

## ON SOME NEW CHARACTERIZATIONS OF NEAR PARACOMPACTNESS AND ASSOCIATED RESULTS

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**Abstract.** Near paracompactness is a concept, in Set Topology, which is weaker than paracompactness; in this paper, several characterizations of this concept have been enunciated and proved. In the process, several tools have been utilized. The main theorem uses the selection theory of Michael.

### 1. Introduction and preliminary results

Singal and Arya [14] introduced, in 1969, the concept of near paracompactness; this concept is weaker than paracompactness and has its own meaningful facets. Many men of topology, since 1969, have studied this concept from different angles. Papers like [6, 9, 10–12] and many others have studied this concept by use of the own techniques of the authors.

The present paper can very well be considered as a continuation of the valuable investigations done so far; but it can very well demand its own originality in the methodology used for its development.

For a topological space  $(X, \tau)$ , the semiregularization topology  $\tau_s$  [2] is a known concept; it is well known that the base  $\mathcal{B}$  for  $\tau_s$  is given by  $\mathcal{B} = \{\text{intcl}U : U \in \tau\}$  ('int' and 'cl' stand for the 'interior' and 'closure' respectively in the space  $(X, \tau)$ ). In building the proofs of most of the results in the paper, we have been aware that near paracompactness of a topological space  $(X, \tau)$  is simply the paracompactness of  $(X, \tau_s)$ .

The characterization of near paracompactness has been the motif although; the notions of cushioned refinement, locally starring and partition of unity have appeared with relevance. However, the most striking characterization has come via the use of the theory of selections due to Michael [8].

Besides characterizations via several appliances, a few properties of near paracompactness have also been demonstrated.

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In what follows, by a space  $X$  we will mean a topological space  $(X, \tau)$  endowed with a topology  $\tau$  (say). A subset  $A$  of a space  $X$  is called regular open if  $A = \text{intcl } A$ ; the complements of such sets are called regular closed. We shall sometime write  $A^*$  for  $\text{intcl } A$ . For a space  $(X, \tau)$ ,  $RO(X)$  and  $RC(X)$  will represent respectively the collections of all regular open and regular closed sets in  $X$ . Since  $\text{intcl} : P(X) \rightarrow P(X)$ , where  $P(X)$  denotes the power set of  $X$ , is an idempotent operator, we have  $RO(X) = \{ \text{intcl } A : A \subseteq X \}$ . For a space  $(X, \tau)$ , the semiregularization space  $X$  endowed with the topology  $\tau_s$ , will be denoted by  $(X)_s$ . A subset  $A$  of  $X$  is called  $\delta$ -open [16] if  $A \in \tau_s$  (i.e., if and only if  $A$  is open in  $(X)_s$ ); the complement of a  $\delta$ -open set is called a  $\delta$ -closed set. We shall need the following lemma often in the sequel.

LEMMA 1.1. *If  $A, B$  are open sets in a space  $(X, \tau)$ , then  $A \cap B \neq \phi \Leftrightarrow A^* \cap B^* \neq \phi$ , and hence  $(X, \tau)$  is  $T_2$  if and only if  $(X)_s$  is  $T_2$ .*

For any open cover  $\mathcal{C}$  of a space  $(X, \tau)$  let us write  $\mathcal{C}^\# = \{A^* : A \in \mathcal{C}\}$ . Now, since for any open set  $A$  in  $X$ ,  $A \subseteq A^*$  holds, it is easy to see that  $\mathcal{C}^\#$  is also a cover of  $X$ , and also that  $\mathcal{C}^\# \subseteq RO(X)$ .

LEMMA 1.2. Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are two open covers of a space  $X$  with  $\mathcal{B} \subseteq RO(X)$ . If  $\mathcal{A}$  is a locally finite refinement of  $\mathcal{B}$ , then  $\mathcal{A}^\#$  is an  $(X)_s$ -locally finite refinement of  $\mathcal{B}$ .

*Proof.* Obviously  $\mathcal{A}^\#$  is a cover of  $X$  and also  $\mathcal{A}^\#$  is a refinement of  $\mathcal{B}$ . The local finiteness of  $\mathcal{A}^\#$  follows in view of Lemma 1.1. ■

DEFINITION 1.3. [3] Suppose  $\mathcal{B} \subseteq P(X)$  and  $A \subseteq X$ . We define  $St(A, \mathcal{B}) = \bigcup \{B \in \mathcal{B} : A \cap B \neq \phi\}$ , called the star of the set  $A$  with respect to the family  $\mathcal{B}$ . For any open cover  $\mathcal{B}$  of  $X$ , we set the notation  $\mathcal{B}^* = \{St(B, \mathcal{B}) : B \in \mathcal{B}\}$ .

LEMMA 1.4. *Suppose  $U$  is an open set in  $X$  and  $\mathcal{A}$  is a family of open subsets of  $X$ . If  $B \in RO(X)$  and  $St(U, \mathcal{A}) \subseteq B$ , then  $St(U^*, \mathcal{A}^\#) \subseteq B$ .*

*Proof.* Let  $\mathcal{C} = \{A : U \cap A \neq \phi, A \in \mathcal{A}\}$  and  $\mathcal{D} = \{A^* : U \cap A^* \neq \phi, A \in \mathcal{A}\}$ . Then  $St(U, \mathcal{A}) = \bigcup \mathcal{C}$  and  $St(U^*, \mathcal{A}^\#) = \bigcup \mathcal{D}$ . Now,  $A \in \mathcal{C} \Leftrightarrow U \cap A \neq \phi \Leftrightarrow U^* \cap A^* \neq \phi$  (by Lemma 1.1)  $\Leftrightarrow A^* \in \mathcal{D}$ . Thus  $\mathcal{D} = \{A^* : A \in \mathcal{C}\}$ . Therefore, for each  $A \in \mathcal{C}$  and  $B \in RO(X)$  which satisfies  $St(U, \mathcal{A}) \subseteq B$  we can easily get that  $A^* \subseteq B$ . Hence,  $\bigcup \{A^* : A \in \mathcal{C}\} \subseteq B$  and this implies that  $St(U^*, \mathcal{A}^\#) \subseteq B$ . ■

LEMMA 1.5. Suppose  $U$  is an open set in  $X$  and  $\mathcal{A}$  is a family of open subsets of  $X$ . If  $B \in RO(X)$  and  $St(U, \mathcal{A}) \subseteq B$ , then  $St(U, \mathcal{A}^\#) \subseteq B$ .

*Proof.* Follows from Lemma 1.4 and the fact that  $U \subseteq U^*$ , for any open set  $U$  in  $X$ . ■

## 2. Near paracompactness and almost regularity

We begin with the well known definition of a paracompact space: A space  $X$  is called paracompact if every open cover of  $X$  has a locally finite open refinement. E. Michael [7] proved the following characterizations of regular paracompact spaces.

THEOREM 2.1. *For a regular space  $X$ , the following are equivalent:*

- (a)  $X$  is paracompact.
- (b) Every open cover of  $X$  has a  $\sigma$ -locally finite open refinement (i.e., each open cover of  $X$  has an open refinement that can be decomposed into countably many locally finite families).
- (c) Every open cover of  $X$  has a locally finite closed refinement.
- (d) Every open cover of  $X$  has a locally finite refinement consisting of sets not necessarily closed or open.

In [13] Singal and Arya introduced the notion of almost regularity and in [14] the same authors initiated the idea of near paracompactness as a generalized concept of paracompactness. In the latter paper they proved some characterizations similar to those obtained by E. Michael in [7], for almost regular spaces.

DEFINITION 2.2. [13] A space  $X$  is called almost regular if for any  $A \in RC(X)$  and any  $x \in X \setminus A$ , there exist disjoint open sets  $U$  and  $V$  in  $X$  such that  $x \in U$  and  $A \subseteq V$ .

DEFINITION 2.3. [14] A space  $X$  is called nearly paracompact if every regular open cover of  $X$  has a locally finite open refinement.

In the next two results we observe that almost regularity and near paracompactness of a space  $X$  are nothing but the regularity and paracompactness respectively of  $(X)_s$ .

THEOREM 2.4. *A space  $X$  is almost regular if and only if  $(X)_s$  is regular.*

*Proof.* First assume that  $X$  is almost regular. Now,  $RC(X)$  is a base for the closed sets in  $(X)_s$ . Thus to show that  $(X)_s$  is regular, it is sufficient to prove that any member  $A$  of  $RC(X)$  and any point  $x \in X \setminus A$  are strongly separated by open sets of  $(X)_s$ . Now by almost regularity of  $X$ , there exist disjoint open sets  $U, V$  in  $X$  such that  $x \in U$  and  $A \subseteq V$ . Then  $x \in U \subseteq U^*$ ,  $A \subseteq V \subseteq V^*$  and in view of Lemma 1.1,  $U^* \cap V^* = \phi$ . This shows that  $U^*$  and  $V^*$  strongly separate  $x$  and  $A$ , where  $U^*, V^*$  are open in  $(X)_s$ .

The converse is clear. ■

NOTE 2.5. It is not difficult to show that a space  $X$  is almost regular if and only if  $(X)_s$  is so. Thus using Corollary 3.1 of [13] one can arrive at the result in Theorem 2.4 above.

THEOREM 2.6. *A space  $X$  is nearly paracompact if and only if  $(X)_s$  is paracompact.*

*Proof.* The proof follows immediately from the fact that for a locally finite open refinement  $\mathcal{B}$  of a regular open cover  $\mathcal{A}$ ,  $\mathcal{B}^\#$  is an  $(X)_s$ -locally finite refinement of  $\mathcal{A}$  ( by Lemma 1.2 ) with  $\mathcal{B}^\# \subseteq RO(X)$ . ■

We now give the following characterizations of near paracompactness for almost regular spaces. These results or similar versions are obtained also by Singal and Arya in [14]. However, we give much simpler proofs as direct consequences of Theorems 2.1, 2.4 and 2.6.

**THEOREM 2.7.** *For an almost regular space  $X$ , the following are equivalent:*

- (a)  $X$  is nearly paracompact.
- (b) Every regular open cover of  $X$  has a  $\sigma$ -locally finite open refinement.
- (c) Every regular open cover of  $X$  has an  $(X)_s$ -locally finite  $\delta$ -closed refinement.
- (d) Every regular open cover of  $X$  has an  $(X)_s$ -locally finite refinement consisting of sets of any type.

*Proof.* Since  $X$  is almost regular,  $(X)_s$  is regular and hence Theorem 2.1 can be applied for the space  $(X)_s$ .

(a)  $\Leftrightarrow$  (b): ‘(a)  $\Rightarrow$  (b)’ is obvious; we only prove (b)  $\Rightarrow$  (a).

Let  $\mathcal{A}$  be a regular open cover of  $X$ . Then by (b), there exists a  $\sigma$ -locally finite open refinement  $\mathcal{B}$  of  $\mathcal{A}$ . Let  $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ , where each  $\mathcal{B}_n$  is a locally finite family. Then  $\mathcal{B}_n^\#$  is an  $(X)_s$ -locally finite family, for all  $n \in \mathbb{N}$  (= the set of naturals) and hence  $\mathcal{B}^\# = \bigcup_{n=1}^{\infty} \mathcal{B}_n^\#$  becomes a  $\sigma$ -locally finite open refinement in  $(X)_s$  for the cover  $\mathcal{A}$ . Thus by Theorem 2.1,  $(X)_s$  becomes paracompact and therefore by Theorem 2.6,  $X$  is nearly paracompact.

(a)  $\Rightarrow$  (c)  $\Rightarrow$  (d): If  $X$  is nearly paracompact, then  $(X)_s$  is paracompact. Then by Theorem 2.1, ‘(a)  $\Rightarrow$  (c)’ follows. The implication ‘(c)  $\Rightarrow$  (d)’ is obvious.

(d)  $\Rightarrow$  (a): If (d) is true, then by Theorem 2.1,  $(X)_s$  is paracompact and hence  $X$  is nearly paracompact.

### 3. Cushioned refinement, locally starring and near paracompactness

**DEFINITION 3.1.** [11] Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are two covers of  $X$  by means of subsets of  $X$ . Then  $\mathcal{A}$  is called a cushioned refinement of  $\mathcal{B}$  if there exists a mapping  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  such that for each subfamily  $\mathcal{A}_0$  of  $\mathcal{A}$ ,  $\text{cl}(\bigcup\{A : A \in \mathcal{A}_0\}) \subseteq \bigcup\{\varphi(A) : A \in \mathcal{A}_0\}$ .  $\mathcal{A}$  is called a  $\sigma$ -cushioned refinement of  $\mathcal{B}$  if  $\mathcal{A}$  can be decomposed into countably many families, each of which is a cushioned refinement of  $\mathcal{B}$ .

Let us recall the following well known result (see Nagata [11]).

**THEOREM 3.2.** *A Hausdorff space  $X$  is paracompact if and only if every open cover of  $X$  has a  $\sigma$ -cushioned open refinement.*

We now have the following analogous characterization for nearly paracompact spaces:

**THEOREM 3.3.** *A Hausdorff space  $X$  is nearly paracompact if and only if every regular open cover of  $X$  has a  $\sigma$ -cushioned open refinement.*

*Proof.* Since  $X$  is Hausdorff, so is  $(X)_s$  (by Lemma 1.1). Thus  $X$  is nearly paracompact  $\Leftrightarrow (X)_s$  is paracompact  $\Rightarrow$  every regular open cover of  $X$  has a  $\sigma$ -cushioned  $(X)_s$ -open (and hence open) refinement (by Theorem 3.2), this is due to the fact that  $RO(X)$  makes an open base for the topology of  $(X)_s$  and that  $\text{cl}A = (X)_s\text{-cl}(A)$  (the closure of  $A$  in the space  $(X)_s$ ) for every open set  $A$  in  $X$ .

The converse follows similarly by using the fact that for any collection  $\{U_\alpha : \alpha \in \Lambda\}$  of open sets, the closure in  $(X)_s$  of  $\bigcup_{\alpha \in \Lambda} \text{intcl } U_\alpha$  equals the closure (in  $X$ ) of  $(\bigcup_{\alpha \in \Lambda} U_\alpha)$ . ■

**DEFINITION 3.4.** [3] Let  $\mathcal{A}$  be a cover of  $X$  by means of subsets of  $X$ . A sequence  $\{\mathcal{A}_n : n \in \mathbb{N}\}$  of open covers of  $X$  is called a locally starring for  $\mathcal{A}$ , if for each  $x \in X$ , there exist an open neighbourhood  $V$  of  $x$  and an  $n \in \mathbb{N}$  such that  $St(V, \mathcal{A}_n) \subseteq A$ , for some  $A \in \mathcal{A}$ .

The following characterization of paracompactness was proved by Arhangel'skii and can be found in [3].

**THEOREM 3.5.** *A Hausdorff space  $X$  is paracompact if and only if for each open cover  $\mathcal{A}$  of  $X$ , there is a sequence  $\{\mathcal{A}_n : n \in \mathbb{N}\}$  of open covers of  $X$  that is locally starring for  $\mathcal{A}$ .*

An analogue of the above theorem for near paracompactness goes as follows:

**THEOREM 3.6.** *A Hausdorff space  $X$  is nearly paracompact if and only if for every regular open cover  $\mathcal{A}$  of  $X$ , there is a sequence  $\{\mathcal{A}_n : n \in \mathbb{N}\}$  of open covers of  $X$  that is locally starring for  $\mathcal{A}$ .*

*Proof.* First let  $X$  be nearly paracompact and let  $\mathcal{A} \subseteq RO(X)$  be a cover of  $X$ . In view of Theorem 2.6 and Lemma 1.1, it follows that  $(X)_s$  is paracompact and  $T_2$  and hence, by Theorem 3.5, there exists a sequence  $\{\mathcal{A}_n : n \in \mathbb{N}\}$  of  $\delta$ -open covers of  $X$  that is locally starring for  $\mathcal{A}$ . Since  $\delta$ -open sets are open in  $X$ , the necessity follows.

Conversely, let  $\mathcal{A} \subseteq RO(X)$  be a cover of  $X$ . By hypothesis, there is a sequence  $\{\mathcal{A}_n : n \in \mathbb{N}\}$  of open covers of  $X$  that is locally starring for  $\mathcal{A}$ . Then  $\{\mathcal{A}_n^\# : n \in \mathbb{N}\}$  is also a sequence of open covers of  $X$  with  $\mathcal{A}_n^\# \subseteq RO(X)$ , for all  $n \in \mathbb{N}$ . We claim that  $\{\mathcal{A}_n^\# : n \in \mathbb{N}\}$  is  $(X)_s$ -locally starring for  $\mathcal{A}$ . For, let  $x \in X$  be arbitrary. Since  $\{\mathcal{A}_n : n \in \mathbb{N}\}$  is locally starring for  $\mathcal{A}$ , there exist an open neighbourhood  $V$  of  $x$  in  $X$  and an  $n \in \mathbb{N}$  such that  $St(V, \mathcal{A}_n) \subseteq A$ , for some  $A \in \mathcal{A}$ . Since  $A \in RO(X)$ , by Lemma 1.4, it follows that  $St(V^*, \mathcal{A}_n^\#) \subseteq A$ , where  $V^*$  is an open neighbourhood of  $x$  in  $(X)_s$ . Therefore,  $\{\mathcal{A}_n^\# : n \in \mathbb{N}\}$  is  $(X)_s$ -locally starring for  $\mathcal{A}$ . Since  $RO(X)$  is an open base for the topology of  $(X)_s$ , by Theorem 3.5 it follows that  $(X)_s$  is paracompact and hence in view of Theorem 2.6,  $X$  becomes nearly paracompact. ■

#### 4. Near paracompactness via partition of unity and selection theory

In [4] there is a nice characterization of paracompactness by using the concept of partition of unity which states that a space  $X$  is paracompact and  $T_2$  if and only if

it is  $T_1$  and every open cover of  $X$  has a locally finite partition of unity subordinate to it. To make things clear we recall the following well known definition.

DEFINITION 4.1. [4] A family  $\{f_\alpha : \alpha \in \Lambda\}$  of continuous maps over a space  $X$  with entries in  $[0, 1]$  is called a partition of unity, if  $\sum_{\alpha \in \Lambda} f_\alpha(x) = 1$ , where  $\sum_{\alpha \in \Lambda} f_\alpha(x) = \sup \{\sum_{\alpha \in \Lambda_0} f_\alpha(x) : \Lambda_0 \text{ is a finite subset of } \Lambda\}$ . If  $\{f_\alpha : \alpha \in \Lambda\}$  is a partition of unity, it is easy to see that  $\{f_\alpha^{-1}((0, 1]) : \alpha \in \Lambda\}$  is an open cover of  $X$ . If, in addition, this open cover is locally finite, then we say that the partition of unity is locally finite. If  $\{f_\alpha^{-1}((0, 1]) : \alpha \in \Lambda\}$  refines a cover  $\mathcal{A}$  of  $X$ , then we say that the partition of unity is subordinate to the cover  $\mathcal{A}$ .

Our intention now is to characterize a nearly paracompact space in terms of partition of unity. For that we recall the following two results which may be found in [4].

THEOREM 4.2. *If an open cover  $\mathcal{A}$  of a space  $X$  has a partition of unity subordinate to it, then  $\mathcal{A}$  has a locally finite open refinement.*

THEOREM 4.3. *If  $X$  is paracompact and Hausdorff, then every open cover of  $X$  has a locally finite partition of unity subordinate to it.*

THEOREM 4.4. *A Hausdorff space  $X$  is nearly paracompact if and only if every regular open cover of  $X$  has a locally finite partition of unity subordinate to it.*

*Proof.* The sufficiency follows by use of Theorem 4.3.

Conversely, let  $\mathcal{A} \subseteq RO(X)$  be a cover of  $X$ . Since  $X$  is nearly paracompact and Hausdorff, in view of Theorem 2.6 and Lemma 1.1 it follows that  $(X)_s$  is paracompact and Hausdorff. Then by Theorem 4.3, there exists a family  $\{f_\alpha : \alpha \in \Lambda\}$  of continuous maps defined over  $(X)_s$  with values in  $[0, 1]$ , which is a partition of unity with the property that  $\{f_\alpha^{-1}((0, 1]) : \alpha \in \Lambda\}$  is locally finite and refines  $\mathcal{A}$ . Since the topology of  $X$  is finer than that of  $(X)_s$ , the identity map  $i : X \rightarrow (X)_s$  is continuous. Thus each  $i \circ f_\alpha$  is a continuous map from  $X$  to  $[0, 1]$  and hence the theorem is proved. ■

Finally we are going to prove the desired characterization of near paracompactness using the theory of selections, first initiated by E. Michael [8]. Before that we need the following definition.

DEFINITION 4.5. [8] *Let  $X$  and  $Y$  be two topological spaces and  $\mathcal{B}$  a family of some nonempty subsets of  $Y$ . Any mapping  $\psi : X \rightarrow \mathcal{B}$  is called a carrier of  $X$  into  $\mathcal{B}$ . A carrier  $\psi : X \rightarrow \mathcal{B}$  is called lower semicontinuous (l.s.c. in short), if for any open set  $V$  in  $Y$ , the set  $\psi^{-1}(V) = \{x \in X : \psi(x) \cap V \neq \emptyset\}$  is open in  $X$ . A continuous map  $f : X \rightarrow Y$  is called a continuous selection (or simply a selection) for  $\psi$ , if  $f(x) \in \psi(x)$ , for all  $x \in X$ .*

The following is a well known characterization of paracompactness due to E. Michael [8].

**THEOREM 4.6.** *A  $T_1$  space  $X$  is paracompact and  $T_2$  if and only if for every Banach space  $Y$  and every l.s.c. carrier  $\psi : X \rightarrow \overline{\text{Conv}}(Y)$ , there exists a continuous selection  $f : X \rightarrow Y$  for  $\psi$ , where  $\overline{\text{Conv}}(Y)$  is the family of all non-void closed convex subsets of  $Y$ .*

We shall try to get an analogous version of Theorem 4.6 for nearly paracompact spaces. For that we set the following definition:

**DEFINITION 4.7.** A carrier  $\psi : X \rightarrow \mathcal{B}$  is called strongly l.s.c., if for any open set  $V$  in  $Y$ , the set  $\psi^{-}(V) = \{x \in X : \psi(x) \cap V \neq \phi\}$  is  $\delta$ -open in  $X$ .

**REMARK 4.8.** Obviously, any strongly l.s.c. carrier is an l.s.c. carrier; that the converse is false is shown by the following example.

**EXAMPLE 4.9.** Let  $\mathbb{R}$  be the set of real numbers. Let  $\tau_1$  and  $\tau_2$  respectively denote the co-countable and the co-finite topologies on  $\mathbb{R}$ . Let  $X = (\mathbb{R}, \tau_1)$  and  $Y = (\mathbb{R}, \tau_2)$ . Let  $\mathcal{B}$  be the family of all singletons of  $Y$ . Then the carrier  $\psi : X \rightarrow \mathcal{B}$  defined by  $\psi(x) = \{x\}$ , for all  $x \in X$ , is l.s.c. but not strongly l.s.c.

**REMARK 4.10.** From the very definition of a strongly l.s.c. carrier, it follows that if  $\psi : X \rightarrow \mathcal{B}$  is a strongly l.s.c. carrier, then  $\psi : (X)_s \rightarrow \mathcal{B}$  is an l.s.c. carrier and conversely.

The following is a necessary and sufficient condition for a carrier to be strongly an l.s.c. carrier.

**LEMMA 4.11.** *A carrier  $\psi : X \rightarrow \mathcal{B}$  is strongly l.s.c. if and only if for every  $x \in X$ ,  $y \in \psi(x)$  and every open neighbourhood  $V$  of  $y$  in  $Y$ , there exists a  $\delta$ -open neighbourhood  $U$  of  $x$  in  $X$  such that  $(z \in U \Rightarrow \psi(z) \cap V \neq \phi)$ .*

*Proof.* Suppose  $\psi$  is a strongly l.s.c. carrier,  $x \in X$ ,  $y \in \psi(x)$  and  $V$  is an open neighbourhood of  $y$  in  $Y$ . Since  $\psi$  is strongly l.s.c.,  $\psi^{-}(V)$  is  $\delta$ -open in  $X$ . Let  $U = \psi^{-}(V)$ . Since  $y \in \psi(x)$  and  $y \in V$ , we have  $\psi(x) \cap V \neq \phi \Rightarrow x \in \psi^{-}(V) = U$ . Thus  $U$  is a  $\delta$ -open neighbourhood of  $x$  in  $X$ . Now,  $z \in U \Rightarrow z \in \psi^{-}(V) \Rightarrow \psi(z) \cap V \neq \phi$ .

Conversely, assume that the given condition is satisfied by a carrier  $\psi : X \rightarrow \mathcal{B}$  and  $V$  is an open set in  $Y$ . To show that  $\psi^{-}(V)$  is  $\delta$ -open in  $X$ , let  $x \in \psi^{-}(V)$ . Then  $\psi(x) \cap V \neq \phi$ . Thus we can choose some  $y \in \psi(x) \cap V$ . Therefore,  $y \in \psi(x)$  and  $V$  is an open neighbourhood of  $y$  in  $Y$ . By hypothesis, there exists a  $\delta$ -open neighbourhood  $U$  of  $x$  in  $X$  such that  $z \in U \Rightarrow \psi(z) \cap V \neq \phi \Rightarrow z \in \psi^{-}(V)$ . Thus  $x \in U \subseteq \psi^{-}(V)$ , where  $U$  is a  $\delta$ -open set in  $X$ . Since  $x \in \psi^{-}(V)$  is arbitrary, it follows that  $\psi^{-}(V)$  is  $\delta$ -open in  $X$  and hence  $\psi$  becomes strongly l.s.c.. ■

We are now in a position to give our final characterization of near paracompactness, one part of which is simple and we shall prove it first.

**THEOREM 4.12.** *If  $X$  is a  $T_2$ , nearly paracompact space and  $\overline{\text{Conv}}(Y)$  is the collection of all nonempty closed, convex subsets of a Banach space  $Y$ , then any strongly l.s.c. carrier  $\psi : X \rightarrow \overline{\text{Conv}}(Y)$  has a continuous selection  $f : X \rightarrow Y$ .*

*Proof.* Since  $X$  is nearly paracompact and  $T_2$ , by Theorem 2.6 and Lemma 1.1 it follows that  $(X)_s$  is paracompact and  $T_2$ . Then by Theorem 4.6 and Remark

4.10, there is a continuous selection  $f : (X)_s \rightarrow Y$  for  $\psi$ . Since the topology of  $X$  is finer than that of  $(X)_s$ ,  $f$  when regarded as a map from  $X$  to  $Y$ , is also continuous and hence the theorem is proved.

We now prove the converse part of the above result:

**THEOREM 4.13.** *If for every Banach space  $Y$  and every strongly l.s.c. carrier  $\psi : X \rightarrow \overline{\text{Conv}}(Y)$ , there is a continuous selection  $f : X \rightarrow Y$  for  $\psi$ , then  $X$  is nearly paracompact.*

*Proof.* Let  $\mathcal{U}$  be a regular open cover of  $X$ . Let  $Y = l_1(\mathcal{U})$ , where  $l_1(\mathcal{U})$  consists of all real-valued functions  $y$  defined over  $\mathcal{U}$  satisfying  $\Sigma\{|y(U)| : U \in \mathcal{U}\} < \infty$ , where as usual,  $\Sigma\{|y(U)| : U \in \mathcal{U}\} = \sup\{\Sigma_{U \in \mathcal{U}_0} |y(U)| : \mathcal{U}_0 \text{ is a finite subset of } \mathcal{U}\}$ . Then  $Y$  is a vector space over the field of real numbers, where the relevant compositions are defined pointwise. Define  $\|\cdot\| : Y \rightarrow \mathbb{R}$  by  $\|y\| = \Sigma\{|y(U)| : U \in \mathcal{U}\}$ , for all  $y \in Y$ . It is easy to verify that  $\|\cdot\|$  defines a norm on  $Y$  under which  $Y$  becomes a Banach space. Let

$$C = \{y \in Y : y(U) \geq 0, \forall U \in \mathcal{U} \text{ and } \Sigma\{|y(U)| : U \in \mathcal{U}\} = 1\}.$$

We show that  $C$  is a closed and convex subset of  $Y$ . Since the convexity of  $C$  is clear, we only prove that  $C$  is a closed subset of  $Y$ . From the definition of  $C$ , it follows that  $C = A \cap \{y \in Y : \|y\| = 1\}$ , where  $A = \{y : y(U) \geq 0, \forall U \in \mathcal{U}\}$ . Since  $\|\cdot\| : Y \rightarrow \mathbb{R}$  is continuous,  $\{y \in Y : \|y\| = 1\}$  is closed in  $Y$ . So, it is sufficient to show that  $A$  is closed in  $Y$ . Let  $z$  be a limit point of  $A$ . We claim that  $z(U) = 0, \forall U \in \mathcal{U}$ . If not, there is some  $U \in \mathcal{U}$  such that  $z(U) < 0$ , i.e.,  $z(U) = -\epsilon$ , for some  $\epsilon > 0$ . Now, for any  $y \in A$ , we have  $\|z - y\| \geq |z(U) - y(U)| = |-\epsilon - y(U)| = \epsilon + y(U) \geq \epsilon$  which implies that  $z$  is not a limit point of  $A$ , a contradiction. Thus,  $z(U) = 0$ , for all  $U \in \mathcal{U}$  and hence  $z \in A$ . Therefore,  $A$  and hence  $C$  becomes closed in  $Y$ . For each  $x \in X$ , let  $C(x) = \{y \in Y : y(U) = 0 \text{ if } x \notin U, \text{ where } U \in \mathcal{U}\}$ . We also show that  $C(x)$  is a closed and convex subset of  $Y$ , for each  $x \in X$ . The convexity being again obvious, we only show that  $C(x)$  is closed in  $Y$ . Let  $z$  be a limit point of  $C(x)$ . We shall show that  $z \in C(x)$ . For that, let  $U \in \mathcal{U}$  be such that  $x \notin U$ . If we can show that  $z(U) = 0$ , then we are done. In order to prove that  $z(U) = 0$ , it is sufficient to show that  $|z(U)| < 1/n$ , for all  $n \in \mathbb{N}$ . So, let  $n \in \mathbb{N}$  be arbitrary. Since  $z$  is a limit point of  $C(x)$ , there is some  $y_n \in C(x)$  such that  $\|z - y_n\| < 1/n$ , i.e.,  $|z(U) - y_n(U)| < 1/n$ . Since  $y_n \in C(x)$  and  $x \notin U$ , we have  $y_n(U) = 0$  and hence  $|z(U)| < 1/n$ . Thus,  $C(x)$  is closed in  $Y$ , for each  $x \in X$ . Since the intersection of two closed and convex subsets is again closed and convex, it follows that  $C \cap C(x)$  is a closed and convex subset of  $Y$ , for each  $x \in X$ . We now show that  $C \cap C(x) \neq \phi$ , for each  $x \in X$ . For each  $U_0 \in \mathcal{U}$ , define  $y_{U_0} : \mathcal{U} \rightarrow \mathbb{R}$  by  $y_{U_0}(U_0) = 1$  and  $y_{U_0}(U) = 0$ , for all  $U \in \mathcal{U}$  with  $U \neq U_0$ . Then  $y_{U_0} \in C$ , for all  $U_0 \in \mathcal{U}$ . Let  $x \in X$  be arbitrary. Since  $\mathcal{U}$  is a cover of  $X$ , there is some  $U_0 \in \mathcal{U}$  such that  $x \in U_0$  and hence it is easy to see that  $y_{U_0} \in C(x)$ . Therefore,  $y_{U_0} \in C \cap C(x)$ . Thus,  $C \cap C(x) \neq \phi$ , for each  $x \in X$  and hence  $C \cap C(x) \in \overline{\text{Conv}}(Y), \forall x \in X$ . Let us consider the carrier  $\psi : X \rightarrow \overline{\text{Conv}}(Y)$  defined by  $\psi(x) = C \cap C(x), \forall x \in X$ . We first show that for any  $y \in C$  and any



$\epsilon > 0$ , there exist  $y' \in C$  and  $U_1, U_2, \dots, U_k \in \mathcal{U}$  satisfying

$$\|y - y'\| < \epsilon, y(U_i) > 0 \ (i = 1, 2, \dots, k) \text{ and } y'(U) = 0, \text{ if } U \neq U_i \ (i = 1, 2, \dots, k) \quad (*)$$

To prove this, let  $y \in C$  and  $\epsilon > 0$ . Since  $y \in C$ , we have  $y(U) \geq 0$ , for all  $U \in \mathcal{U}$  and  $\sum\{y(U) : U \in \mathcal{U}\} = 1$ . So we can choose  $U_1, U_2, \dots, U_k \in \mathcal{U}$  such that  $y(U_1) + y(U_2) + \dots + y(U_k) > 1 - \epsilon/2$  with  $y(U_i) > 0$  ( $i = 1, 2, \dots, k$ ). Let  $\delta = y(U_1) + y(U_2) + \dots + y(U_k)$ . Then  $\delta \leq 1$  and  $2(1 - \delta) < \epsilon$ . Let us define  $y' : \mathcal{U} \rightarrow \mathbb{R}$  as follows :  $y'(U) = 0$ , if  $U \neq U_i$  ( $i = 1, 2, \dots, k$ );  $y'(U_i) = y(U_i)$  ( $i = 2, 3, \dots, k$ ) and  $y'(U_1) = y(U_1) + 1 - \delta$ . Since  $y(U) \geq 0$ , for all  $U \in \mathcal{U}$  and  $\delta \leq 1$ , it follows that  $y'(U) \geq 0, \forall U \in \mathcal{U}$ . Also,

$$\begin{aligned} \sum\{y'(U) : U \in \mathcal{U}\} &= y'(U_1) + y'(U_2) + \dots + y'(U_k) \\ &= y(U_1) + 1 - \delta + y(U_2) + \dots + y(U_k) \\ &= y(U_1) + y(U_2) + \dots + y(U_k) + 1 - \delta \\ &= \delta + 1 - \delta = 1. \end{aligned}$$

Hence  $y' \in C$ . Now,

$$\begin{aligned} \|y - y'\| &= \sum\{|y(U) - y'(U)| : U \in \mathcal{U}\} \\ &= |y(U_1) - y'(U_1)| + \sum_{j=2}^k |y(U_j) - y'(U_j)| \\ &\quad + \sum\{y(U) : U \neq U_i, i = 1, 2, \dots, k\} \\ &= (1 - \delta) + 0 + (1 - \delta) \\ &= 2(1 - \delta) < \epsilon. \end{aligned}$$

This shows that  $y'$  is the desired member of  $C$ .

To show that  $\psi$  is strongly l.s.c., let  $x \in X$ ,  $y \in \psi(x)$  and  $S_\epsilon(y)$  be an  $\epsilon$ -neighbourhood of  $y$  in  $Y$ , where  $\epsilon > 0$ . Since  $\psi(x) = C \cap C(x)$ , we have  $y \in C$ .

So, we can choose  $y' \in C$  and  $U_1, U_2, \dots, U_k \in \mathcal{U}$  satisfying the conditions of (\*). Since  $y(U_i) > 0$  ( $i = 1, 2, \dots, k$ ) and  $y \in \psi(x) \subseteq C(x)$ , it follows that  $x \in U_i$  for  $i = 1, 2, \dots, k$ . Let  $U_0 = U_1 \cap U_2 \cap \dots \cap U_k$ . Then  $U_0$  is a  $\delta$ -open neighbourhood of  $x$  in  $X$ . Let  $z \in U_0$  be arbitrary. If we can show that  $\psi(z) \cap S_\epsilon(y) \neq \phi$ , then by Lemma 4.11, it will follow that  $\psi$  is strongly l.s.c. We claim that  $y' \in C(z)$ . For, let  $U \in \mathcal{U}$  be such that  $z \notin U$ . Since  $z \in U_0$ ,  $U \neq U_i$  ( $i = 1, 2, \dots, k$ ) and hence by (\*)  $y'(U) = 0$  i.e.,  $y' \in C(z)$ . Since  $y' \in C$  as well, we have  $y' \in C \cap C(z) = \psi(z)$ . Also, by (\*),  $\|y - y'\| < \epsilon$  i.e.,  $y' \in S_\epsilon(y)$ . Therefore,  $y' \in \psi(z) \cap S_\epsilon(y)$  and hence  $\psi(z) \cap S_\epsilon(y) \neq \phi$ . Thus  $\psi$  becomes strongly l.s.c. So, there is a continuous selection  $f : X \rightarrow Y$  for  $\psi$ .

For each  $U \in \mathcal{U}$ , we define  $f_U : X \rightarrow \mathbb{R}$  by the rule  $f_U(x) = (f(x))(U)$ , for all  $x \in X$ . To show that  $f_U$  is continuous for each  $U \in \mathcal{U}$ , we first show that for each  $U \in \mathcal{U}$ , the map  $T_U : Y \rightarrow \mathbb{R}$  defined by  $T_U(y) = y(U)$ , for all  $y \in Y$ , is continuous. It is easy to see that  $T_U$  is a linear functional on  $Y$ , for each  $U \in \mathcal{U}$ . Now, for any  $y \in Y$ , we have  $|T_U(y)| = |y(U)| \leq \|y\|$ . Since  $T_U$  is linear, it

follows that  $T_U$  is continuous, for each  $U \in \mathcal{U}$ . Since  $f : X \rightarrow Y$  is continuous and  $T_U : Y \rightarrow \mathbb{R}$  is continuous, it follows that  $T_U \circ f : X \rightarrow \mathbb{R}$  is continuous, for each  $U \in \mathcal{U}$ . But, for any  $x \in X$ , we have  $(T_U \circ f)(x) = T_U(f(x)) = (f(x))(U) = f_U(x)$ , i.e.,  $T_U \circ f = f_U$ . Thus,  $f_U$  is continuous, for each  $U \in \mathcal{U}$ . Finally, we show that  $\{f_U : U \in \mathcal{U}\}$  is a partition of unity subordinate to  $\mathcal{U}$ . Since  $f$  is a selection for  $\psi$ ,  $f(x) \in \psi(x) \subseteq C$ , for all  $x \in X$ . Therefore, we have  $(f(x))(U) \geq 0$ , for all  $U \in \mathcal{U}$  and  $\sum\{(f(x))(U) : U \in \mathcal{U}\} = 1$  i.e.,  $f_U(x) \geq 0$ , for all  $U \in \mathcal{U}$  and  $\sum\{f_U(x) : U \in \mathcal{U}\} = 1$ . Thus,  $\{f_U : U \in \mathcal{U}\}$  is a partition of unity. To show that this partition of unity is subordinate to  $\mathcal{U}$ , we must show that  $\{f_U^-((0, 1]) : U \in \mathcal{U}\}$  refines  $\mathcal{U}$ . For that, let  $x \in f_U^-((0, 1])$ , where  $U \in \mathcal{U}$ . Then  $f_U(x) > 0$  i.e.,  $(f(x))(U) > 0$ . Since  $f(x) \in \psi(x) \subseteq C(x)$ , it follows that  $x \in U$ . Therefore,  $f_U^-((0, 1]) \subseteq U$ , for all  $U \in \mathcal{U}$ . Thus  $\{f_U : U \in \mathcal{U}\}$  is a partition of unity subordinate to  $\mathcal{U}$  and hence, by Theorem 4.2,  $\mathcal{U}$  has a locally finite open refinement. Therefore,  $X$  becomes nearly paracompact. ■

Combining the last two theorems, we have the following characterization of nearly paracompact spaces.

**THEOREM 4.14.** *A Hausdorff space  $X$  is nearly paracompact if and only if for every Banach space  $Y$  and every strongly l.s.c. carrier  $\psi : X \rightarrow \text{Conv}(Y)$ , there is a continuous selection  $f : X \rightarrow Y$  for  $\psi$ .*

## 5. Some properties of nearly paracompact spaces

In this section we wish to obtain some properties of nearly paracompact spaces, especially with regard to the concepts of near compactness and near Lindelöfness, the latter concepts being introduced by Singal and Mathur in [15] and Singal and Arya in [14] respectively.

**DEFINITION 5.1.** [15] A space  $X$  is called nearly compact if every regular open cover of  $X$  has a finite subcover.

**REMARK 5.2.** It is clear that a space  $X$  is nearly compact if and only if  $(X)_s$  is compact.

The following theorem was proved in [14]; however, we supply here a very short and alternative proof.

**THEOREM 5.3.** *The product of a nearly paracompact space  $X$  with a nearly compact space  $Y$  is nearly paracompact.*

*Proof.* If  $X$  is nearly paracompact and  $Y$  is nearly compact, then  $(X)_s$  is paracompact and  $(Y)_s$  is compact. Therefore,  $(X)_s \times (Y)_s$  is paracompact (see [3]). But  $(X)_s \times (Y)_s = (X \times Y)_s$  (see [5]). Thus  $X \times Y$  is nearly paracompact. ■

**DEFINITION 5.4.** [3] A space  $X$  is called  $\sigma$ -compact, if it is equal to a union of countably many compact subsets of it.

**THEOREM 5.5.** *If  $X$  is Hausdorff and nearly paracompact, and  $Y$  is an almost regular  $\sigma$ -compact space, then  $X \times Y$  is nearly paracompact.*

*Proof.* Since  $X$  is Hausdorff and nearly paracompact,  $(X)_s$  is Hausdorff and paracompact. Since  $Y$  is almost regular, by Theorem 2.4,  $(Y)_s$  is regular. Also, since  $Y$  is  $\sigma$ -compact and the topology of  $(Y)_s$  is weaker than that of  $Y$ ,  $(Y)_s$  is  $\sigma$ -compact. Thus  $(X)_s \times (Y)_s = (X \times Y)_s$  is paracompact and hence  $X \times Y$  becomes nearly paracompact. ■

**DEFINITION 5.6.** A space  $X$  is called almost separable if there exists a countable subset  $D$  of  $X$  such that  $A \cap D \neq \phi, \forall A \in RO(X) \setminus \{\phi\}$ .

**REMARK 5.7.** Obviously, a separable space is almost separable; but the converse is false. For, if  $X$  is an uncountable set which is endowed with the topology of countable complements, then  $X$  is almost separable but it is not separable.

**DEFINITION 5.8.** [14] A space  $X$  is called nearly Lindelöf, if every regular open cover of  $X$  has a countable subcover.

**REMARK 5.9.** Obviously, a Lindelöf space is nearly Lindelöf but the converse is false. For, take any uncountable cardinal number  $\alpha$  and consider a set  $X$  with  $|X| = 2^\alpha$ . Let  $\tau = \{A \subseteq X : |X \setminus A| \leq \alpha\} \cup \{\phi\}$ . Then  $\tau$  defines a topology on  $X$ , and  $X$  endowed with this topology is nearly Lindelöf but not Lindelöf.

**REMARK 5.10.** It is easy to see that  $X$  is almost separable ( nearly Lindelöf ) if and only if  $(X)_s$  is separable ( resp. Lindelöf ).

**THEOREM 5.11.** *An almost separable, nearly paracompact space is nearly Lindelöf.*

*Proof.* Since  $X$  is almost separable and nearly paracompact, by Remark 5.10 and earlier result,  $(X)_s$  is separable and paracompact, and hence  $(X)_s$  is Lindelöf. Thus  $X$  is nearly Lindelöf. ■

**THEOREM 5.12.** *An almost regular, nearly Lindelöf space is nearly paracompact.*

*Proof.* If  $X$  is almost regular and nearly Lindelöf, then  $(X)_s$  is regular and Lindelöf and hence  $(X)_s$  is paracompact. Thus  $X$  is nearly paracompact. ■

**THEOREM 5.13.** A nearly paracompact, countably compact space  $X$  is nearly compact.

*Proof.* As  $X$  is nearly paracompact,  $(X)_s$  is paracompact. Since  $X$  is countably compact and the topology of  $(X)_s$  is weaker than that of  $X$ , it follows that  $(X)_s$  is also countably compact. Thus  $(X)_s$  is paracompact and countably compact which implies that  $(X)_s$  is compact (see [3]). Hence  $X$  becomes nearly compact. ■

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