\in R$. In particular, for $x_1=y_1=0$,

$$[[[\sum_{i=0}^{r-1}(ux)^i(d(u)x)(ux)^{n+r-1-i},ux]_s,[uy,d(u)y]_t],uz]=0 \tag{11}$$

which is a non-trivial GPI for R, because u and $d(u)$ are linearly C-independent, a contradiction. ■

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. If $d(I)I=0$, then by Lemma 2.1 we obtain our conclusion. So, let $d(I)I\neq 0$. By Lemma 2.5, R is a GPI-ring, so is Q [7]. By [21], Q is a primitive ring with $H=Soc(Q)\neq 0$. Moreover, we may assume that $[IH,IH]IH\neq 0$, otherwise by [7], $[IQ,IQ]IQ=0$, which is a contradiction. We may also assume that $d(IH)IH\neq 0$, otherwise by Lemma 2.1, d is an inner derivation induced by an element $b\in Q$ such that $bIH=0$ that is $bI=0$, implying $d(I)I=0$, a contradiction.

Let $a\in IH$. Since H is a regular ring, there exists $e^2=e\in H$ such that $eH=aH$. Then $e\in IH$ and $a=ea$. By our assumption and by [12, Theorem 2], we may also assume that $[[[d(x^r)x^n,x^r]_s,[y,d(y)]_t],z]$ is an identity for IQ. In particular, $[[[d(x^r)x^n,x^r]_s,[y,d(y)]_t],z]$ is an identity for IH and so for eH. Replacing x with e, y with $ey(1-e)$ and z with $ey(1-e)$, it follows that, for all $y\in H$,

$$0=[[[d(e)e^n,e]_s,[ey(1-e),d(ey(1-e))]_t],ey(1-e)]. \tag{12}$$

Let $V=[[d(e)e^n,e]_s,[ey(1-e),d(ey(1-e))]_t]$. We have the facts that for any idempotent e, $d(x(1-e))e=-x(1-e)d(e)$, $(1-e)d(ex)=(1-e)d(e)ex$ and $ed(e)e=0$ and hence we compute

$$\begin{aligned}
Ve&=[[d(e)e^n,e]_s,[ey(1-e),d(ey(1-e))]_t]e\\
&=[d(e)e^n,e]_s[ey(1-e),d(ey(1-e))]_te-[ey(1-e),d(ey(1-e))]_t[d(e)e^n,e]_se\\
&=[d(e)e^n,e]_s\sum_{j=0}^{t}(-1)^j\binom{t}{j}d(ey(1-e))^jey(1-e)d(ey(1-e))^{t-j}e\\
&\qquad-\sum_{j=0}^{t}(-1)^j\binom{t}{j}d(ey(1-e))^jey(1-e)d(ey(1-e))^{t-j}[d(e)e^n,e]_se\\
&=0-\sum_{j=0}^{t}(-1)^j\binom{t}{j}(-ey(1-e)d(e))^jey(1-e)(d(e)ey(1-e))^{t-j}d(e)e\\
&=-2^t(ey(1-e)d(e))^{t+1}e
\end{aligned} \tag{13}$$

and

$$\begin{aligned}(1-e)V &= (1-e)[[d(e)e^n, e]_s, [ey(1-e), d(ey(1-e))]_t]\\ &= (1-e)d(e)e[ey(1-e), d(ey(1-e))]_t\\ &\quad - (1-e)[ey(1-e), d(ey(1-e))]_t[d(e)e^n, e]_s\\ &= (1-e)d(e)e\sum_{j=0}^{t}(-1)^j\binom{t}{j}d(ey(1-e))^j ey(1-e)d(ey(1-e))^{t-j}\\ &\quad - (1-e)\sum_{j=0}^{t}(-1)^j\binom{t}{j}d(ey(1-e))^j ey(1-e)d(ey(1-e))^{t-j}[d(e)e^n, e]_s\\ &= (1-e)d(e)e\sum_{j=0}^{t}(-1)^j\binom{t}{j}(-ey(1-e)d(e))^j ey(1-e)(d(e)ey(1-e))^{t-j} - 0\\ &= 2^t((1-e)d(e)ey)^{t+1}(1-e). \qquad (14)\end{aligned}$$

Thus (12) gives

$$\begin{aligned}0 &= [V, ey(1-e)]\\ &= Vey(1-e) - ey(1-e)V\\ &= -2^t(ey(1-e)d(e))^{t+1}ey(1-e) - 2^t ey((1-e)d(e)ey)^{t+1}(1-e)\\ &= -2^{t+1}(ey(1-e)d(e))^{t+1}ey(1-e). \qquad (15)\end{aligned}$$

Multiplying on the left by $(1-e)d(e)$ and on the right by $d(e)ey$ and using char $(R) \neq 2$, the above equation gives $((1-e)d(e)ey)^{t+2} = 0$ for all $y \in H$. By Levitzki's lemma [14, Lemma 1.1], $(1-e)d(e)eH = 0$. By primeness of H, $(1-e)d(e)e = 0$. This implies $(1-e)d(e) = (1-e)d(e^2) = (1-e)d(e)e = 0$. Thus $d(e) = ed(e) \in eH \subseteq IH$. Now $d(a) = d(ea) = d(e)ea + ed(ea) \in IH$. Hence, $d(IH) \subseteq IH$. Since $d(l_H(IH)) \subseteq l_H(IH)$ holds, d naturally induces a derivation δ on the prime ring $\overline{IH} = \frac{IH}{IH \cap l_H(IH)}$ defined by $\delta(\overline{x}) = \overline{d(x)}$ for $x \in IH$, where $l_H(IH)$ denotes the left annihilator of IH in H. Thus by assumption we have

$$[[\delta(\overline{x}^r)\overline{x}^n, \overline{x}^r]_s, [\overline{y}, \delta(\overline{y})]_t, \overline{z}] = 0$$

for all $\overline{x}, \overline{y}, \overline{z} \in \overline{IH}$. By Theorem 2.3, we have either $\delta = 0$ or $\overline{IH}$ is commutative. Therefore, we have that either $d(IH)IH = 0$ or $[IH, IH]IH = 0$. In both cases, we have contradictions. This completes the proof of the theorem. ■

Corollary 2.6. *Let R be a prime ring of char $(R) \neq 2$, d a nonzero derivation of R and I a nonzero right ideal of R such that $[d(x^r)x^n, x^r]_s = 0$ for all $x \in I$, where $n \geq 0$, $s \geq 0$, $r \geq 1$ are fixed integers. If $[I, I]I \neq 0$, then $d(I)I = 0$.*

Corollary 2.7. *Let R be a prime ring of char $(R) \neq 2$, d a nonzero derivation of R and I a nonzero right ideal of R such that $[x, d(x)]_t = 0$ for all $x \in I$, where $t \geq 1$ is a fixed integer. If $[I, I]I \neq 0$, then $d(I)I = 0$.*

3. The case: R a semiprime ring

In this section we extend Theorems 2.2 and 2.3 to the case of semiprime ring. Let R be a semiprime ring and U be its right Utumi quotient ring. The center of U is called extended centroid of R and is denoted by C. It is well known fact that any derivation of a semiprime ring R can be uniquely extended to a derivation of its right Utumi quotient ring U and so any derivation of R can be defined on the whole of U [19, Lemma 2]. Let $M(C)$ be the set of all maximal ideals of C. Now by the standard theory of orthogonal completions for semiprime rings (see [19, p. 31-32]), we have the following lemma.

Lemma 3.1. [1, Lemma 1 and Theorem 1] *Let R be a 2-torsion free semiprime ring and P a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P \in M(C)$ with U/PU 2-torsion free$\} = 0$.*

Theorem 3.2. *Let R be a 2-torsion free semiprime ring, d a non-zero derivation of R such that $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t]^m = 0$ for all $x, y \in R$, where $n, s, t \geq 0$ and $m, r \geq 1$ are fixed integers. Then d maps R into its centre.*

Proof. By assumption and by [19, Theorem 3], we can write $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t]^m = 0$ for all $x, y \in U$. Note that U is also a 2-torsion free semiprime ring. Let $P \in M(C)$ such that U/PU is 2-torsion free. Then by Lemma 3.1, PU is a prime ideal of U invariant under d. Set $\overline{U} = U/PU$. Then derivation d canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. Therefore, $[[\overline{d}(\overline{x}^r)\overline{x}^n, \overline{x}^r]_s, [\overline{y}, \overline{d}(\overline{y})]_t]^m = 0$ for all $\overline{x}, \overline{y} \in \overline{U}$. By Theorem 2.2, either $\overline{d} = 0$ or $[\overline{U}, \overline{U}] = 0$ i.e., $d(U) \subseteq PU$ or $[U, U] \subseteq PU$. In any case $d(U)[U, U] \subseteq PU$ for any $P \in M(C)$. By Lemma 3.1, $\bigcap\{PU \mid P \in M(C)$ with U/PU 2-torsion free $\} = 0$. Thus $d(U)[U, U] = 0$. Without loss of generality, we have $d(R)[R, R] = 0$. This implies $d(R)R[R, R] = 0$ and so $[R, d(R)]R[R, d(R)] = 0$. Since R is semiprime, we have $[R, d(R)] = 0$, that is, $d(R) \subseteq Z(R)$, as desired. ■

By a similar proof, Theorem 2.3 can be extended to semiprime ring as follows:

Theorem 3.3. *Let R be a 2-torsion free semiprime ring, d a non-zero derivation of R such that $[[d(x^r)x^n, x^r]_s, [y, d(y)]_t] \in Z(R)$ for all $x, y \in R$, where $n, s, t \geq 0$ and $r \geq 1$ are fixed integers. Then d maps R into its centre.*

Corollary 3.4. *Let R be a 2-torsion free semiprime ring, d a non-zero derivation of R such that $[d(x^r)x^n, x^r]_s = 0$ for all $x \in R$, where $n, s \geq 0$ and $r \geq 1$ are fixed integers. Then d maps R into its center.*

Corollary 3.5. *Let R be a 2-torsion free semiprime ring, d a non-zero derivation of R such that $[x, d(x)]_t = 0$ for all $x \in R$, where $t \geq 0$ is a fixed integer. Then d maps R into its center.*

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