МАТЕМАТИЧКИ ВЕСНИК
65, 3 (2013), 394–402
September 2013

оригинални научни рад
research paper

UNIVALENCE CONDITIONS OF GENERAL INTEGRAL OPERATOR

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Abstract. In this paper, we obtain new univalence conditions for the integral operator

$$I_{\xi}^{\alpha_i,\beta_i}(f_1,\dots,f_n)(z) = \left[\xi\int_0^z t^{\xi-1}\left(f_1'(t)\right)^{\alpha_1}\left(\frac{f_1(t)}{t}\right)^{\beta_1}\cdots\left(f_n'(t)\right)^{\alpha_n}\left(\frac{f_n(t)}{t}\right)^{\beta_n}dt\right]^{\frac{1}{\xi}}$$

of analytic functions defined in the open unit disc.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

Very recently, Frasin [13] introduced and studied the following general integral operator

Definition 1.1. Let $\alpha_i, \beta_i \in \mathbb{C}$ for all $i = 1,\dots,n$, $n \in \mathbb{N}$. We let $I_{\xi}^{\alpha_i,\beta_i} : \mathcal{A}^n \to \mathcal{A}$ to be the integral operator defined by

$$I_{\xi}^{\alpha_i,\beta_i}(f_1,\dots,f_n)(z) = \left[\xi\int_0^z t^{\xi-1}\left(f_1'(t)\right)^{\alpha_1}\left(\frac{f_1(t)}{t}\right)^{\beta_1}\cdots\left(f_n'(t)\right)^{\alpha_n}\left(\frac{f_n(t)}{t}\right)^{\beta_n}dt\right]^{\frac{1}{\xi}}, \tag{1.1}$$

where $\xi \in \mathbb{C}\backslash\{0\}$ and $f_i \in A$ for all $i = 1,\dots,n$.

Here and throughout in the sequel every many-valued function is taken with the principal branch.

2010 Mathematics Subject Classification: 30C45
Keywords and phrases: Analytic function; univalent function; integral operator.

Remark 1.2. Note that the integral operator $I_\xi^{\alpha_i,\beta_i}(f_1,\ldots,f_n)(z)$ generalizes the following operators introduced and studied by several authors:

(1) For $\xi = 1$, we obtain the integral operator

$$I^{\alpha_i,\beta_i}(f_1,\ldots,f_n)(z) = \int_0^z (f_1'(t))^{\alpha_1}\left(\frac{f_1(t)}{t}\right)^{\beta_1}\cdots(f_n'(t))^{\alpha_n}\left(\frac{f_n(t)}{t}\right)^{\beta_n} dt$$

introduced and studied by Frasin [14].

(2) For $\xi = 1$ and $\alpha_i = 0$ for all $i = 1,\ldots,n$, we obtain the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\beta_1}\cdots\left(\frac{f_n(t)}{t}\right)^{\beta_n} dt$$

introduced and studied by Breaz and Breaz [3].

(3) For $\xi = 1$ and $\beta_i = 0$ for all $i = 1,\ldots,n$, we obtain the integral operator

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1}\cdots(f_n'(t))^{\alpha_n}\, dt$$

introduced and studied by Breaz et al. [6].

(4) For $\xi = 1$, $n = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $f_1 = f$, we obtain the integral operator

$$F_{\alpha,\beta}(z) = \int_0^z (f'(t))^{\alpha}\left(\frac{f(t)}{t}\right)^{\beta} dt \qquad (\alpha,\beta\in\mathbb{R})$$

studied in [9] (see also [10]).

(5) For $\xi = 1$, $n = 1$, $\alpha_1 = 0$, $\beta_1 = \beta$ and $f_1 = f$, we obtain the integral operator

$$F_\beta(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\beta} dt$$

studied in [7]. In particular, for $\beta = 1$, we obtain Alexander integral operator introduced in [1]

$$I(z) = \int_0^z \frac{f(t)}{t}\, dt$$

(6) For $\xi = 1$, $n = 1$, $\beta_1 = 0$, $\alpha_1 = \alpha$ and $f_1 = f$, we obtain the integral operator

$$G_\alpha(z) = \int_0^z (f'(t))^{\alpha}\, dt$$

studied in [21] (see also [25]).

Many authors studied the problem of integral operators which preserve the class $\mathcal{S}$ (see, for example, [2, 4, 5, 7, 8, 12, 19, 22, 23, 24, 27]).

In particular, Pfaltzgraff [25] and Kim and Merkes [16], have obtained the following univalence conditions for the functions $\int_0^z (f'(t))^{\alpha}\, dt$ and $\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$, respectively.

Theorem 1.3. [25] *If* $f \in \mathcal{A}$ *and* $\alpha \in \mathbb{C}$ *with* $|\alpha| \le 1/4$*, then the function* $\int_0^z (f'(t))^\alpha \, dt$ *is in the class* $\mathcal{S}$*.*

Theorem 1.4. [16] *If* $f \in \mathcal{A}$ *and* $\alpha \in \mathbb{C}$ *with* $|\alpha| \le 1/4$*, then the function* $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ *is in the class* $\mathcal{S}$*.*

In the present paper, we obtain univalence conditions for the integral operator $I_\xi^{\alpha_i,\ \beta_i}(f_1,\ldots,f_n)(z)$ defined by (1.1).

In order to derive our main results, we have to recall here the following lemmas.

Lemma 1.5. [18] *If* $f \in \mathcal{A}$ *satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}\right| < \frac{5}{4}\left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathcal{U}), \tag{1.2}$$

then f *is univalent and starlike in* $\mathcal{U}$*.*

Lemma 1.6. [15, 26] *If* $f \in \mathcal{A}$ *satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| < \frac{1}{2} \qquad (z \in \mathcal{U}), \tag{1.3}$$

then $\left|\frac{zf'(z)}{f(z)}-1\right| < 1$ $(z \in \mathcal{U})$*.*

Lemma 1.7. [11] *If* $f \in \mathcal{S}$ *then*

$$\left|\frac{zf'(z)}{f(z)}\right| < \frac{1+|z|}{1-|z|} \qquad (z \in \mathcal{U}). \tag{1.4}$$

Lemma 1.8. [20] *Let* $\delta \in \mathbb{C}$ *with* $\operatorname{Re}(\delta) > 0$*. If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1-|z|^{2\operatorname{Re}(\delta)}}{\operatorname{Re}(\delta)}\left|\frac{zf''(z)}{f'(z)}\right| \le 1,$$

for all $z \in \mathcal{U}$*, then, for any complex number* ξ*, with* $\operatorname{Re}(\xi) \ge \operatorname{Re}(\delta)$*, the integral operator*

$$F_\xi(z) = \left\{\xi \int_0^z t^{\xi-1} f'(t)\, dt\right\}^{\frac{1}{\xi}}$$

is in the class $\mathcal{S}$*.*

2. Main results

We begin by proving the following theorem.

Theorem 2.1. *Let* $\alpha_i, \beta_i \in \mathbb{C}$ *for all* $i = 1,\ldots,n$ *and each* $f_i \in \mathcal{A}$ *satisfies the condition* (1.2)*. If*

$$\sum_{i=1}^{n}(29\,|\alpha_i| + 16\,|\beta_i|) \le \begin{cases} 4\operatorname{Re}\delta, & \text{if } \operatorname{Re}\delta \in (0,1) \\ 4, & \text{if } \operatorname{Re}\delta \in [1,\infty), \end{cases} \tag{2.1}$$

then, for any complex number ξ*, with* $\operatorname{Re}(\xi) \ge \operatorname{Re}(\delta) > 0$*, the integral operator* $I_\xi^{\alpha_i,\ \beta_i}(f_1,\ldots,f_n)(z)$ *defined by* (1.1) *is in the class* $\mathcal{S}$*.*

Proof. Define a regular function $h(z)$ by

$$h(z)=\int_0^z \prod_{i=1}^{n}\left(f_i'(t)\right)^{\alpha_i}\left(\frac{f_i(t)}{t}\right)^{\beta_i}dt.$$

Then it is easy to see that

$$h'(z)=\prod_{i=1}^{n}\left(f_i'(z)\right)^{\alpha_i}\left(\frac{f_i(z)}{z}\right)^{\beta_i} \tag{2.2}$$

and $h(0)=h'(0)-1=0$. Differentiating both sides of (2.2) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}\right)+\sum_{i=1}^{n}\beta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)$$

and so

$$\frac{zh''(z)}{h'(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\beta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right).$$

From Lemma 1.5, it follows that

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\le \frac{5}{4}\sum_{i=1}^{n}|\alpha_i|\left|\frac{zf_i'(z)}{f_i(z)}\right|+\sum_{i=1}^{n}|\beta_i|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\sum_{i=1}^{n}|\alpha_i|\\ &\le \frac{5}{4}\sum_{i=1}^{n}|\alpha_i|\left[\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+1\right]+\sum_{i=1}^{n}|\beta_i|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\sum_{i=1}^{n}|\alpha_i|\\ &\le \sum_{i=1}^{n}\left[\left(\frac{5}{4}|\alpha_i|+|\beta_i|\right)\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right]+\frac{9}{4}\sum_{i=1}^{n}|\alpha_i|\\ &\le \sum_{i=1}^{n}\left[\left(\frac{5}{4}|\alpha_i|+|\beta_i|\right)\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)\right]+\frac{9}{4}\sum_{i=1}^{n}|\alpha_i|. \end{aligned}\tag{2.3}$$

Multiplying both sides of (2.3) by $\frac{1-|z|^{2\mathrm{Re}\,\delta}}{\mathrm{Re}\,\delta}$, from (1.4), we get

$$\begin{aligned}&\frac{1-|z|^{2\mathrm{Re}\,\delta}}{\mathrm{Re}\,\delta}\left|\frac{zh''(z)}{h'(z)}\right|\\ &\le \frac{1-|z|^{2\mathrm{Re}\,\delta}}{\mathrm{Re}\,\delta}\sum_{i=1}^{n}\left(\frac{5}{4}|\alpha_i|+|\beta_i|\right)\left(\frac{2}{1-|z|}\right)+\frac{9(1-|z|^{2\mathrm{Re}\,\delta})\sum_{i=1}^{n}|\alpha_i|}{4\mathrm{Re}\,\delta}.\end{aligned}\tag{2.4}$$

Suppose that $\mathrm{Re}\,\delta\in(0,1)$. Define a function $\Phi:(0,1)\to\mathbb{R}$ by

$$\Phi(x)=1-a^{2x}\qquad(0<a<1).$$

Then Φ is an increasing function and consequently for $|z|=a;\ z\in\mathcal{U}$, we obtain

$$1-|z|^{2\mathrm{Re}\,\delta}<1-|z|^2 \tag{2.5}$$

for all $z\in\mathcal{U}$.

We thus find from (2.4) and (2.5) that

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{\sum_{i=1}^{n}(5|\alpha_i|+4|\beta_i|)}{\operatorname{Re}\delta} + \frac{9\sum_{i=1}^{n}|\alpha_i|}{4\operatorname{Re}\delta} = \frac{\sum_{i=1}^{n}(29|\alpha_i|+16|\beta_i|)}{4\operatorname{Re}\delta}$$

for all $z \in \mathcal{U}$.

Using the hypothesis (2.1) for $\operatorname{Re}\delta \in (0,1)$, we readily get

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \le 1.$$

Now if $\operatorname{Re}\delta \in [1,\infty)$, we define a function $\Psi : [1,\infty) \to \mathbb{R}$ by

$$\Psi(x) = \frac{1-a^{2x}}{x} \qquad (0 < a < 1).$$

We observe that the function Ψ is decreasing and consequently for $|z| = a;\ z \in \mathcal{U}$, we have

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \le 1-|z|^2 \tag{2.6}$$

for all $z \in \mathcal{U}$. It follows from (2.4) and (2.6) that

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \le \sum_{i=1}^{n}(\frac{29}{4}|\alpha_i|+4|\beta_i|).$$

Using once again the hypothesis (2.1) when $\operatorname{Re}\delta \in [1,\infty)$, we easily get

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \le 1.$$

Finally by applying Lemma 1.8, we conclude that the integral operator $I_{\xi}^{\alpha_i,\,\beta_i}(f_1,\dots,f_n)(z)$ defined by (1.2) is in the class S. ■

Letting $n = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $f_1 = f$ in Theorem 2.1, we have

Corollary 2.2. *Let $\alpha, \beta \in \mathbb{C}$ and $f \in \mathcal{A}$ satisfy the condition* (1.2). *If*

$$29|\alpha| + 16|\beta| \le \begin{cases} 4\operatorname{Re}\delta, & \text{if } \operatorname{Re}\delta \in (0,1) \\ 4, & \text{if } \operatorname{Re}\delta \in [1,\infty), \end{cases}$$

then, for any complex number ξ, with $\operatorname{Re}(\xi) \ge \operatorname{Re}(\delta) > 0$, the integral operator

$$I_{\xi}^{\alpha,\beta}(z) = \left[\xi \int_0^z t^{\xi-\beta-1}\,(f'(t))^{\alpha}\,(f(t))^{\beta}\,dt\right]^{\frac{1}{\xi}}$$

is in the class $\mathcal{S}$.

Letting $\beta = 0$ in Corollary 2.2, we have

Corollary 2.3. *Let $\alpha \in \mathbb{C}$ and $f \in \mathcal{A}$ satisfy the condition* (1.2). *If*

$$|\alpha| \leq \begin{cases} \frac{4\mathrm{Re}\,\delta}{29}, & \text{if } \mathrm{Re}\,\delta \in (0,1) \\ \frac{4}{29}, & \text{if } \mathrm{Re}\,\delta \in [1,\infty), \end{cases}$$

then, for any complex number ξ, with $\mathrm{Re}(\xi) \geq \mathrm{Re}(\delta) > 0$, the integral operator

$$I_\xi^\alpha(z) = \left[\xi \int_0^z t^{\xi-1} \left(f'(t)\right)^\alpha \, dt\right]^{\frac{1}{\xi}}$$

is in the class $\mathcal{S}$.

Letting $\alpha = 0$ in Corollary 2.2, we have

Corollary 2.4. *Let $\beta \in \mathbb{C}$ and $f \in \mathcal{A}$ satisfy the condition* (1.2). *If*

$$|\beta| \leq \begin{cases} \frac{\mathrm{Re}\,\delta}{4}, & \text{if } \mathrm{Re}\,\delta \in (0,1) \\ \frac{1}{4}, & \textit{if } \mathrm{Re}\,\delta \in [1,\infty), \end{cases}$$

then, for any complex number ξ, with $\mathrm{Re}(\xi) \geq \mathrm{Re}(\delta) > 0$, the integral operator

$$I_\xi^\beta(z) = \left[\xi \int_0^z t^{\xi-1} \left(\frac{f(t)}{t}\right)^\beta \, dt\right]^{\frac{1}{\xi}}$$

is in the class $\mathcal{S}$.

Letting $\xi = \delta = 1$ in Corollary 2.3, we have

Corollary 2.5. *If $f \in \mathcal{A}$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 4/29 \approx 0.137$, then the function $\int_0^z (f'(t))^\alpha \, dt$ is in the class $\mathcal{S}$.*

Remark 2.6. If we let $\xi = \delta = 1$ in Corollary 2.4, then we have Theorem 1.4.

Next, we obtain the following univalence condition for the integral operator $I_\xi^{\alpha_i,\,\beta_i}(f_1, \ldots, f_n)(z)$ defined by (1.1) when $\beta_i = 1 - \alpha_i$ for all $i = 1, \ldots, n$.

Theorem 2.7. *Let $\alpha_i, \beta_i \in \mathbb{C}$ for all $i = 1, \ldots, n$ and each $f_i \in \mathcal{A}$ satisfy the condition* (1.3). *If $\mathrm{Re}\,\delta \geq n$, $n \in \mathbb{N}$, $\delta \in \mathbb{C}$ with*

$$\sum_{i=1}^n |\alpha_i| \leq 2\mathrm{Re}\,\delta - 2n \tag{2.7}$$

then, for any complex number ξ, with $\mathrm{Re}(\xi) \geq \mathrm{Re}(\delta) > 0$, the integral operator

$$I_\xi^{\alpha_i}(f_1, \ldots, f_n)(z) = \left[\xi \int_0^z t^{\xi-1} \prod_{i=1}^n \left(t\frac{f_i'(t)}{f_i(t)}\right)^{\alpha_i} \left(\frac{f_i(t)}{t}\right) dt\right]^{\frac{1}{\xi}} \tag{2.8}$$

is in the class $\mathcal{S}$.

Proof. Define a regular function $G(z)$ by

$$G(z) = \int_0^z \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t}\right)^{1-\alpha_i} dt. \tag{2.9}$$

. Then it follows from (2.9) that

$$\frac{zG''(z)}{G'(z)} = \sum_{i=1}^{n} \alpha_i \left(1 + \frac{zf_i''(z)}{f_i'(z)} - \frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{2.10}$$

Using Lemma 1.6, from (2.10), we have

$$\left|\frac{zG''(z)}{G'(z)}\right| \leq \frac{1}{2}\sum_{i=1}^{n} |\alpha_i| + n \tag{2.11}$$

Multiply both sides of (2.11) by $\frac{1-|z|^{2\mathrm{Re}\,\delta}}{\mathrm{Re}\,\delta}$, we obtain

$$\frac{1-|z|^{2\mathrm{Re}\,\delta}}{\mathrm{Re}\,\delta}\left|\frac{zG''(z)}{G'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\,\delta}}{\mathrm{Re}\,\delta}\left(\frac{1}{2}\sum_{i=1}^{n} |\alpha_i| + n\right)$$
$$\leq \frac{1}{2\mathrm{Re}\,\delta}\left(\sum_{i=1}^{n} |\alpha_i| + 2n\right),$$

which, in the light of the hypothesis (2.7) yields

$$\frac{1-|z|^{2\mathrm{Re}\,\delta}}{\mathrm{Re}\,\delta}\left|\frac{zG''(z)}{G'(z)}\right| \leq 1.$$

Finally by applying Lemma 1.8, we conclude that the integral operator $I_\xi^{\alpha_i}(f_1,\ldots,f_n)(z)$ defined by (2.8) is in the class $\mathcal{S}$. ■

Letting $n=1$, $\alpha_1=\alpha$ and $f_1=f$ in Theorem 2.7, we have

COROLLARY 2.8. *Let $f \in \mathcal{A}$ satisfies the condition* (1.3), $\alpha, \delta \in \mathbb{C}$ *and* $\mathrm{Re}\,\delta \geq 1$. *If*

$$|\alpha| \leq 2\mathrm{Re}\,\delta - 2$$

then, for any complex number ξ, with $\mathrm{Re}(\xi) \geq \mathrm{Re}(\delta)$, *the integral operator*

$$I_\xi^{\alpha}(z) = \left[\xi \int_0^z t^{\xi+\alpha-2} (f'(t))^{\alpha} (f(t))^{1-\alpha}\, dt\right]^{\frac{1}{\xi}}$$

is in the class $\mathcal{S}$.

Letting $\alpha = 0$ in Corollary 2.8, we have

COROLLARY 2.9. *Let $f \in \mathcal{A}$ satisfies the condition* (1.3). *If* $\delta \in \mathbb{C}$, $\mathrm{Re}\,\delta \geq 1$ *then, for any complex number ξ, with* $\mathrm{Re}(\xi) \geq \mathrm{Re}(\delta)$, *the integral operator*

$$I_\xi(z) = \left[\xi \int_0^z t^{\xi-2} f(t)\, dt\right]^{\frac{1}{\xi}}$$

is in the class $\mathcal{S}$.

Letting $\alpha = 1$ in Corollary 2.8, we have

Corollary 2.10. *Let $f \in \mathcal{A}$ satisfies the condition* (1.3)*. If $\delta \in \mathbb{C}$, $\operatorname{Re}\delta \geq 3/2$ then, for any complex number ξ, with $\operatorname{Re}(\xi) \geq \operatorname{Re}(\delta)$, the integral operator*

$$F_\xi(z) = \left[\xi \int_0^z t^{\xi-1} f'(t)\, dt\right]^{\frac{1}{\xi}}$$

is in the class $\mathcal{S}$.

Acknowledgement. The authors would like to thank the referee for his/her helpful comments and suggestions.

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(received 23.08.2011; in revised form 21.02.2012; available online 10.06.2012)

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