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WEIGHTED HANKEL OPERATORS AND MATRICES

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Abstract. In this paper, the notions of weighted Hankel matrix along with weighted Hankel operator S_ϕ^β, with $\phi \in L^\infty(\beta)$ on the space $L^2(\beta)$, $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ being a sequence of positive numbers with $\beta_0 = 1$, are introduced. It is proved that an operator on $L^2(\beta)$ is a weighted Hankel operator on $L^2(\beta)$ if and only if its matrix is a weighted Hankel matrix. Various properties of the weighted Hankel operators S_ϕ^β on $L^2(\beta)$ are also discussed.

1. Preliminaries and introduction

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for some $r > 0$. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the formal Laurent series (whether or not the series converges for any values of z). Define $\|f\|_\beta$ as

$$\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2.$$

The space $L^2(\beta)$ consists of all $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ for which $\|f\|_\beta < \infty$. The space $L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n \, \bar{b}_n {\beta_n}^2,$$

for $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$. $H^2(\beta)$ is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

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$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. Also, $L^\infty(\beta) \subseteq L^2(\beta)$. $H^\infty(\beta)$ denotes the set of formal Power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. We refer to [13] as well as the references therein, for the details of the spaces $L^2(\beta)$, $H^2(\beta)$, $L^\infty(\beta)$ and $H^\infty(\beta)$. If $\beta_n = 1$ for each $n \in \mathbb{Z}$, and the functions under considerations are complex-valued measurable functions defined over the unit circle $\mathbb{T}$ then these spaces coincide with the classical spaces $L^2(\mathbb{T})$, $H^2(\mathbb{T})$, $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$. If $\phi \in L^\infty(\beta)$ is given by $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, then we use the symbols $\overline{\phi}, \phi^*$ and $\widetilde{\phi}$ to represent the expressions $\overline{\phi}(z) = \sum_{n=-\infty}^{\infty} \overline{a}_n z^{-n}$, $\phi^*(z) = \sum_{n=-\infty}^{\infty} \overline{a}_n z^n$ and $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$ respectively.

Laurent operators or multiplication operators $M_\phi(f \mapsto \phi f)$ on $L^2(\mathbb{T})$ play a vital role in the theory of operators with their tendency of inducing various classes of operators. In the year 1911, O. Toeplitz [14] introduced the Toeplitz operators given as $T_\phi = PM_\phi$, where P is an orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ and much later in 1964, Brown and Halmos [4] studied algebraic properties of these operators. In 1980, Power [11] studied the Hankel operators $S_\phi = PJM_\phi$ defined on the space $H^2(\mathbb{T})$, where J is the reflection operator. Many more classes of operators are introduced and studied by mathematicians using Laurent operators, like, Slant Toeplitz operators, k^{th}-order slant Toeplitz operators, Slant Hankel operators, k^{th}-order slant Hankel operators, Essentially slant Hankel operators (see [1, 3, 8, 10] and the references therein). An alternative for the Fourier transforms is the wavelet transforms, which have many applications in data compression and to solve the differential equations. An interesting reference connecting the spectral properties of the slant Toeplitz operators with the smoothness of wavelets is [6].

The study of Laurent operators was extended to $L^2(\beta)$ by Shield [13] in the year 1974. The study over $L^2(\beta)$ is more interesting as well as demandable because of the tendency of these spaces to cover Bergman spaces, Hardy spaces and Dirichlet spaces (see [13]). Reference [13] provides a nice survey over the historical growth, details and applications of these spaces and also provides a comprehensive study of Laurent operators M_ϕ^β on $L^2(\beta)$ with the symbol $\phi \in L^\infty(\beta)$. We call this operator as weighted Laurent operator. In the present paper we consider these spaces with $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ as a sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for some $r > 0$.

Let $P^\beta : L^2(\beta) \mapsto H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. In the year 2005, Lauric [9] discussed the notion of weighted Toeplitz operators $T_\phi^\beta = P^\beta M_\phi^\beta$ on $H^2(\beta)$ and obtain a description of its commutant. Authors in [5], introduced the weighted Hankel operators H_ϕ^β on $H^2(\beta)$ and also provided their connection with weighted Toeplitz operators. Recently Arora and Kathuria [3] has extended the study to slant weighted Toeplitz operators.

In [7], the notion of Laurent matrix is discussed and Laurent operators on $L^2(\mathbb{T})$ are charcterized in terms of the Laurent matrices. In this paper we extend the definition of weighted Hankel operator to the space $L^2(\beta)$ and also introduce the notion of weighted Hankel matrix. Our theorem (Theorem 2.2) in the first section of the paper, extends the following result of Halmos ([7, Problem 193]) to these new notions.

THEOREM 1.1. *An operator A on $L^2(\mathbb{T})$ is a Laurent operator on $L^2(\mathbb{T})$ if and only if it's matrix with respect to the orthonormal basis $\{e_n(z) = z^n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ is a Laurent matrix.*

In the second section of the paper, an attempt is made to study the compactness, hyponormality and normality of the weighted Hankel operators on $L^2(\beta)$ and it is also obtained that if the product of two weighted Hankel operators is a weighted Hankel operator then the product must be zero.

2. Operators and matrices

For $\phi \in L^\infty(\beta)$ with formal Laurent series expression $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, the weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is given by

$$H_\phi^\beta e_j = \frac{1}{\beta_j}\sum_{n=0}^{\infty} a_{-n-j}\beta_{-n}e_n,\ j \geq 0.$$

However, this definition of weighted Hankel operator can be extended to the space $L^2(\beta)$ as follows.

A weighted Hankel operator S_ϕ^β on $L^2(\beta)$ is given by

$$S_\phi^\beta e_j = \frac{1}{\beta_j}\sum_{n=-\infty}^{\infty} a_{-n-j}\beta_{-n}e_n$$

for $j \in \mathbb{Z}$. The weighted Laurent operator $M_\phi^\beta(f \to \phi f)$, $\phi \in L^\infty(\beta)$ is a bounded operator with $\|M_\phi^\beta\| = \|\phi\|_\infty$ (see [13]). It is easily seen that $S_\phi^\beta = J^\beta M_\phi^\beta$ and $M_\phi^\beta = J^\beta S_\phi^\beta$, where $J^\beta : L^2(\beta) \to L^2(\beta)$ denotes the reflection operator defined as $Je_n = e_{-n}$ and $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ is the orthonormal basis of $L^2(\beta)$. This insures that S_ϕ^β is a bounded operator on $L^2(\beta)$. Also, as desired, $H_\phi^\beta = P^\beta J^\beta M_\phi^\beta\big|_{H^2(\beta)} = P^\beta S_\phi^\beta\big|_{H^2(\beta)}$. The reflection operator J^β on $L^2(\beta)$ is a nice example of the weighted Hankel operator on $L^2(\beta)$ induced by $\phi \equiv 1$. It is self-adjoint and self invertible i.e. $(J^\beta)^{-1} = J^\beta$.

The matrix representation of S_ϕ^β with respect to the orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ is given on the next page.

As the operators have a close connection with the matrices and in order to find a matrix having the tendency to characterize the weighted Hankel operator we introduce the following notion of weighted Hankel matrix.

DEFINITION 2.1. A doubly infinite matrix $[\alpha_{ij}]_{i,j\in\mathbb{Z}}$ is said to be a weighted Hankel matrix with respect to a sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ of positive real numbers if

$$\frac{\beta_j}{\beta_{-i}}\alpha_{ij} = \frac{\beta_{j+1}}{\beta_{-(i-1)}}\alpha_{i-1j+1},$$

for each $i, j \in \mathbb{Z}$.

$$
\left[\begin{array}{cccc|ccccc}
 & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
\cdots & a_3\frac{\beta_2}{\beta_{-1}} & a_2\frac{\beta_2}{\beta_0} & & a_1\frac{\beta_2}{\beta_1} & a_0\frac{\beta_2}{\beta_2} & a_{-1}\frac{\beta_2}{\beta_3} & a_{-2}\frac{\beta_2}{\beta_4} & \cdots \\
\cdots & a_2\frac{\beta_1}{\beta_{-1}} & a_1\frac{\beta_1}{\beta_0} & & a_0\frac{\beta_1}{\beta_1} & a_{-1}\frac{\beta_1}{\beta_2} & a_{-2}\frac{\beta_1}{\beta_3} & a_{-3}\frac{\beta_1}{\beta_4} & \cdots \\
\cdots & a_1\frac{\beta_0}{\beta_{-1}} & a_0\frac{\beta_0}{\beta_0} & & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & a_{-3}\frac{\beta_0}{\beta_3} & a_{-4}\frac{\beta_0}{\beta_4} & \cdots \\
\hline
\cdots & a_0\frac{\beta_{-1}}{\beta_{-1}} & a_{-1}\frac{\beta_{-1}}{\beta_0} & & a_{-2}\frac{\beta_{-1}}{\beta_1} & a_{-3}\frac{\beta_{-1}}{\beta_2} & a_{-4}\frac{\beta_{-1}}{\beta_3} & a_{-5}\frac{\beta_{-1}}{\beta_4} & \cdots \\
\cdots & a_{-1}\frac{\beta_{-2}}{\beta_{-1}} & a_{-2}\frac{\beta_{-2}}{\beta_0} & & a_{-3}\frac{\beta_{-2}}{\beta_1} & a_{-4}\frac{\beta_{-2}}{\beta_2} & a_{-5}\frac{\beta_{-2}}{\beta_3} & a_{-6}\frac{\beta_{-2}}{\beta_4} & \cdots \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]
$$

Shield [13, Theorem 3(a)], proved that a bounded operator T on $L^2(\beta)$ is a weighted Laurent operator if and only if it commutes with the weighted Laurent operator M_z^β and this result provides an important tool for our study. Now using this result, we present a characterization of the weighted Hankel operators on $L^2(\beta)$ in terms of weighted Hankel matrix.

THEOREM 2.2. *An operator A on $L^2(\beta)$ is a weighted Hankel operator on $L^2(\beta)$ if and only if its matrix with respect to the orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$ is a weighted Hankel matrix with respect to the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$.*

Proof. The proof of the necessary part follows directly from the matrix of the weighted Hankel operator S_ϕ^β on $L^2(\beta)$. We prove the sufficient part only. For, let A be an operator on $L^2(\beta)$ with its matrix $[\alpha_{ij}]_{i,j\in\mathbb{Z}}$ with respect to the orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ satisfying

$$\frac{\beta_j}{\beta_{-i}}\alpha_{ij} = \frac{\beta_{j+1}}{\beta_{-(i-1)}}\alpha_{i-1j+1}$$

for each $i, j \in \mathbb{Z}$. Now a straightforward computation shows that

$$
\begin{aligned}
\langle J^\beta A M_z^\beta e_j, e_i\rangle &= \langle A M_z^\beta e_j, e_{-i}\rangle \\
&= \frac{\beta_{j+1}}{\beta_j}\langle A e_{j+1}, e_{-i}\rangle = \frac{\beta_{j+1}}{\beta_j}\alpha_{-i\ j+1} \\
&= \frac{\beta_i}{\beta_j}\frac{\beta_{j+1}}{\beta_i}\alpha_{-i\ j+1} = \frac{\beta_i}{\beta_{i-1}}\alpha_{-i+1\ j}.
\end{aligned}
$$

Similarly, using the fact that $M_z^{\beta *}e_i = \frac{\beta_i}{\beta_{i-1}}e_{i-1}$ we can see that $\langle M_z^\beta J^\beta A\, e_j, e_i\rangle = \frac{\beta_i}{\beta_{i-1}}\alpha_{-i+1\ j}$. Consequently, $J^\beta A$ commutes with the weighted Laurent operator M_z^β and hence $J^\beta A = M_\phi^\beta$ for some $\phi \in L^\infty(\beta)$. This gives $A = J^\beta M_\phi^\beta = S_\phi^\beta$, $\phi \in L^\infty(\beta)$. This completes the proof. ∎

It is now natural from Theorem 2.2 to expect that the adjoint of a weighted Hankel operator need not be a weighted Hankel operator on $L^2(\beta)$. This can even be verified through the following example.

EXAMPLE 2.3. Consider the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$, where

$$\beta_n = \begin{cases} 1 & \text{if } n \leq 0 \\ 2^n & \text{if } n \geq 1. \end{cases}$$

Let $\phi(z) = z^2$. Now, for $r = 1/2$, β satisfies the conditions $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n < 0$. Also, $\phi \in L^\infty(\beta)$. But the matrix of $S_\phi^{\beta *}$ in not a weighted Hankel matrix as, if we take $i = -1$ and $j = -1$ then the condition

$$\frac{\beta_j}{\beta_{-i}} \langle S_\phi^{\beta *} e_j, e_i \rangle = \frac{\beta_{j+1}}{\beta_{-(i-1)}} \langle S_\phi^{\beta *} e_{j+1}, e_{i-1} \rangle$$

implies that $1 = 4$. Hence $S_\phi^{\beta *}$ is not a weighted Hankel operator on $L^2(\beta)$.

If $\phi \equiv c$ is a constant, then the adjoint of weighted Hankel operator is a weighted Hankel operator, in fact, $S_\phi^{\beta *} = \overline{c} J^\beta$. Now we find a characterization for the general case as follows.

THEOREM 2.4. *If ϕ is a non-constant function in $L^\infty(\beta)$, then the adjoint $S_\phi^{\beta *}$ of a weighted Hankel operator S_ϕ^β on $L^2(\beta)$ is a weighted Hankel operator on $L^2(\beta)$ if and only if $\beta_n = 1$ for each $n \in \mathbb{Z}$.*

Proof. Let $\beta_n = 1$ for each $n \in \mathbb{Z}$ and $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$. Now, it is a matter of study over the classical space $L^2(\mathbb{T})$ and we find that $\overline{\phi}, \phi^* \in L^\infty(\beta)$, $M_\phi^{\beta *} = M_{\overline{\phi}}^\beta$ and $J^\beta M_{\phi *}^\beta e_j(z) = (\sum_{n=-\infty}^{\infty} \overline{a}_n z^{-n}) z^{-j} = M_{\overline{\phi}}^\beta J^\beta e_j(z)$. Hence $S_\phi^{\beta *} = M_\phi^{\beta *} J^\beta = M_{\overline{\phi}}^\beta J^\beta = J^\beta M_{\phi *}^\beta = S_{\phi *}^\beta$.

For the converse, suppose $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ is a non-constant function such that $S_\phi^{\beta *}$ is a weighted Hankel operator. Choose an integer $j_0 \neq 0$ such that $a_{j_0} \neq 0$. If $[\gamma_{ij}]_{i,j\in\mathbb{Z}}$ denotes the matrix of $S_\phi^{\beta *}$ with respect to the orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$ then for each $i, j \in \mathbb{Z}$, $\gamma_{ij} = \overline{a}_{-i-j} \frac{\beta_{-j}}{\beta_i}$ so that

$$\frac{\beta_i}{\beta_{-j}} \gamma_{ij} = \frac{\beta_{i-k}}{\beta_{-(j+k)}} \gamma_{i-k\ j+k} \tag{2.4.1}$$

for each $k \in \mathbb{Z}$.

If $S_\phi^{\beta *}$ is a weighted Hankel operator on $L^2(\beta)$, then using Theorem 2.2,

$$\frac{\beta_j}{\beta_{-i}} \gamma_{ij} = \frac{\beta_{j+k}}{\beta_{-(i-k)}} \gamma_{i-k\ j+k} \tag{2.4.2}$$

for each $i, j, k \in \mathbb{Z}$. Now $\gamma_{0\ -j_0} = \overline{a}_{j_0} \frac{\beta_{j_0}}{\beta_0} \neq 0$ and using (2.4.2), we find that $\gamma_{-k\ -j_0+k} \neq 0$ for each $k \in \mathbb{Z}$. Now using (2.4.1) and (2.4.2) for $i = 0$ and $j = -j_0$, we have

$$\frac{\beta_0^2}{\beta_{j_0}\beta_{-j_0}} = \frac{\beta_k \beta_{-k}}{\beta_{-j_0+k}\beta_{j_0-k}} \tag{2.4.3}$$

for each $k \in \mathbb{Z}$. If we put $k = j_0$ in (2.4.3), we get $\beta_{j_0}\beta_{-j_0} = 1$ and hence $\beta_{j_0} = \beta_{-j_0} = 1$. Now let $\beta_{j_0} = \beta_{-j_0} = \beta_{2j_0} = \beta_{-2j_0} = \cdots = \beta_{mj_0} = \beta_{-mj_0} = 1$. Now on substituting $k = (m+1)j_0$ in (2.4.3), it gives $\beta_{(m+1)j_0} = \beta_{-(m+1)j_0} = 1$. The result follows from here by applying the principle of mathematical induction and using the fact that $\beta_n \leq \beta_{n+1}$ for $n \geq 0$ and $\beta_n \leq \beta_{n-1}$ for $n \leq 0$. ∎

One can easily check that if $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a semi-dual sequence i.e. $\beta_{-n} = \beta_n$ for each n then $M_z^\beta J^\beta e_j = J^\beta M_{z^{-1}}^\beta e_j$ for each $j \in \mathbb{Z}$. This observation helps to conclude the following.

THEOREM 2.5. *If $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a semi-dual sequence then a necessary and sufficient condition for an operator A on $L^2(\beta)$ to be a weighted Hankel operator is that $M_z^\beta A = AM_{z^{-1}}^\beta$.*

Proof. If $A = S_\phi^\beta$ then

$$\begin{aligned} M_z^\beta A &= M_z^\beta J^\beta M_\phi^\beta = J^\beta M_{z^{-1}}^\beta M_\phi^\beta \\ &= J^\beta M_\phi^\beta M_{z^{-1}}^\beta = S_\phi^\beta M_{z^{-1}}^\beta. \end{aligned}$$

Conversely, if $M_z^\beta A = AM_{z^{-1}}^\beta$ then on pre multiplying by J^β, we get

$$M_{z^{-1}}^\beta J^\beta A = J^\beta A M_{z^{-1}}^\beta,$$

which on pre and post multiplying both sides by M_z^β gives

$$J^\beta A M_z^\beta = M_z^\beta J^\beta A.$$

Therefore, $J^\beta A$ is a weighted Laurent operator, symbolically $J^\beta A = M_\phi^\beta$ for some $\phi \in L^\infty(\beta)$. Hence $A = J^\beta M_\phi^\beta = S_\phi^\beta$. ∎

It is worth noticing that weighted Hankel operators are linear with respect to their symbols. Thus, if we denote the class of all weighted Hankel operators on $L^2(\beta)$ by w_H^β, then w_H^β is a linear subspace of $\mathfrak{B}(L^2(\beta))$, the space of all bounded operators on $L^2(\beta)$. Furthermore, we show that it is closed.

THEOREM 2.6. *w_H^β is a closed (strongly) linear subspace of $\mathfrak{B}(L^2(\beta))$.*

Proof. Suppose $A \in \mathfrak{B}(L^2(\beta))$ and a sequence $\{\phi_n\}$ in $L^\infty(\beta)$ are such that $S_{\phi_n}^\beta \to A$ as $n \to \infty$. Evidently,

$$\begin{aligned} J^\beta A M_z^\beta &= \lim_{n\to\infty} J^\beta S_{\phi_n}^\beta M_z^\beta = \lim_{n\to\infty} (J^\beta)^2 M_{\phi_n}^\beta M_z^\beta \\ &= \lim_{n\to\infty} M_z^\beta M_{\phi_n}^\beta = \lim_{n\to\infty} M_z^\beta J^\beta S_{\phi_n}^\beta = M_z^\beta J^\beta A. \end{aligned}$$

This gives $J^\beta A = M_\phi^\beta$ for some $\phi \in L^\infty(\beta)$ so that $A = J^\beta M_\phi^\beta = S_\phi^\beta$. This completes the proof. ∎

3. Properties of weighted Hankel operator

This section is devoted to study some basic algebraic properties of the weighted Hankel operators on $L^2(\beta)$. It is shown that there is a dearth of compact weighted Hankel operators, in fact, the only compact weighted Hankel operator on $L^2(\beta)$ is the zero operator. We also discuss the case for self-adjoint weighted Hankel operator. Product of weighted Hankel operators with the weighted Laurent as well as weighted Hankel operators are calculated.

The following lemma is a common knowledge among the readers who study the Laurent operator. Unfortunately, we have not been able to find a reference providing the proof in the case of weighted Laurent operator. We include the proof of the result here.

Lemma 3.1. *The weighted Laurent operator M_ϕ^β on $L^2(\beta)$ is compact if and only if $\phi = 0$.*

Proof. Nothing is needed to prove the if part. We prove the converse part only. For, let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in $L^\infty(\beta)$ be such that M_ϕ^β is a compact operator. Then, it maps weakly convergent sequences to strongly once. Hence, for each $j \in \mathbb{Z}$, $|\langle M_\phi^\beta e_n, e_{n+j}\rangle| \le \|M_\phi^\beta e_n\| \to 0$ as $n \to \infty$. But,

$$|\langle M_\phi^\beta e_n, e_{n+j}\rangle| = \Big|\sum_{m=-\infty}^{\infty} a_m \frac{\beta_{m+n}}{\beta_n}\langle e_{m+n}, e_{n+j}\rangle\Big| = |a_j|\frac{\beta_{n+j}}{\beta_n}.$$

Therefore, if $j \ge 0$, then

$$|a_j| \le |a_j|\frac{\beta_{n+1}}{\beta_n}\frac{\beta_{n+2}}{\beta_{n+1}}\cdot\ldots\cdot\frac{\beta_{n+j}}{\beta_{n+j-1}} \to 0$$

as $n \to \infty$. As a consequence, $a_j = 0$ for each $j \ge 0$. In case $j < 0$, then for any natural number n such that $n + j \ge 0$,

$$|a_j| r^{j+1} \le |a_j|\frac{\beta_{n-1}}{\beta_n}\frac{\beta_{n-2}}{\beta_{n-1}}\cdot\ldots\cdot\frac{\beta_{n+j}}{\beta_{n+j+1}}.$$

This implies that $a_j = 0$ for each $j < 0$. Hence, $\phi = 0$. ∎

Theorem 3.2. *The weighted Hankel operator S_ϕ^β on $L^2(\beta)$ is compact if and only if $\phi = 0$.*

Proof. The operator S_ϕ^β is compact if and only if $J^\beta S_\phi^\beta = M_\phi^\beta$ is compact. The latter is compact if and only if $\phi = 0$. ∎

We recall that for ϕ given by $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, the symbol $\widetilde{\phi}$ mean the expression $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$. It is easy to see that if $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a semi-dual sequence and $\phi \in L^\infty(\beta)$ then $\widetilde{\phi} \in L^\infty(\beta)$.

Lemma 3.3. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a semi-dual sequence and $\phi \in L^\infty(\beta)$. Then*

(1) $J^\beta M_\phi^\beta = M_{\widetilde{\phi}}^\beta J^\beta$.

(2) *The product $M_\phi^\beta S_\psi^\beta$ is a weighted Hankel operator on $L^2(\beta)$. Moreover, $M_\phi^\beta S_\psi^\beta = S_{\widetilde{\phi}\psi}^\beta$.*

(3) *The product of two weighted Hankel operators on $L^2(\beta)$ is a weighted Laurent operator on $L^2(\beta)$.*

Proof. Being $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ a semi-dual sequence, we have $\beta_n = \beta_{-n}$ for each $n \in \mathbb{Z}$ and hence

$$\begin{aligned} S_\phi^\beta e_j(z) = J^\beta M_\phi^\beta e_j(z) &= \frac{1}{\beta_j}\sum_{n=-\infty}^{\infty} a_n \beta_{n+j} e_{-n-j}(z) \\ &= \frac{1}{\beta_j}(\sum_{n=-\infty}^{\infty} a_n z^{-n}) z^{-j} = M_{\widetilde{\phi}}^\beta J^\beta e_j(z). \end{aligned}$$

This means that $S_\phi^\beta = M_{\widetilde{\phi}}^\beta J^\beta$ and hence $J^\beta M_\phi^\beta = M_{\widetilde{\phi}}^\beta J^\beta$.

Now proof of (2) follows as $\widetilde{\widetilde{\phi}} = \phi$ and hence $M_\phi^\beta S_\psi^\beta = M_\phi^\beta J^\beta M_\psi^\beta = J^\beta M_{\widetilde{\phi}}^\beta M_\psi^\beta = J^\beta M_{\widetilde{\phi}\psi}^\beta = S_{\widetilde{\phi}\psi}^\beta$.

If S_ϕ^β and S_ψ^β are two weighted Hankel operators on $L^2(\beta)$ then

$$S_\phi^\beta S_\psi^\beta = (J^\beta M_\phi^\beta)(J^\beta M_\psi^\beta) = (M_{\widetilde{\phi}}^\beta J^\beta)(J^\beta M_\psi^\beta) = M_{\widetilde{\phi}}^\beta M_\psi^\beta = M_{\widetilde{\phi}\psi}^\beta.$$

This completes the proof of (3). ∎

Proposition 3.4. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a semi-dual sequence and $\phi, \psi \in L^\infty(\beta)$. Then following are equivalent:*

(1) *$M_\phi^\beta S_\psi^\beta$ is a compact operator.*

(2) *$S_\phi^\beta S_\psi^\beta$ is a compact operator.*

(3) $\widetilde{\phi}\psi = 0$.

Proof. Equivalency of (1) and (3) follows from Theorem 3.2 and Lemma 3.3(2) and equivalency of (2) and (3) is immediate from Lemma 3.1 and Lemma 3.2(3). ∎

Now we are in a stage to state the following result which follow immediately from the preceding analysis.

Proposition 3.5. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a semi-dual sequence and $\phi, \psi \in L^\infty(\beta)$. Then*

(1) $S_\psi^\beta M_\phi^\beta = M_{\widetilde{\phi}}^\beta S_\psi^\beta$.

(2) *$\sigma(S_\phi^\beta) = \sigma(S_{\widetilde{\phi}}^\beta)$, where $\sigma(S_\phi^\beta)$ denotes the spectrum of the operator S_ϕ^β.*

Now we investigate the situations under which the weighted Hankel operators on $L^2(\beta)$ are self-adjoint.

THEOREM 3.6. *Let $\phi = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ be a non-constant function in $L^\infty(\beta)$. Then the weighted Hankel operator S_ϕ^β on $L^2(\beta)$ is self-adjoint if and only if each $\beta_n = 1$ and a_n is a real number.*

Proof. If part follows directly from the definition, as for each $j \in \mathbb{Z}$, $S_\phi^\beta e_j = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{-n-j}\beta_{-n} e_n$ and $S_\phi^{\beta *} e_j = \beta_{-j} \sum_{n=-\infty}^{\infty} \frac{\overline{a}_{-n-j}}{\beta_n} e_n$.

Conversely, suppose that S_ϕ^β on $L^2(\beta)$ is a self-adjoint operator. Then for each $j \in \mathbb{Z}$, $S_\phi^\beta e_j = S_\phi^{\beta *} e_j$ which implies that

$$\frac{a_{-n-j}\beta_{-n}}{\beta_j} = \frac{\overline{a}_{-n-j}\beta_{-j}}{\beta_n} \tag{3.6.1}$$

for each $n, j \in \mathbb{Z}$. Being ϕ is non-constant, there must exist a non-zero integer k such that $a_{-k} \neq 0$. Now we use the induction to conclude that $\beta_{mk} = \beta_{-mk} = 1$ for each $m \geq 1$. Equation (3.6.1) with $j = 0$ and $n = k$ becomes $\beta_k \beta_{-k} = 1$, which in turn implies $\beta_k = \beta_{-k} = 1$. Assume that $\beta_k = \beta_{-k} = \beta_{2k} = \beta_{-2k} = \cdots = \beta_{pk} = \beta_{-pk} = 1$. Now on substituting $j = -pk$ and $n = (p+1)k$ in (3.6.1), we get

$$\frac{a_{-(p+1)k+pk}\beta_{-(p+1)k}}{\beta_{-pk}} = \frac{\overline{a}_{-(p+1)k+pk}\beta_{pk}}{\beta_{(p+1)k}},$$

which, on taking the modulus both sides, gives $\beta_{(p+1)k} = \beta_{-(p+1)k} = 1$. Therefore, we have $\beta_n = 1$ for each $n \in \mathbb{Z}$. Now go back to (3.6.1), we find that $a_{-n-j} = \overline{a}_{-n-j}$ for each $n, j \in \mathbb{Z}$. This provides that each a_n is a real number. ∎

Normal operators are always hyponormal, however the next result provides a condition for the converse to hold for the weighted Hankel operators on $L^2(\beta)$.

THEOREM 3.7. *Let $\phi = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ be a non-constant function in $L^\infty(\beta)$. Then the hyponormal weighted Hankel operator S_ϕ^β, induced by ϕ, on $L^2(\beta)$ is normal if and only if $\beta_n = 1$ for each $n \in \mathbb{Z}$.*

Proof. Let the hyponormal weighted Hankel operator S_ϕ^β on $L^2(\beta)$ is normal. Then for each $j \geq 0$, $\|S_\phi^{\beta *} e_j\| = \|S_\phi^\beta e_j\|$ which implies that

$$\beta_{-j}^2 \sum_{n=-\infty}^{\infty} \frac{|a_{-n-j}|^2}{{\beta_n}^2} = \frac{1}{{\beta_j}^2} \sum_{n=-\infty}^{\infty} |a_{-n-j}|^2 \beta_{-n}^2. \tag{3.7.1}$$

For $j = 0$, this gives $\sum_{n=-\infty}^{\infty} (|a_{-n}|^2 \beta_{-n}^2 - \frac{|a_{-n}|^2}{{\beta_n}^2}) = 0$ and hence, being each term in the bracket is positive, we get

$$\frac{|a_{-n}|^2}{{\beta_n}^2} = |a_{-n}|^2 \beta_{-n}^2 \tag{3.7.2}$$

for each $n \in \mathbb{Z}$.

First we consider the situation, when there exist some non-zero integer p such that $a_p \neq 0$ and $a_n = 0$ for $|n| > |p|$. Accordingly, for $p > 0$, $\phi(z) = a_{-p}z^{-p} + a_{-p+1}z^{-p+1} + \cdots + a_{-1}z^{-1} + a_0 + a_1z^1 + \cdots + a_pz^p$. We use the mathematical induction to conclude that $\beta_n = 1$ for each $n \in \mathbb{Z}$. Equation (3.7.2) for $n = p$ gives $\beta_p\beta_{-p} = 1$, which means $\beta_n = 1$ for $|n| \leq p$. Suppose that $\beta_{mp}\beta_{-mp} = 1$ for $m > 1$. Then equation (3.7.1) for $j = mp$ provides

$$\begin{cases} \sum\limits_{n=-(m+1)p}^{-(m-1)p} (|a_{-n-(m+1)p}|^2\beta_{-n}^2 - \frac{|a_{-n-(m+1)p}|^2}{\beta_n^2}) = 0 & \text{if } p > 0 \\ \sum\limits_{n=-(m-1)p}^{-(m+1)p} (|a_{-n-(m+1)p}|^2\beta_{-n}^2 - \frac{|a_{-n-(m+1)p}|^2}{\beta_n^2}) = 0 & \text{if } p < 0. \end{cases}$$

In any case, we can conclude that $\beta^2_{(m+1)p}\beta^2_{-(m+1)p} = 1$. Therefore $\beta_{mp}\beta_{-mp} = 1$ for every $m \geq 1$ and hence $\beta_n = 1$ for each $n \in \mathbb{Z}$.

If the first situation fails to occur then for each $n \geq 0$, we can find $n_0 > n$ such that $a_{n_0} \neq 0$ or $a_{-n_0} \neq 0$. In any case, from equation (3.7.2) we can obtain $\frac{|a_{-n_0}|^2}{{\beta_{n_0}}^2} = |a_{-n_0}|^2\beta^2_{-n_0}$, which provides $\beta_{n_0} = \beta_{-n_0} = 1$. Hence, $\beta_m = 1$ for $|m| \leq n_0$. Thus $\beta_n = 1$ for each $n \in \mathbb{Z}$. This completes the proof of the necessary part.

For the converse, we suppose that $\beta_n = 1$ for each $n \in \mathbb{Z}$. Now, if $f(z) = \sum_{n=-\infty}^{\infty} b_nz^n \in L^2(\beta)$ then $f^*(z) = \sum_{n=-\infty}^{\infty} \overline{b}_nz^n \in L^2(\beta)$. Moreover, simple computations show that $\|S_\phi^{\beta*}f\| = \|S_\phi^\beta f^*\|$ and $\|S_\phi^{\beta*}f^*\| = \|S_\phi^\beta f\|$. Now hyponormality of S_ϕ^β means that $\|S_\phi^{\beta*}f\| = \|S_\phi^\beta f\|$, which implies that S_ϕ^β is a normal operator. ∎

Now we proceed to answer the question: When is the product of weighted Hankel operators on $L^2(\beta)$ a weighted Hankel operator?

LEMMA 3.8. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a semi-dual sequence and $\phi \in L^\infty(\beta)$. Then $M_\phi^\beta J^\beta$ is a weighted Laurent operator on $L^2(\beta)$ if and only if $\phi = 0$.*

Proof. If $M_\phi^\beta J^\beta$ is a weighted Laurent operator on $L^2(\beta)$ then it commutes with M_z^β. As a fact, we have $\langle M_z^\beta M_\phi^\beta J^\beta e_j, e_i\rangle = \langle M_\phi^\beta J^\beta M_z^\beta e_j, e_i\rangle$, which turn out as $\frac{\beta_j}{\beta_{-j}}a_{i+j-1} = \frac{\beta_{j+1}}{\beta_{-j-1}}a_{i+j+1}$. As β is a semi-dual sequence, this implies that $a_{i+j-1} = a_{i+j+1}$ for each $i, j \in \mathbb{Z}$. It is easy to conclude from here that $a_{2n} = a_0$ and $a_{2n+1} = a_1$ for each $n \in \mathbb{Z}$. As $L^\infty(\beta) \subseteq L^2(\beta)$, so we find that $a_n \to 0$ as $n \to \infty$ and this yields that each $a_n = 0$. Hence, $\phi = 0$. The converse is obvious. ∎

Using this lemma, we can prove the following.

THEOREM 3.9. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a semi-dual sequence and $\phi \in L^\infty(\beta)$. Then the product of two weighted Hankel operators on $L^2(\beta)$ is a weighted Hankel operator on $L^2(\beta)$ if and only if the product is the zero operator.*

Proof. Let the product $S^\beta_\phi S^\beta_\psi$ of two weighted Hankel operators S^β_ϕ and S^β_ψ on $L^2(\beta)$ is a weighted Hankel operator say S^β_ξ for some $\xi \in L^\infty(\beta)$. Now using Lemma 3.2(1),

$$J^\beta M^\beta_\xi = S^\beta_\xi = S^\beta_\phi S^\beta_\psi = J^\beta M^\beta_{\phi\tilde{\psi}} J^\beta.$$

Thus, $S^\beta_\phi S^\beta_\psi$ is a weighted Hankel operator (S^β_ξ) if and only if $M^\beta_{\phi\tilde{\psi}} J^\beta$ is a weighted Laurent operator (M^β_ξ). Using Lemma 3.6, the latter holds if and only if $\phi\widetilde{\psi} = 0$. This completes the proof. ∎

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