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ON CERTAIN UNIVALENT CLASS ASSOCIATED WITH FUNCTIONS OF NON-BAZILEVIČ TYPE

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Abstract. In this work, we study certain differential inequalities and first order differential subordinations. As their applications, we obtain some sufficient conditions for univalence, which generalize and refine some previous results.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let f be analytic in U, g analytic and univalent in U and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in U, we shall mean $f(U) \subset g(U)$.

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

The function $f \in \mathcal{A}$ is called Φ-like if

$$\Re\Big\{\frac{zf'(z)}{\Phi(f(z))}\Big\} > 0, \quad z \in U.$$

This concept was introduced by Brickman [2] and established that a function $f \in \mathcal{A}$ is univalent if and only if f is Φ-like for some Φ.

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DEFINITION 1. Let Φ be analytic function in a domain containing $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(U) - \{0\}$. Let $q(z)$ be a fixed analytic function in U, $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z), \quad z \in U.$$

Ruscheweyh [12] investigated this general class of Φ-like functions.

In the present paper, we consider another new class $H^\mu\big(\lambda; \Phi_1(f(z)), \Phi_2(f(z))\big)$ involving two different types of Φ-like functions, Φ_1 and Φ_2, which are defined by

$$(1+\lambda)\frac{zf'(z)}{\Phi_1(f(z))}\Big(\frac{z}{f(z)}\Big)^\mu - \lambda\frac{zf'(z)}{\Phi_2(f(z))}\Big(\frac{z}{f(z)}\Big)^\mu \prec F(z), \tag{1}$$

where $\mu, \lambda \in \mathbb{R}$, F is the conformal mapping of the unit disk U with $F(0) = 1$ and Φ_1 and Φ_2 satisfy Definition 1.1.

REMARK 1. As special cases of the class $H^\mu\big(\lambda; \Phi_1(f(z)), \Phi_2(f(z))\big)$ and for different type of F, are the following well known classes: $H^0\big(0; \Phi(f(z))\big)$ (see [12]); $H^\mu\big(0; z\big)$ (see [11]); $H^\mu\big(\lambda; zf'(z), f(z)\big)$ (see [15]) when $F(z) := \frac{1+Az}{1+Bz}$. Also this class reduces to the classes of starlike functions, convex functions and close-to-convex functions.

Recently, many authors studied the non-Bazilevič type of functions (see [5, 6, 7, 16, 17]). In order to obtain our results, we need the following lemmas.

LEMMA 1. [8] *Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*

1. *$Q(z)$ is starlike univalent in U, and*
2. *$\Re\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z \in U$.*

If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

DEFINITION 2. [9] Denote by $\mathbf{Q}$ the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where $E(f) := \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

LEMMA 2. [3] *Let $q(z)$ be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*

1. *$zq'(z)\varphi(q(z))$ is starlike univalent in U, and*
2. *$\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ for $z \in U$.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap \mathbf{Q}$, with $p(U) \subseteq D$ and $\vartheta(p(z)) + zp'(z)\varphi(z)$ is univalent in U and $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$ then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

2. The class $H^{\mu}\big(\lambda; \Phi_1(f(z)), \Phi_2(f(z))\big)$

In this section we introduce subordination results and the sufficient conditions for functions f to be in the class $H^{\mu}\big(\lambda; \Phi_1(f(z)), \Phi_2(f(z))\big)$.

Theorem 1. *Let q, $q(z) \neq 0$, be a univalent function in U, and $g(z) \neq 0$ be analytic in $\mathbb{C}$ such that for nonnegative real numbers μ and ν*

$$\Re\Big\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\Big\} > \max\Big\{0, \Big(\frac{\mu}{\nu}\Big)\Re\Big(q(z)[1 + \frac{g'(z)}{g(z)}(\frac{q(z)}{q'(z)} + \frac{\nu z}{\mu q(z)})]\Big)\Big\}. \quad (2)$$

If $p(z) \neq 0$, $z \in U$ satisfies the differential subordination

$$g(z)\Big[\mu p(z) + \nu\frac{zp'(z)}{p(z)}\Big] \prec g(z)\Big[\mu q(z) + \nu\frac{zq'(z)}{q(z)}\Big], \quad (3)$$

then $p \prec q$ and q is the best dominant.

Proof. Define the functions θ and ϕ as follows:

$$\theta(w(z)) := \mu w(z)g(z) \quad \text{and} \quad \phi(w(z)) := \frac{\nu g(z)}{w(z)}.$$

Obviously, the functions θ and ϕ are analytic in domain $D = \mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ in D. Now, define the functions Q and h as follows:

$$Q(z) := zq'(z)\phi(q(z)) = \nu g(z)\frac{zq'(z)}{q(z)},$$

$$h(z) := \theta(q(z)) + Q(z) = \mu q(z)g(z) + \nu g(z)\frac{zq'(z)}{q(z)}.$$

Then in view of condition (2), we obtain Q is starlike in U and $\Re\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z \in U$. Furthermore, in view of condition (3) we have

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)).$$

Therefore, the proof follows from Lemma 1. ∎

As an application of Theorem 1, we pose the sufficient condition for functions in $H^{\mu}\big(\lambda; \Phi_1(f(z)), \Phi_2(f(z))\big)$. We have the following result:

Corollary 1. *If $f(z) \in \mathcal{A}$ satisfies the conditions* (2) *and* (3) *for some g in Theorem 1, then $f \in H^{\mu}\big(\lambda; \Phi_1(f(z)), \Phi_2(f(z))\big)$.*

3. Sandwich theorem

By employing the concept of the superordination (Lemma 2), we state the sandwich theorem containing functions $f \in \mathcal{A}$.

Theorem 2. *Let $q(z)$ be convex univalent in the unit disk U. Suppose that g is analytic in the unit disk such that*

1. $\nu g(z)\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U*, and*

2. $\frac{\mu}{\nu}\Re\{q(z)q'(z)\} > 0$ *for* $z \in U$*.*

If $p(z) \in \mathcal{H}[q(0),1] \cap \mathbf{Q}$*, with* $p(U) \subseteq D$ *and* $g(z)\big[\mu p(z) + \nu\frac{zp'(z)}{p(z)}\big]$ *is univalent in* U *and*

$$g(z)\Big[\mu q(z) + \nu\frac{zq'(z)}{q(z)}\Big] \prec g(z)\Big[\mu p(z) + \nu\frac{zp'(z)}{p(z)}\Big]$$

then $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. Define functions θ and ϕ as follows:

$$\vartheta(w(z)) := \mu w(z)g(z) \quad \text{and} \quad \varphi(w(z)) := \frac{\nu g(z)}{w(z)}.$$

Obviously, the functions ϑ and φ are analytic in domain $D = \mathbb{C}\setminus\{0\}$ and $\varphi(w) \neq 0$ in D. Hence the assumptions of Lemma 2 are satisfied. ■

Combining Theorem 1 and Theorem 3 we get the following sandwich theorem:

Theorem 3. *Let* $q_1(z), q_2 \neq 0$ *be convex and univalent in* U *respectively. Suppose that* g *is analytic in* U *such that*

1. $\nu g(z)\frac{zq_1'(z)}{q_1(z)}$ *is starlike univalent in* U*, and*

2. $\frac{\mu}{\nu}\Re\{q_1(z)q_1'(z)\} > 0$ *for* $z \in U$ *and*

$$\Re\Big\{1+\frac{zq_2''(z)}{q_2'(z)}-\frac{zq_2'(z)}{q_2(z)}\Big\} > \max\Big\{0, \Big(\frac{\mu}{\nu}\Big)\Re\Big(q_2(z)[1+\frac{g'(z)}{g(z)}(\frac{q_2(z)}{q_2'(z)}+\frac{\nu z}{\mu q_2(z)})]\Big)\Big\}. \quad (4)$$

If $p(z) \neq 0 \in \mathcal{H}[q(0),1] \cap \mathbf{Q}$*, with* $p(U) \subseteq D$ *and* $g(z)\big[\mu p(z) + \nu\frac{zp'(z)}{p(z)}\big]$ *is univalent in* U *and*

$$g(z)\Big[\mu q_1(z) + \nu\frac{zq_1'(z)}{q_1(z)}\Big] \prec g(z)\Big[\mu p(z) + \nu\frac{zp'(z)}{p(z)}\Big] \prec g(z)\Big[\mu q_2(z) + \nu\frac{zq_2'(z)}{q_2(z)}\Big]$$

then

$$q_1(z) \prec p(z) \prec q_2(z), \quad (z \in U)$$

and $q_1(z), q_2(z)$ *are the best subordinant and the best dominant respectively.*

By letting $p(z) := \frac{zf'(z)}{f(z)}$ in Theorem 3, we have

Corollary 2. *Let the conditions of Theorem 3 on the functions* q_1 *and* q_2 *hold. If for* $f \in \mathcal{A}$*,* $\frac{zf'(z)}{f(z)} \neq 0 \in \mathcal{H}[q(0),1] \cap \mathbf{Q}$*, with* $(\frac{zf'}{f})(U) \subseteq D$ *and* $g(z)\big[(\mu-\nu)\frac{zf'(z)}{f(z)} + \nu(1+\frac{zf''(z)}{f'(z)})\big]$ *is univalent in* U *and*

$$g(z)\Big[\mu q_1(z) + \nu\frac{zq_1'(z)}{q_1(z)}\Big] \prec g(z)\Big[(\mu-\nu)\frac{zf'(z)}{f(z)} + \nu(1+\frac{zf''(z)}{f'(z)})\Big]$$
$$\prec g(z)\Big[\mu q_2(z) + \nu\frac{zq_2'(z)}{q_2(z)}\Big]$$

then

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \quad (z \in U) \tag{5}$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that Ali et al. [1] have used the results of Bulboacă [3] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy (5).

By assuming $p(z) := \frac{f(z)}{zf'(z)}$ in Theorem 3, we obtain

Corollary 3. *Let the conditions of Theorem 3 on the functions q_1 and q_2 hold. If for $f \in \mathcal{A}$, $\frac{f(z)}{zf'(z)} \neq 0 \in \mathcal{H}[q(0),1] \cap \mathbf{Q}$, with $(\frac{f}{zf'}(U) \subseteq D$ and $g(z)\big[\mu \frac{f(z)}{zf'(z)} + \nu(\frac{zf'(z)}{f(z)} - 1 - \frac{zf''(z)}{f'(z)})\big]$ is univalent in U and*

$$g(z)\Big[\mu q_1(z) + \nu \frac{zq_1'(z)}{q_1(z)}\Big] \prec g(z)\Big[\mu \frac{f(z)}{zf'(z)} + \nu(\frac{zf'(z)}{f(z)} - 1 - \frac{zf''(z)}{f'(z)})\Big]$$
$$\prec g(z)\Big[\mu q_2(z) + \nu \frac{zq_2'(z)}{q_2(z)}\Big]$$

then

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z), \quad (z \in U) \tag{6}$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that Shanmugam et al. [13] posed sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy (6).

Again by considering $p(z) := \frac{z^2 f'(z)}{f^2(z)}$ in Theorem 3, we find

Corollary 4. *Let the conditions of Theorem 3 on the functions q_1 and q_2 hold. If for $f \in \mathcal{A}$, $\frac{z^2 f'(z)}{f^2(z)} \neq 0 \in \mathcal{H}[q(0),1] \cap \mathbf{Q}$, with $\frac{z^2 f'}{f^2}(U) \subseteq D$ and $g(z)\big[\mu \frac{z^2 f'(z)}{f^2(z)} + \nu(\frac{zf''(z)}{f'(z)} + 2 - 2\frac{zf'(z)}{f(z)})\big]$ is univalent in U and*

$$g(z)\Big[\mu q_1(z) + \nu \frac{zq_1'(z)}{q_1(z)}\Big] \prec g(z))\Big[\mu \frac{z^2 f'(z)}{f^2(z)} + \nu(\frac{zf''(z)}{f'(z)} + 2 - 2\frac{zf'(z)}{f(z)})\Big]$$
$$\prec g(z)\Big[\mu q_2(z) + \nu \frac{zq_2'(z)}{q_2(z)}\Big]$$

then

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z), \quad (z \in U) \tag{7}$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that Shanmugam et al. [13] estimated sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy (7).

Furthermore, by letting $p(z) := \frac{z(f*g)'(z)}{\Phi(f*g)(z)}$ in Theorem 3, we pose

Corollary 5. *Let the conditions of Theorem 3 on the functions* q_1 *and* q_2 *hold. If for* $f \in \mathcal{A}$, $\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \neq 0 \in \mathcal{H}[q(0),1] \cap \mathbf{Q}$, *with* $(\frac{z(f*g)'}{\Phi(f*g)})(U) \subseteq D$ *and*

$$g(z)\Big[\mu\frac{z(f*g)'(z)}{\Phi(f*g)(z)} - \nu(\frac{zf''(z)}{f'(z)} + 1 - \frac{z\Phi'(f*g)(z)}{\Phi(f*g)(z)})\Big]$$

is univalent in U *and*

$$g(z)\Big[\mu q_1(z) + \nu\frac{zq_1'(z)}{q_1(z)}\Big] \prec g(z)\Big[\mu\frac{z(f*g)'(z)}{\Phi(f*g)(z)} - \nu(\frac{zf''(z)}{f'(z)} + 1 - \frac{z\Phi'(f*g)(z)}{\Phi(f*g)(z)})\Big] \prec g(z)\Big[\mu q_2(z) + \nu\frac{zq_2'(z)}{q_2(z)}\Big]$$

then

$$q_1(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q_2(z), \quad (z \in U) \tag{8}$$

and $q_1(z), q_2(z)$ *are the best subordinant and the best dominant respectively.*

Note that Shanmugam et al. [14] posed sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy (8).

Finally, by setting $p(z) := (\frac{H_m^l[\alpha_1]f(z)}{z})^\delta$, where $f \in \mathcal{A}$ and $H_m^l[\alpha_1]$ is the Dziok-Srivastava linear operator [4], in Theorem 3, we have

Corollary 6. *Let the conditions of Theorem 3 on the functions* q_1 *and* q_2 *hold. If for* $f \in \mathcal{A}$, $(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta \neq 0 \in \mathcal{H}[q(0),1] \cap \mathbf{Q}$, *with* $\big((\frac{H_m^l[\alpha_1]f}{z})^\delta\big)(U) \subseteq D$ *and*

$$g(z)\Big[\mu(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta - \nu\delta z(\frac{z}{H_m^l[\alpha_1]f(z)} - 1)\Big]$$

is univalent in U *and*

$$g(z)\Big[\mu q_1(z) + \nu\frac{zq_1'(z)}{q_1(z)}\Big] \prec g(z)\Big[\mu(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta - \nu\delta z(\frac{z}{H_m^l[\alpha_1]f(z)} - 1)\Big] \prec g(z)\Big[\mu q_2(z) + \nu\frac{zq_2'(z)}{q_2(z)}\Big]$$

then

$$q_1(z) \prec (\frac{H_m^l[\alpha_1]f(z)}{z})^\delta \prec q_2(z), \quad (z \in U) \tag{9}$$

and $q_1(z), q_2(z)$ *are the best subordinant and the best dominant respectively.*

Note that Murugusundaramoorthy and Magesh [10] introduced sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy (9).

Corollary 7. *Let the assumptions of Theorem 3 on the function*

$$p(z) := (1+\lambda)\frac{zf'(z)}{\Phi_1(f(z))}\Big(\frac{z}{f(z)}\Big)^\mu - \lambda\frac{zf'(z)}{\Phi_2(f(z))}\Big(\frac{z}{f(z)}\Big)^\mu$$

hold. Then

$$q_1(z) \prec (1+\lambda)\frac{zf'(z)}{\Phi_1(f(z))}\Big(\frac{z}{f(z)}\Big)^{\mu} - \lambda\frac{zf'(z)}{\Phi_2(f(z))}\Big(\frac{z}{f(z)}\Big)^{\mu} \prec q_2(z), \quad (z \in U) \quad (10)$$

and $q_1(z), q_2(z)$ *are the best subordinant and the best dominant respectively.*

REFERENCES

[**1**] R.M. Ali, V. Ravichandran, M. Hussain Khan, K.G. Subramanian, *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci. **15** (2005), 87–94.

[**2**] L. Brickman, *Φ-like analytic functions*, I, Bull. Amer. Math. Soc. **79** (1973), 555–558.

[**3**] T. Bulboaca, *Classes of first-order differential superordinations*, Demonstr. Math. **35** (2002), 287–292.

[**4**] J. Dziok, H.M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transforms Spec. Funct. **14** (2003), 7–18.

[**5**] S.P. Goyal, Rakesh Kumar, *Subordination and superordination results of non-Bazilevič functions involving Dziok-Srivastava Operator*, Int. J. Open Problems Complex Analysis **2** (2010), 1–14.

[**6**] S.P. Goyal, P. Goswami, H. Silverman, *Subordination and Superordination results for a class of analytic multivalent functions*, Internat. J. Math. Math. Sci. **2008** (2008), 1–12.

[**7**] R.W. Ibrahim, M. Darus, N. Tuneski, *On subordinations for classes of non-Bazilevič type*, Annales Univ. Mariae Curie-Sklodowska, **2** (2010), 49–60.

[**8**] S.S. Miller, P.T. Mocanu, *Differential Subordinantions: Theory and Applications*, Pure and Applied Mathematics, No. 225, Dekker, New York, 2000.

[**9**] S.S. Miller, P.T.Mocanu, *Subordinants of differential superordinations*, Complex Variables **48** (2003), 815–826.

[**10**] G. Murugusundaramoorthy, N. Magesh, *Differential subordinations and superordinations for analytic functions defined by the Dziok-Srivastava linear operator*, J. Inequal. Pure Appl. Math. **7** (2006), 1– 20.

[**11**] M. Obradović, *A class of univalent functions*, Hokkaido Math. J. **7** (1998), 329–335.

[**12**] St. Ruscheweyh, *A subordination theorem for Φ-like functions*, J. London Math. Soc. **2** (1976), 275–280.

[**13**] T.N. Shanmugam, V. Ravichandran, S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions*, Aust. J. Math. Anal. Appl. **3** (2006), 1–8.

[**14**] T.N. Shanmugam, S. Sivasubramanian, M. Darus, *Subordination and superordination for Φ−like functions*, J. Inequal. Pure Appl. Math. **8** (2007), 1–6.

[**15**] Z. Wang, C. Gao, M. Liao, *On certain generalized class of non- Bazilevič functions*, Acta Math. Academia Paedagogicae Nyiregyhaziensis **21** (2005), 147–154.

[**16**] Z.-G. Wang, Y.-P. Jiang, *A new generalized class of non-Bazilevič functions defined by Briot-Bouquet differential subordination*, Indian J. Math. **50** (2008), 245–255.

[**17**] Z.-G. Wang, H.-T. Wang, Y. Sun, *A class of multivalent non-Bazilevič functions involving the Cho-Kwon-Srivastava operator*, Tamsui Oxf. J. Math. Sci. **26** (2010), 1–19.

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