МАТЕМАТИЧКИ ВЕСНИК
65, 3 (2013), 364–372
September 2013

оригинални научни рад
research paper

RELATIVE EXT GROUPS OF ABELIAN CATEGORIES

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Abstract. In this article, we characterize the relative Ext groups of abelian categories relative to a fixed precovering class $\mathcal{F}$ and give some examples.

1. Introduction

Let R be a ring with 1. The class of projective R-modules is a cornerstone in classical homological algebra. Recall that every R-module M admits a projective resolution:

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

which is exact with all P_i projective. Moreover, any such resolution is unique up to homotopy. Then one can use projective resolution of M to define derived functors $Ext_R^{i\geq 1}(-,\ N)$ for any R-module N. It is well-known that for any R-module M and any integer $n \geq 0$, the following are equivalent:

(1) M admits a projective resolution of the form:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

(2) $Ext_R^k(M,\ N) = 0$ for any R-module N and $k \geq n+1$;

(3) $Ext_R^{n+1}(M,\ N) = 0$ for any R-module N;

(4) Every projective resolution of M has a projective nth syzygy.

Recall that M and M' are said to be projectively equivalent if $M \oplus P \cong M' \oplus P'$ for some projective R-modules P and P'. We denote the projective equivalence class of M by $[M]$. Let $\theta : P \to M$ be an epimorphism with P projective. There is a so-called Schanuel class $\mathcal{S}([M]) := [Ker\theta]$. One can consider the n-fold compositions

2010 Mathematics Subject Classification: 18G15, 18G25

Keywords and phrases: Relative Ext group; precovering; $\mathcal{F}$-resolution.

The first author is partially supported by the natural foundation of Jiangxi Agricultural University (No. 09003320) and the second author is partially supported by the Fundamental Research Funds for the Central Universities No. zyz2011081.

of $\mathcal{S}$ for $n \geq 0$. Therefore, the following result (see [1, Chapter V, Proposition 2.1]) is well-known:

For any R-module M and any integer $n \geq 0$, the following are equivalent:

(1) $Ext_R^{n+1}(M,\ N) = 0$ for any R-module N;

(2) M admits a projective resolution of the form:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

(3) $\mathcal{S}^n([M]) = [0]$.

The study of relative homological algebra was initiated by Butler and Horrocks and Eilenberg and Moore, and has been revitalized recently by a number of authors, for instance, Enochs and Jenda and Avramov and Martsinkovsky. Let $\mathcal{F}$ be a class of R-modules. Enochs et. al. defined $\mathcal{F}$-precover and $\mathcal{F}$-precovering, cf., [3, Definition 5.1.1]. Therefore, one can define $\mathcal{F}$-resolution of M, cf., [3, Definition 8.1.2]. Moreover, such $\mathcal{F}$-resolution of M is unique up to homotopy. Then one can define the relative derived functor $Ext^i_{\mathcal{F}}(-,\ N)$ for each $i \geq 0$ and R-module N. Sather-Wagstaff et al. [9] compared the relative cohomology theories with respect to semidualizing modules.

In this paper, we generalize the result of [1, Chapter V, Proposition 2.1] to any abelian category relative to some epic precovering class. Our main result is Theorem 2.8.

2. Main results

Notation 2.1. In this note, $\mathcal{A}$ is an abelian category. $\mathcal{P} = \mathcal{P}(\mathcal{A})$ and $\mathcal{I} = \mathcal{I}(\mathcal{A})$ are the subcategories of projective and injective objects in $\mathcal{A}$ respectively. $\mathcal{F}$ is a precovering class of $\mathcal{A}$, cf., Definition 2.3, which contains 0 and is closed under isomorphisms and finite direct sums.

Definition 2.2. An $\mathcal{A}$-complex is a sequence of homomorphisms in $\mathcal{A}$

$$A = \cdots \xrightarrow{\partial^A_{n+1}} A_n \xrightarrow{\partial^A_n} A_{n-1} \xrightarrow{\partial^A_{n-1}} \cdots$$

such that $\partial^A_{n-1}\partial^A_n = 0$ for all n. The nth homology object of $\mathcal{A}$ is $H_n(A) = Ker(\partial^A_n)/Im(\partial^A_{n+1})$. A is exact the case $Ker(\partial^A_n) = Im(\partial^A_{n+1})$.

Definition 2.3. Let $\mathcal{A}$ be an abelian category and $\mathcal{F}$ be a class of objects of $\mathcal{A}$. A morphism $\varphi:\ F \to M$ of $\mathcal{A}$ is called an $\mathcal{F}$-precover of M if $F \in \mathcal{F}$ and $Hom(F',\ F) \to Hom(F',\ M) \to 0$ is exact for all $F' \in \mathcal{F}$. φ is called an epic $\mathcal{F}$-precover of M if it is an $\mathcal{F}$-precover and is an epimorphism. If every object admits an (epic) $\mathcal{F}$-precover, then we say $\mathcal{F}$ is an (epic) precovering class. An augmented $\mathcal{F}$-resolution of an object M is a complex

$$\mathbf{X}^+ = \cdots \longrightarrow X_n \xrightarrow{d_n} \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{\gamma} M \longrightarrow 0$$

with all $X_i \in \mathcal{F}$ such that $Hom(F',\ X^+)$ is exact for any $F' \in \mathcal{F}$. Clearly, if $\mathcal{F}$ is precovering, every object M has an augmented $\mathcal{F}$-resolution. M is said to have a special $\mathcal{F}$-precover if there is an exact sequence

$$0 \longrightarrow C \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $F \in \mathcal{F}$ and $Ext^1(\mathcal{F},\ C) = 0$. It is clear that M has an epic $\mathcal{F}$-precover if it has a special $\mathcal{F}$-precover.

LEMMA 2.4. [3, Ex. 8.1.2, p. 169] *Let $\mathcal{F}$ be a precovering. Consider the diagram*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & & \big\downarrow f & & \\ \cdots & \longrightarrow & F_1' & \longrightarrow & F_0' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

where the rows are $\mathcal{F}$-resolutions of M and M' respectively. Then $f : M \to M'$ induces a chain map of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow f_1 & & \big\downarrow f_0 & & \big\downarrow f & & \\ \cdots & \longrightarrow & F_1' & \longrightarrow & F_0' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

which is unique up to homotopy.

DEFINITION 2.5. Let $\mathcal{F}$ be a precovering class and N be any object of $\mathcal{A}$. By Lemma 2.4, one can well-define the nth derived functor of $Hom(-,\ N)$ relative to $\mathcal{F}$. We write it by $Ext^n_{\mathcal{F}}(-,\ N)$.

LEMMA 2.6. [3, Lemma 8.6.3] *If $F \to M$ and $G \to M$ are $\mathcal{F}$-precovers with kernels K and L respectively, then $K \oplus G \cong L \oplus F$.*

DEFINITION 2.7. Let K and K' be two objects in $\mathcal{A}$. They are called $\mathcal{F}$-equivalent denoted $K \equiv_{\mathcal{F}} K'$, if there exist $F,\ G \in \mathcal{F}$ such that $K \oplus G \cong K' \oplus F$. We use $[K]$ to denote the $\mathcal{F}$-equivalence class containing K. By Lemma 2.6, the kernels of any $\mathcal{F}$-precovers of M are $\mathcal{F}$-equivalent. Let $\varphi :\ F \to M$ be any $\mathcal{F}$-precover of M. Then $[Ker\varphi]$ is a well-defined class depending only on M. So we write $\mathcal{S}_{\mathcal{F}}(M) = [Ker\varphi]$. By the analogy of [3, Lemma 8.6.2], $\mathcal{S}_{\mathcal{F}}(M)$ only depends on the $\mathcal{F}$-equivalence class of M. So there induces well-defined map:

$$\mathcal{S}_{\mathcal{F}} : \mathcal{A}/\equiv_{\mathcal{F}} \to \mathcal{A}/\equiv_{\mathcal{F}} .$$

We set $\mathcal{S}^0_{\mathcal{F}} = id$ and $\mathcal{S}^n_{\mathcal{F}}$ the n-fold compositions of $\mathcal{S}_{\mathcal{F}}$ for any positive integer $n > 0$.

THEOREM 2.8. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{F}$ be an epic precovering class closed under direct summands. Then for any positive integer $n \geq 0$ the following are equivalent:*

(1) $Ext_{\mathcal{F}}^{n+1}(M,\ N) = 0$ *for any object* N *of* $\mathcal{A}$*;*

(2) *there exists an augmented* $\mathcal{F}$*-resolution of the form*

$$0 \to F_n \to \cdots \to F_0 \to M \to 0;$$

(3) $\mathcal{S}_{\mathcal{F}}^{n}([M]) = [0]$.

Proof. (2) ⇒ (1) : Clear.

(1) ⇒ (2) : If $n = 0$, $Ext_{\mathcal{F}}^{1}(M,\ N) = 0$ for any object N in $\mathcal{A}$. We claim that M is in $\mathcal{F}$. Since $\mathcal{F}$ is an epic precovering class, there is a short exact sequence,

$$\dagger:\ 0 \longrightarrow K = Ker\,\varphi \xrightarrow{\ \mu\ } F_0 \xrightarrow{\ \varphi\ } M \longrightarrow 0,$$

which is $Hom(\mathcal{F},\ -)$-exact. By the long exact sequence of relative Ext groups, cf. [3,Theorem 8.2.3],

$$Hom_{\mathcal{F}}(F_0,\ K) \to Hom_{\mathcal{F}}(K,\ K) \to Ext_{\mathcal{F}}^{1}(M,\ K) = 0,$$

there is a morphism $\nu \in Hom_{\mathcal{F}}(F_0,\ K)$ such that $\nu\mu = 1_K$. Therefore, † is split, and $K \oplus M \cong F_0$. Since $\mathcal{F}$ is closed under direct summands, $M \in \mathcal{F}$. Hence, M has an $\mathcal{F}$-resolution,

$$0 \longrightarrow M \xrightarrow{\ 1_M\ } M \longrightarrow 0.$$

If $n > 0$, $Ext_{\mathcal{F}}^{n+1}(M,\ N) = 0$ for any object N in $\mathcal{A}$. Since $\mathcal{F}$ is an epic precovering class, there is a short exact sequence,

$$\flat:\ 0 \longrightarrow K = Ker\,\varphi \xrightarrow{\ \mu\ } F_0 \xrightarrow{\ \varphi\ } M \longrightarrow 0,$$

which is $Hom_{\mathcal{F}}(\mathcal{F},\ -)$-exact with $F_0 \in \mathcal{F}$. By the dimensional shifting Theorem, cf. [3, Corollary 8.2.4], $Ext_{\mathcal{F}}^{(n-1)+1}(K,\ N) = Ext_{\mathcal{F}}^{n+1}(M,\ N) = 0$. So by inductive hypothesis, K admits an an $\mathcal{F}$-resolution,

$$\natural:\ 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow K \longrightarrow 0.$$

Pasting the sequences ♭ and ♮, we get the desired $\mathcal{F}$-resolution of M.

(2) ⇒ (3) : We begin with the base case $n = 0$. There is an $\mathcal{F}$-resolution of M,

$$0 \longrightarrow F_0 \xrightarrow{\ \partial_0\ } M \longrightarrow 0.$$

So

$$0 \longrightarrow Hom_{\mathcal{F}}(\mathcal{F},\ F_0) \xrightarrow{\ (\partial_0)_*\ } Hom_{\mathcal{F}}(\mathcal{F},\ M) \longrightarrow 0$$

is exact. Then we claim that ∂_0 is monic. Noting that $\mathcal{F}$ is an epic precovering class, then ∂_0 is epic. Consider the exact sequence

$$0 \longrightarrow D = Ker\,\partial_0 \longrightarrow F_0 \xrightarrow{\ \partial_0\ } M \longrightarrow 0.$$

So the sequence

$$0 \longrightarrow Hom_{\mathcal{F}}(\mathcal{F},\ D) \longrightarrow Hom_{\mathcal{F}}(\mathcal{F},\ F_0) \xrightarrow{\ (\partial_0)_*\ } Hom_{\mathcal{F}}(\mathcal{F},\ M) \longrightarrow 0$$

is exact. Thus $Hom_{\mathcal{F}}(\mathcal{F},\ Ker\partial_0) = 0$. It follows that $0 \to Ker\partial_0$ is a precovering of $Ker\partial_0$, which is an isomorphism because $\mathcal{F}$ is epic. Thus $M \in \mathcal{F}$ and then $\mathcal{S}^0_{\mathcal{F}}([M]) = [0]$.

Now we assume that $n > 0$. Let

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

be an $\mathcal{F}$-resolution of M. Splice it into two complexes,

$$\sharp:\ 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow Ker\,\partial_0 \longrightarrow 0,$$

and

$$0 \longrightarrow Ker\,\partial_0 \longrightarrow F_0 \xrightarrow{\partial_0} M \longrightarrow 0.$$

Then $\sharp$ is an augmented $\mathcal{F}$-resolution of $Ker\partial_0$. By induction, $\mathcal{S}^n_{\mathcal{F}}([M]) = \mathcal{S}^{n-1}_{\mathcal{F}}(\mathcal{S}_{\mathcal{F}}([M])) = \mathcal{S}^{n-1}_{\mathcal{F}}([Ker\,\partial_0]) = [0]$.

(3) $\Rightarrow$ (2) : If $\mathcal{S}^0_{\mathcal{F}}([M]) = [0]$, $M \oplus F' \cong 0 \oplus F \cong F$ for some $F',\ F \in \mathcal{F}$, and $M \in \mathcal{F}$. Therefore, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{1_M} M \longrightarrow 0,$$

which is an augmented $\mathcal{F}$-resolution of M. Now we assume that $\mathcal{S}^n_{\mathcal{F}}([M]) = [0]$, $n > 0$. Let

$$F_0 \xrightarrow{\partial_0} M$$

be an $\mathcal{F}$-precover of M. So $\mathcal{S}^{n-1}_{\mathcal{F}}([Ker\partial_0]) = \mathcal{S}^n_{\mathcal{F}}([M]) = [0]$. By the induction hypothesis, there exists an $\mathcal{F}$-resolution of $Ker\partial_0$,

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow Ker\,\partial_0 \longrightarrow 0.$$

Therefore, M has an $\mathcal{F}$-resolution of M,

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0. \qquad \blacksquare$$

Now we give some classes of modules which have the hypotheses of Theorem 2.8.

EXAMPLE 2.9. (1) By [5, Corollary 3.4.4], the class of all modules of projective dimension less than or equal to a fixed natural number s is an epic precovering class, and it is closed under direct summands.

(2) If R is an m-Gorenstein ring, by [3, Theorem 11.5.1] and [6, Theorem 2.5], the class of all Gorenstein projective R-modules is an epic precovering class, and is closed under direct summands.

(3) If R is an m-Gorenstein and coherent ring, by [3, Theorem 11.7.3] and [6, Theorem 3.7], the class of all Gorenstein flat R-modules [2] is an epic precovering class, and is closed under direct summands.

(4) Let R be a commutative ring with a semidualizing module C and any R-module has finite $\mathcal{G}_C$-projective dimension; for more details cf. [10]. By [10,

Theorem 3.6 and Theorem 2.8], the class of all $\mathcal{G}_C$-projective R-modules is an epic precovering class, and is closed under direct summands.

(5) Let R be a commutative coherent ring with a semidualizing module C and any R-module has finite $\mathcal{G}_C$-flat dimension. the class of all $\mathcal{G}_C$-flat R-modules is an epic precovering class, and is closed under direct summands.

Let $\mathcal{F}$ and $\mathcal{C}$ be classes of $\mathcal{A}$. If every object of $\mathcal{C}$ admits an (epic) $\mathcal{F}$-precover, we say $\mathcal{F}$ is an (epic) precovering class of $\mathcal{C}$.

THEOREM 2.10. *Let $\mathcal{F}$ be an epic precovering class of $\mathcal{C}$ closed under direct summands. If any exact sequence $0 \to K \to F \to M \to 0$ with $F \in \mathcal{F}$ and $M \in \mathcal{C}$ implies that $K \in \mathcal{C}$, then for any object $M \in \mathcal{C}$ and any positive integer $n \geq 0$ the following are equivalent:*

(1) *$Ext_{\mathcal{F}}^{n+1}(M,\ N) = 0$ for any object N of $\mathcal{A}$;*

(2) *there exists an augmented $\mathcal{F}$-resolution of the form*

$$0 \to F_n \to \cdots \to F_0 \to M \to 0;$$

(3) *$\mathcal{S}_{\mathcal{F}}^n([M]) = [0]$.*

Proof. A slight modification of the proof of Theorem 2.8 gives this result. ■

Now we assume that R is a local Cohen-Mocaulay ring of Krull dimension d admits a dualizing module Ω. We use $\mathcal{J}_0(R)$ to denote the class of R-modules M such that $Tor_i(\Omega,\ M) = Ext^i(\Omega,\ Hom(\Omega,\ M)) = 0$ for all $i \geq 1$ and such that the natural map $\Omega \otimes Hom(\Omega,\ M) \to N$ is an isomorphism. Let $\mathcal{W}$ denote the class of R-modules W such that $W \cong \Omega \otimes P$ for some projective R-module P. Clearly, $\mathcal{W} \subseteq \mathcal{J}_0(R)$. One can refer to Takahashi and White's paper [8] for the case with respect to semidualizing module.

EXAMPLE 2.11. (1) By [4, Theorem 3.11], $\mathcal{W}$ is an epic precovering class of $\mathcal{J}_0(R)$, and it is closed under direct summands by the analogy of [7, Proposition 5.5]. Any exact sequence $0 \to K \to F \to M \to 0$ with $F \in \mathcal{W}$ and $M \in \mathcal{J}_0(R)$ implies that $K \in \mathcal{J}_0(R)$, since $\mathcal{J}_0(R)$ is closed under the kernels of epimorphism.

(2) We use $\mathcal{G}$ to denote the class of Ω-Gorenstein projective R-modules. By [4, Theorem 3.5], $\mathcal{G} \subseteq \mathcal{J}_0(R)$. By [4, Theorem 3.11], every R-module $M \in \mathcal{J}_0(R)$ has an epic $\mathcal{G}$-precover. From [4, Corollary 3.9], $\mathcal{G}$ is closed under direct summands. Any exact sequence $0 \to K \to F \to M \to 0$ with $F \in \mathcal{G}$ and $M \in \mathcal{J}_0(R)$ implies that $K \in \mathcal{J}_0(R)$, since $\mathcal{J}_0(R)$ is closed under the kernels of epimorphism.

The next result gives conditions on a class $\mathcal{G}$ guaranteeing that the hypotheses of Theorem 2.8 are satisfied.

THEOREM 2.12. *Let $\mathcal{F}$ be an epic precovering class, and $\mathcal{G}$ be the subcategory of $\mathcal{F}$ such that $Ext^1(\mathcal{G},\ \mathcal{F}) = 0$ and each object F in $\mathcal{F}$ has a special $\mathcal{G}$-precover. If $\mathcal{G}$ is closed under direct summands, then $\mathcal{G}$ is an epic precovering class.*

Proof. We just claim that M has a special $\mathcal{G}$-precover for any $M \in \mathcal{A}$. Since $\mathcal{F}$ is an epic precovering class, there is a $Hom_{\mathcal{A}}(\mathcal{F},\ -)$-exact exact sequence,

$$\imath:\ 0 \longrightarrow M' \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $F \in \mathcal{F}$. By the hypothesis, F has a special $\mathcal{G}$-precover, that is, there is a short exact sequence

$$\jmath:\ 0 \longrightarrow F' \longrightarrow G \longrightarrow F \longrightarrow 0$$

with $Ext^1(\mathcal{G},\ F') = 0$ and $G \in \mathcal{G}$. Consider the pull-back whose lower row is $\imath$ and middle column is $\jmath$,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F' & = & F' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M' & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathcal{G} \subseteq \mathcal{F}$, $\imath$ is also $Hom_{\mathcal{A}}(\mathcal{G},\ -)$-exact. By the hypothesis that $Ext^1(\mathcal{G},\ \mathcal{F}) = 0$, $Ext^1(\mathcal{G},\ F) = 0$. According to the long exact sequence for $\imath$,

$$\cdots \to Hom(\mathcal{G},\ F) \to Hom(\mathcal{G},\ M) \to Ext^1(\mathcal{G},\ M') \to Ext^1(\mathcal{G},\ F) = 0,$$

$Ext^1(\mathcal{G},\ M') = 0$. Since $Ext^1(\mathcal{G},\ F') = 0$, by the long exact sequence again for the left column of the pull-back above,

$$0 = Ext^1(\mathcal{G},\ F') \to Ext^1(\mathcal{G},\ U) \to Ext^1(\mathcal{G},\ M') = 0,$$

$Ext^1(\mathcal{G},\ U) = 0$. ∎

The next result gives conditions on a class ${}^{\perp}\mathcal{G}$ guaranteeing that the hypotheses of Theorem 2.8 are satisfied.

Theorem 2.13. *Let $\mathcal{F}$ be a monic precovering class, and $\mathcal{G}$ be a subcategory of $\mathcal{F}$ such that $Ext^{i\geq 1}(\mathcal{F},\ \mathcal{G}) = 0$. If for each object F in $\mathcal{F}$ there is a short exact sequence,*

$$0 \longrightarrow F' \longrightarrow G \longrightarrow F \longrightarrow 0$$

with $F' \in ({}^{\perp}\mathcal{G})^{\perp}$ and $G \in \mathcal{G}$, and if $\mathcal{G}$ is closed under direct summands, for any $M \in \mathcal{A}$ such that $Ext^{i\geq 1}(M,\ \mathcal{G}) = 0$, then ${}^{\perp}\mathcal{G}$ is an epic precovering class.

Proof. We just need to prove that M has a special $\mathcal{G}$-precover by Theorem 2.8.

Consider the pull-back, where the lower row is $Hom(-,\ \mathcal{F})$-exact,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F' & = & F' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since $\mathcal{G} \subseteq \mathcal{F}$, he lower row of the pull-back above is $Hom(-,\ \mathcal{G})$-exact. By the hypothesis that $Ext^{i\geq 1}(M,\ \mathcal{G}) = 0$ and $Ext^{i\geq 1}(F,\ \mathcal{G}) = 0$. According to the long exact sequence for lower row of the pull-back above, $Ext^{i\geq 1}(M',\ \mathcal{G}) = 0$. Again using the long exact sequence for middle row of the pull-back above,

$$0 = Ext^1(G,\ \mathcal{G}) \to Ext^1(U,\ \mathcal{G}) \to Ext^2(M',\ \mathcal{G}) = 0,$$

$Ext^1(U,\ \mathcal{G}) = 0$. Thus

$$0 \longrightarrow F' \longrightarrow U \longrightarrow M \longrightarrow 0$$

is a special ${}^{\perp}\mathcal{G}$-precover of M. ■

Acknowledgement. The authors wish to express their gratitude to the referee for his/her careful reading and comments which improve the presentation of this article.

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(received 13.08.2011; in revised form 08.02.2012; available online 10.06.2012)

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